Title: A Structure for Quantum Measurements and Observables

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Abstract:

A STRUCTURE FOR QUANTUM MEASUREMENTS AND OBSERVABLES

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1. Introduction

Structures are very important in mathematics and physics. For example, the symmetries of a physical system have the structures of a group. Knowing this structure allows us to invoke the powerful results of group theory. These results provide us with a deeper understanding of the symmetries and properties of the original physical system. This work presents an additive and product structure for quantum measurements.

The additive structure generalizes the orthosum of effects in effect algebras and preserves sums of expectations. The additive structure also provides a natural order for measurements and it is shown that an initial interval of measurements forms an effect algebra. In certain cases, this effect algebra retains the properties of the original effect algebra on which the measurements are defined. A sequential product of measurements is also introduced. Observables are a special case of measurements and a new order for Hilbert space observables is defined. We point out that a structure for quantum operations is unclear and needs further development.

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2. Definitions

The simplest type of measurement is a yes-no (or 1-0) measurement or effect. More general measurements and observables can be constructed using these effects. The set of effects for a quantum system can be organized into a mathematical structure called an effect algebra. We now review the definition of an effect algebra and the related concepts of generalized effect algebras and orthoalgebras. The main algebraic operations in these structures is an **orthosum** $a \oplus b$ which is a partial binary operation on the set of effects. If $a \oplus b$ is defined we write $a \perp b$ and say that a and b are **orthogonal**. Roughly speaking, $a \oplus b$ corresponds to a parallel combination of the two effects a and b.

A **generalized effect algebra** is an algebraic system $(E, 0, \oplus)$ where E is a set, $0 \in E$ and $\oplus$ is a partial binary operation on E satisfying:

(GEA1) If $a \perp b$ then $b \perp a$ and $b \oplus a = a \oplus b$.

(GEA2) If $b \perp c$ and $a \perp (b \oplus c)$ then $a \perp b$, $c \perp (a \oplus b)$ and

$$(a \oplus b) \oplus c = a \oplus (b \oplus c).$$

(GEA3) $0 \perp a$ for all $a \in E$ and $0 \oplus a = a$.

(GEA4) If $a \oplus b = a \oplus c$ then $b = c$.

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(OA) If $a \perp a$ then $a = 0$.

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An **orthoalgebra** is an effect algebra that satisfies (OA).

Lemma 2.1. Every effect algebra is a generalized effect algebra and every orthoalgebra is a generalized orthoalgebra.

For a generalized effect algebra E, we define $a \leq b$ if there exists a $c \in E$ such that $a \oplus c = b$. This unique c is denoted by $c = b \ominus a$.

Lemma 2.2. If E is a generalized effect algebra, then $(E, \leq)$ is a poset and $0 \leq a$ for every $a \in E$.

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Theorem 2.3. If E is a generalized effect algebra (orthoalgebra) and $a \in E$ with $a \neq 0$, then $\{[0, a], 0, a, \oplus_a\}$ is an effect algebra (orthoalgebra). Moreover, the order on $[0, a]$ is the restriction of the order on E to $[0, a]$.

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An **effect algebra** is an algebraic system $(E, 0, 1, \oplus)$ where E is a set, $0, 1 \in E$ with $0 \neq 1$ and $\oplus$ is a partial binary operation on E that satisfies (GEA1), (GEA2) and

(EA1) For every $a \in E$ there exists a unique $a' \in E$ such that $a \perp a'$ and $a \oplus a' = 1$.

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An **orthoalgebra** is an effect algebra that satisfies (OA).

Lemma 2.1. Every effect algebra is a generalized effect algebra and every orthoalgebra is a generalized orthoalgebra.

For a generalized effect algebra E, we define $a \leq b$ if there exists a $c \in E$ such that $a \oplus c = b$. This unique c is denoted by $c = b \ominus a$.

Lemma 2.2. If E is a generalized effect algebra, then $(E, \leq)$ is a poset and $0 \leq a$ for every $a \in E$.

Let E be a generalized effect algebra and let $a \in E$ with $a \neq 0$. In the interval $[0, a] = \{b \in E: b \leq a\}$ defined $b \oplus_a c = b \oplus c$ whenever $b \perp c$ and $b \oplus c \leq a$.

Theorem 2.3. If E is a generalized effect algebra (orthoalgebra) and $a \in E$ with $a \neq 0$, then $\{[0, a], 0, a, \oplus_a\}$ is an effect algebra (orthoalgebra). Moreover, the order on $[0, a]$ is the restriction of the order on E to $[0, a]$.

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If E and F are effect algebras, a map $\phi \colon E \to F$ is **additive** if $a, b \in E$ with $a \perp b$ implies $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. An additive map $\phi \colon E \to F$ satisfying $\phi(a) = 1$ is a **morphism**. A **monomorphism** is a morphism $\phi \colon E \to F$ such that $\phi(a) \perp \phi(b)$ implies $a \perp b$. A surjective monomorphism is an **isomorphism**.

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In order to describe series combinations of effects we introduce a sequential product $a \circ b$. For a binary operation $a \circ b$, if $a \circ b = b \circ a$ we write $a \mid b$. A **sequential effect algebra** (SEA) is a system $(E, 0, 1, \oplus, \circ)$ where $(E, 0, 1, \oplus)$ is an effect algebra and $\circ$ is a binary operation on E satisfying:

(SEA1) $\phi(b) = a \circ b$ is additive for every $a \in E$.

(SEA2) $1 \circ a = a$.

(SEA3) If $a \circ b = 0$ then $a \mid b$.

(SEA4) If $a \mid b$ then $a \mid b'$ and $a \circ (b \circ c) = (a \circ b) \circ c$ for every $c \in E$.

(SEA5) If $c \mid a$ and $c \mid b$ then $c \mid (a \circ b)$ and $c \mid (a \oplus b)$.

$\mathcal{E}(\mathcal{H})$ is SEA under the sequential product $A \circ B = A^{1/2}BA^{1/2}$.

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3. Orthosums of Measurements

$\mathcal{B}(\mathbb{R}) = (\mathcal{B}(\mathbb{R}), \emptyset, \mathbb{R}, \oplus)$ is a σ-effect algebra in which $\Delta_1 \perp \Delta_2$ if $\Delta_1 \cap \Delta_2 = \emptyset$ in which case $\Delta_1 \oplus \Delta_2 = \Delta_1 \cup \Delta_2$. In this section $\mathcal{E}$ will denote a σ-effect algebra. A **measurement** X on $\mathcal{E}$ is an effect-valued measure on $\mathcal{B}(\mathbb{R})$. That is, $X: \mathcal{B}(\mathbb{R}) \to \mathcal{E}$ is a σ-morphism in the sense that $X(\mathbb{R}) = 1$ and

$$X\left(\bigoplus_{i=1}^{\infty} \Delta_i\right) = \bigoplus_{i=1}^{\infty} X(\Delta_i)$$

$X(\Delta)$ is the effect observed when X has a value in $\Delta \in \mathcal{B}(\mathbb{R})$. We denote the set of measurements on $\mathcal{E}$ by $\mathcal{M}(\mathcal{E})$. A **finite measurement** X has a finite set of values

$$\Lambda(X) = \{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbb{R}$$

such that $X(\{\lambda_i\}) = a_i \in \mathcal{E}$ where $a_i \neq 0$ and $\oplus a_i = 1$. Then for any $\Delta \in \mathcal{B}(\mathbb{R})$,

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If $X \perp Y$ we define $X \oplus Y : \mathcal{B}(\mathbb{R}) \to \mathcal{E}$ by

$$(X \oplus Y)(\Delta) = X(\Delta) \oplus Y(\Delta) \quad \text{if } 0 \notin \Delta$$

$$(X \oplus Y)(\Delta) = (X \oplus Y)(\Delta \setminus \{0\}) \oplus p(X,Y)' \quad \text{if } 0 \in \Delta$$

Theorem 3.1. If $X \perp Y$, then $X \oplus Y \in \mathcal{M}(\mathcal{E})$ and $\left(\mathcal{M}(\mathcal{E}), \widehat{0}, \oplus\right)$ is a generalized effect algebra.

Denote the set of states on $\mathcal{E}$ by $\Omega(\mathcal{E})$. For $s \in \Omega(\mathcal{E})$, $X \in \mathcal{M}(\mathcal{E})$, $s[X(\Delta)]$ is the probability X has a value in Δ when the system is in state s. The **distribution of X in the state** s is the probability measure $s \circ X : \mathcal{B}(\mathbb{R}) \to [0,1]$. The **expectation of X in the state** s is

$$E_s(X) = \int_{\mathbb{R}} \lambda s[X(d\lambda)]$$

if the integral exists. The next result shows that the orthosum on $\mathcal{M}(\mathcal{E})$ has two important properties; it extends the orthosum on $\mathcal{E}$ and provides additive expectations.

Theorem 3.2. (i) If $a \perp b$ then $\widehat{a} \oplus \widehat{b} = (a \oplus b)^\wedge$. (ii) If $X \perp Y$ and $E_s(X)$, $E_s(Y)$ exist, then $E_s(X \oplus Y) = E_s(X) + E_s(Y)$.

Theorem 3.3. For $X, Y \in \mathcal{M}(\mathcal{E})$, $X \leq Y$ iff $X(\Delta) \leq Y(\Delta)$ for every $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$.

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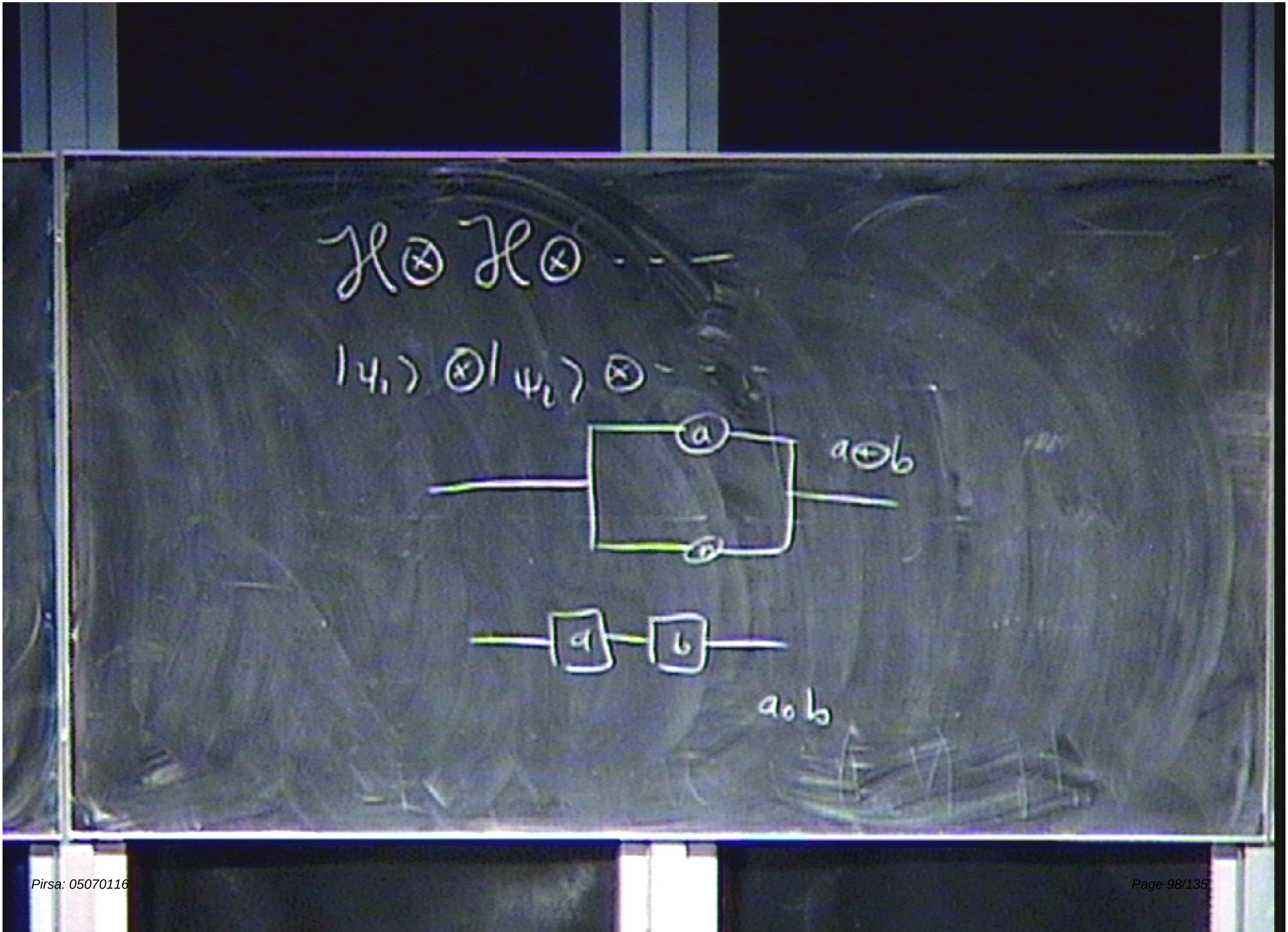
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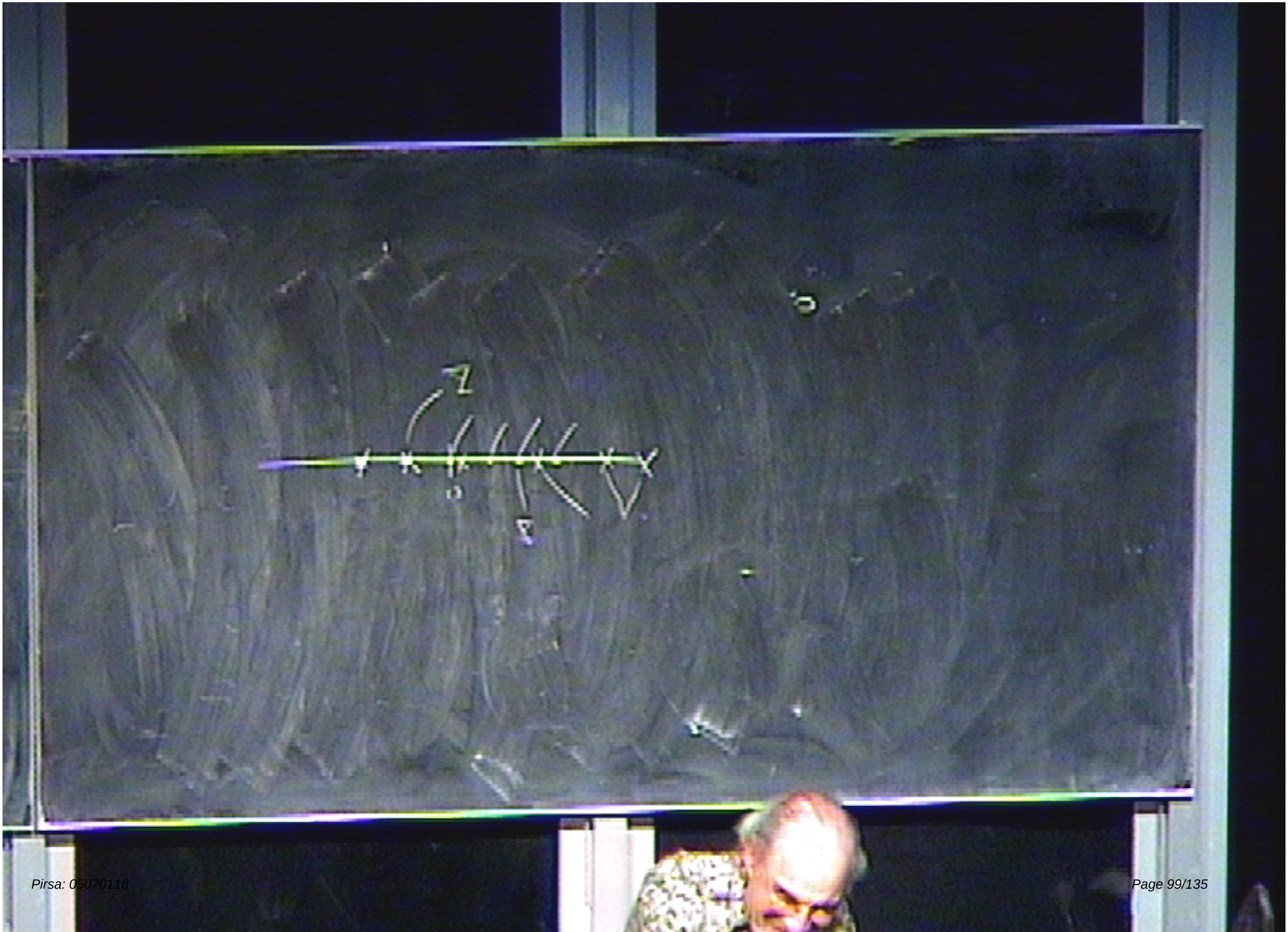

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6. Finite Dimensional Observables

In this section we assume that $\dim \mathcal{H} = n < \infty$. In this case we can obtain stronger results. If $A, B \in \mathcal{S}(\mathcal{H})$ and $B \leq A$ then by Theorem 5.2, A and B are simultaneously diagonalizable and can be represented by matrices $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, $B = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$, where $\alpha_i = \lambda_i$ whenever $\alpha_i \neq 0$. We say that $A \in \mathcal{S}(\mathcal{H})$ is **nondegenerate** if A has no repeated nonzero eigenvalues.

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Even when $\dim \mathcal{H} < \infty$, $A \vee B$ need not exist. However, we have the following:

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