Title: Discrete phase space based on finite fields

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Abstract:

Discrete Phase Space Based on Finite Fields

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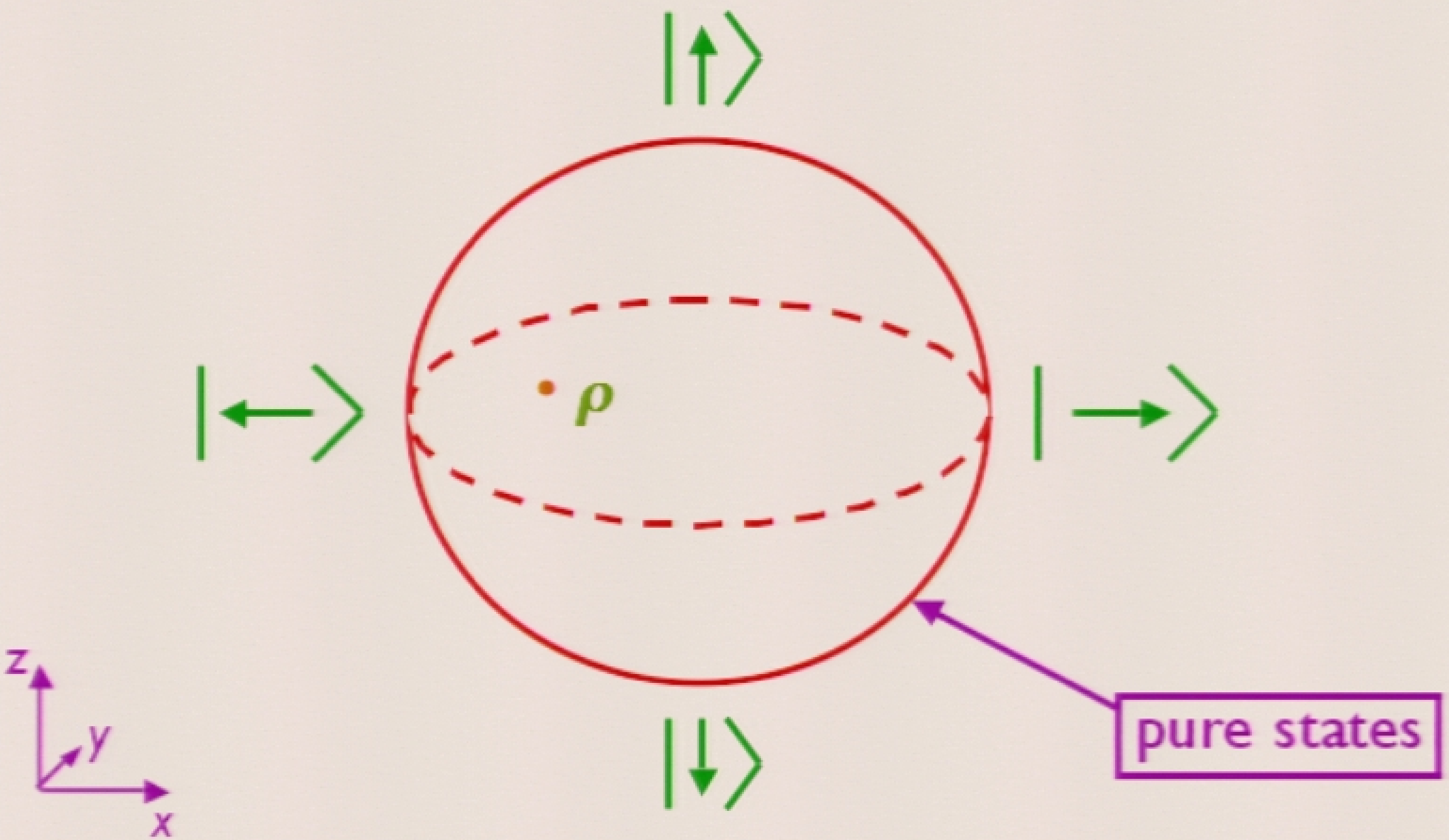
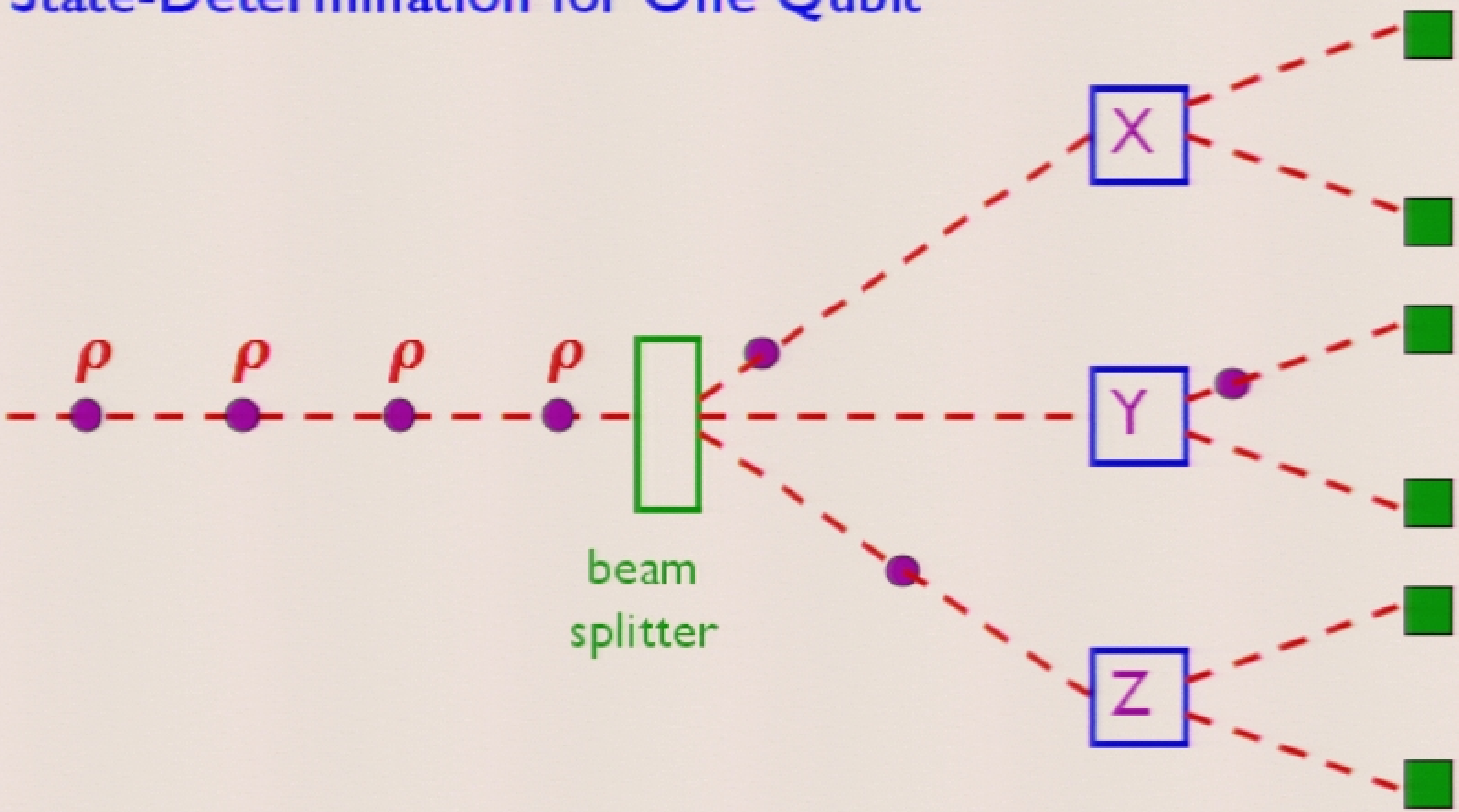
I. State Determination
II. The Wigner Function
III. Discrete Phase Space

General State of One Qubit

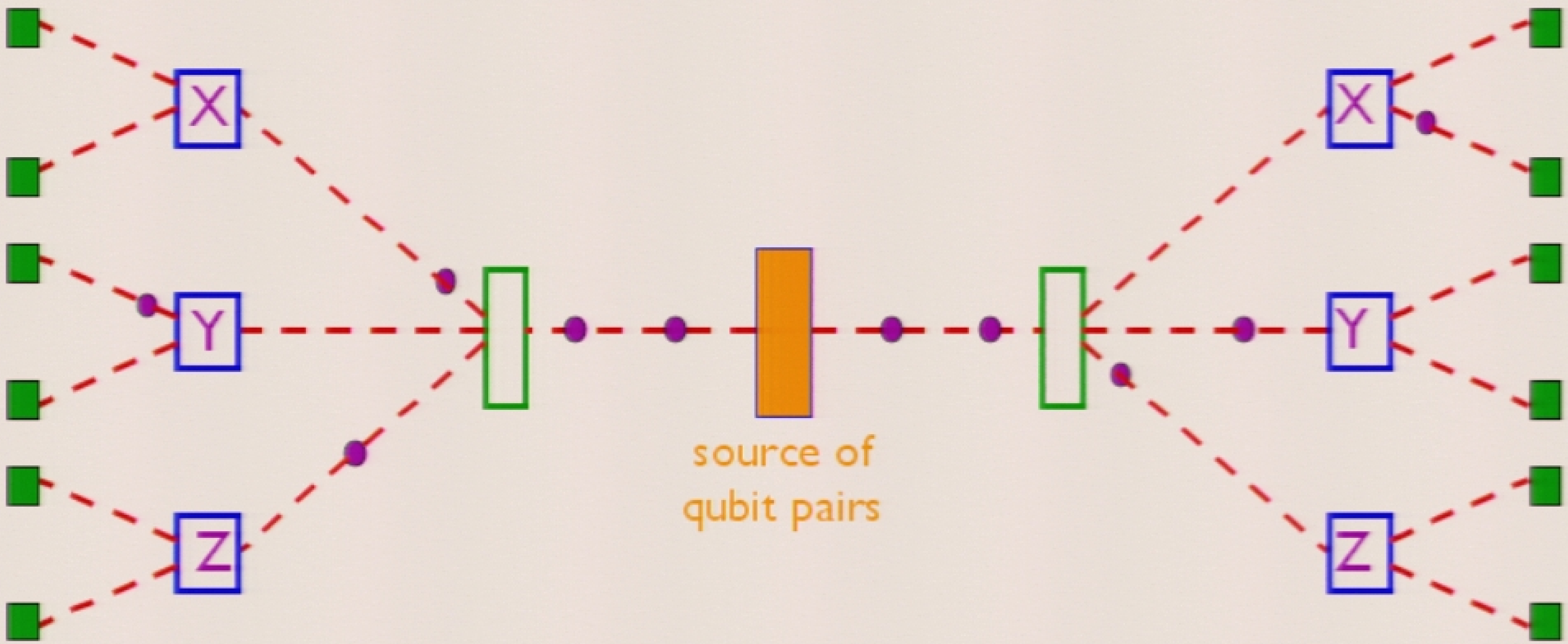


State-Determination for One Qubit



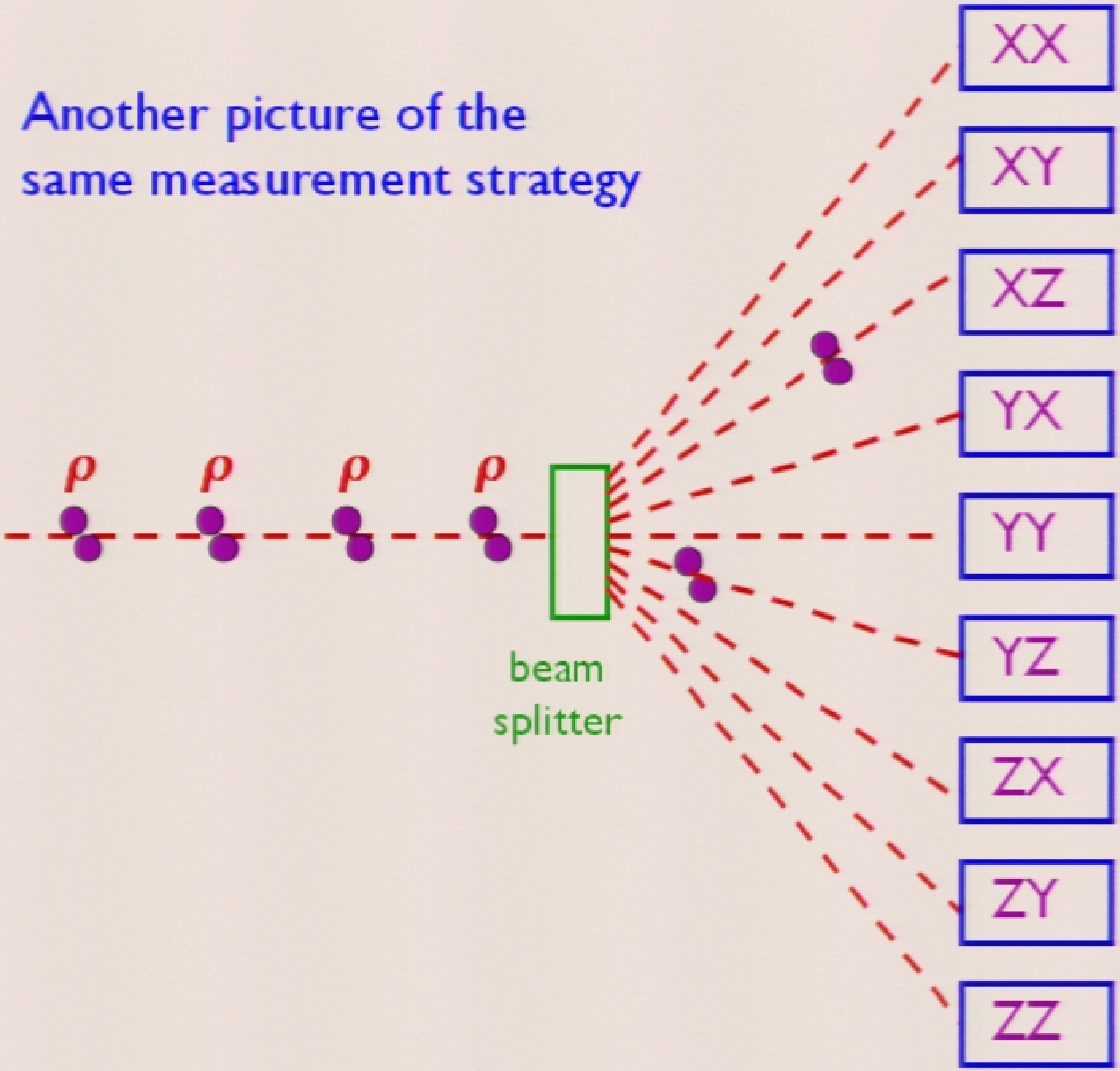
Each measurement defines an orthogonal basis, and the three bases are "mutually unbiased."

State-Determination for a Pair of Qubits



We are in effect making 9 measurements on pairs:
XX, XY, XZ, YX, YY, YZ, ZX, ZY, ZZ

Another picture of the same measurement strategy



Can We Be More "Efficient"?

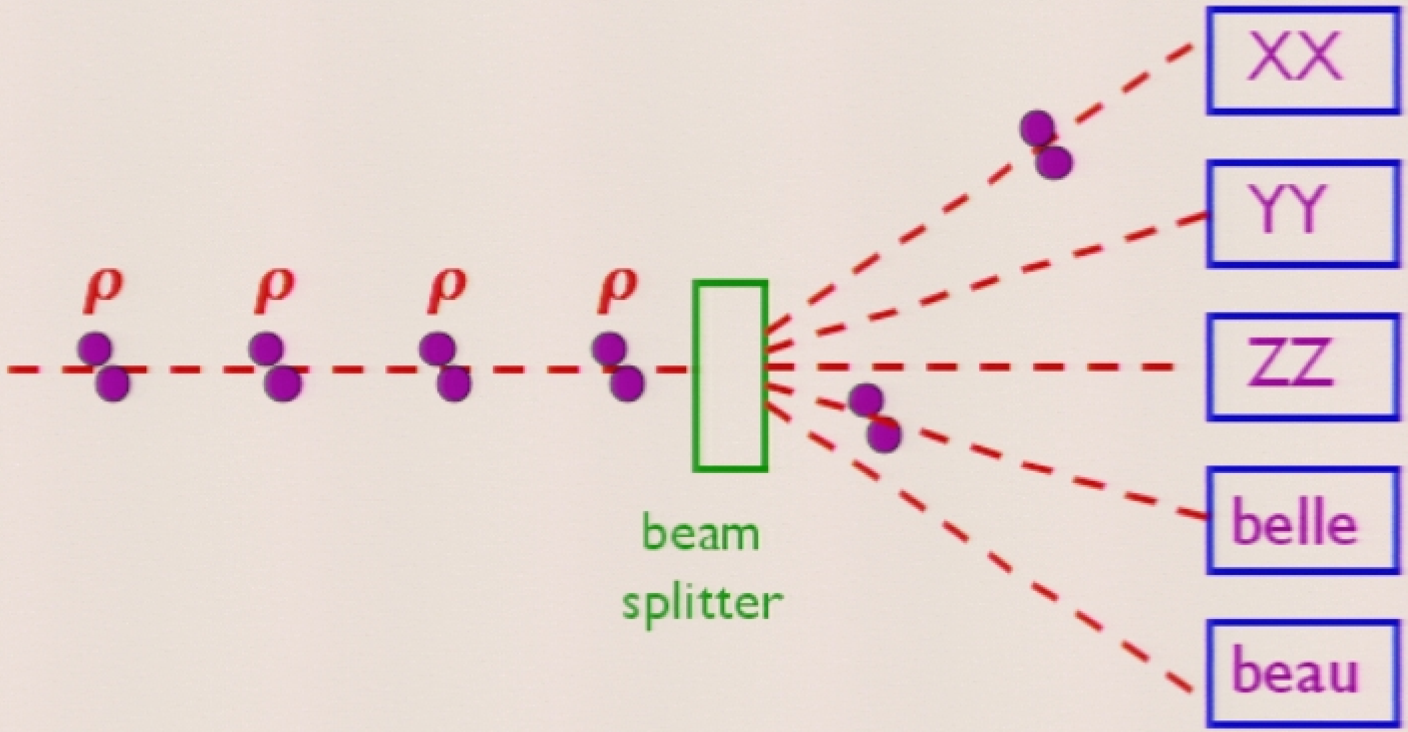
- A density matrix for 2 qubits contains $4^2-1=15$ real parameters.
- A four-outcome measurement provides $4 - 1 = 3$ independent probabilities.
- So we need $15/3 = 5$ different measurements.

Ideally these measurements will be mutually unbiased:

Each eigenstate of one of them should be an *equal-magnitude* superposition of the eigenstates of any other one.

There *Do* Exist Five Mutually Unbiased Bases

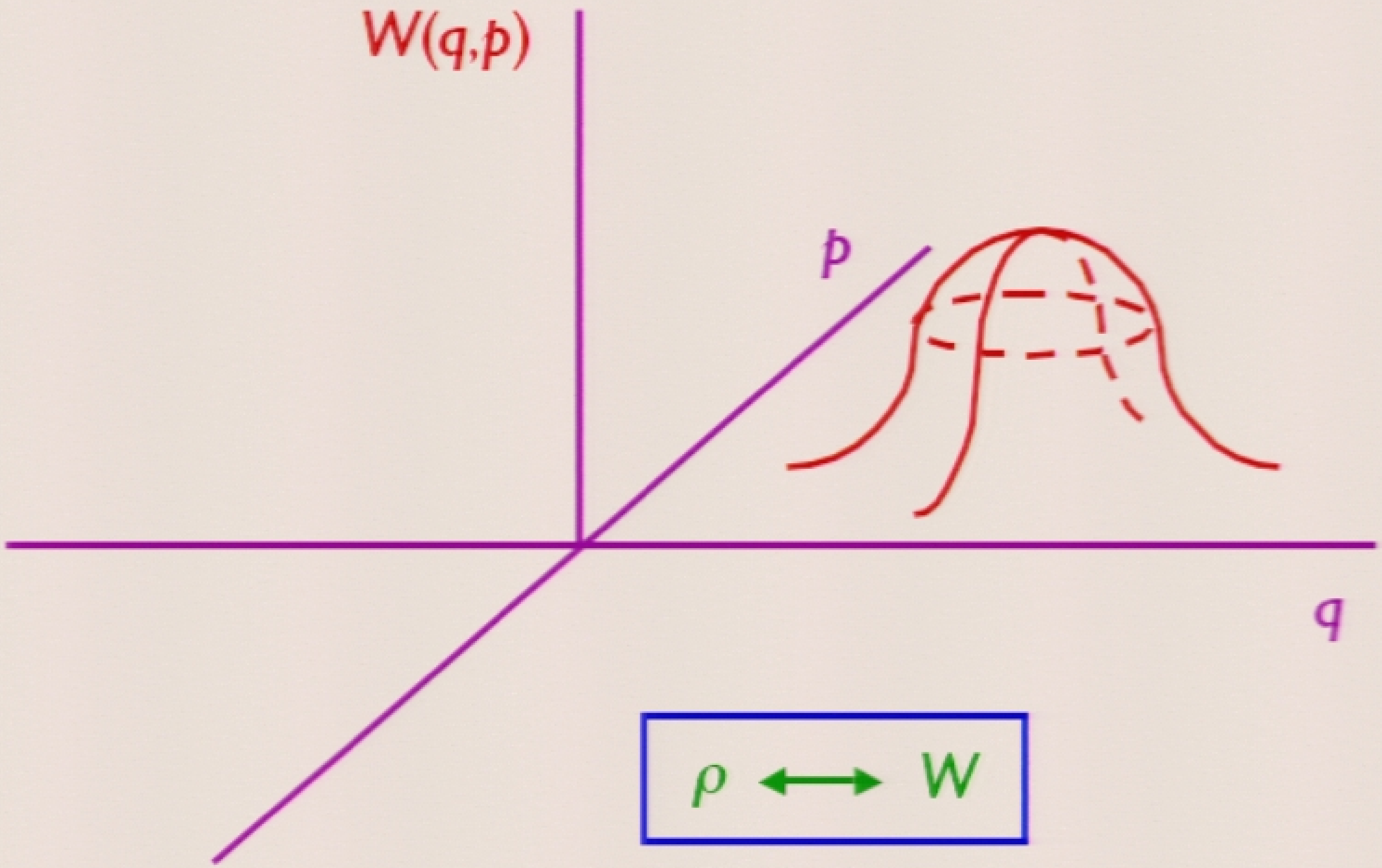
(Wootters & Fields 1987, Calderbank *et al.* 1997, Lawrence *et al.* 2001, Bandyopadhyay *et al.* 2001, Pittenger & Rubin 2003)



belle: $|\uparrow\rightarrow\rangle + i|\downarrow\leftarrow\rangle$, $|\uparrow\rightarrow\rangle - i|\downarrow\leftarrow\rangle$, $|\uparrow\leftarrow\rangle + i|\downarrow\rightarrow\rangle$, $|\uparrow\leftarrow\rangle - i|\downarrow\rightarrow\rangle$

beau: $|\uparrow \text{in}\rangle + |\downarrow \text{out}\rangle$, $|\uparrow \text{in}\rangle - |\downarrow \text{out}\rangle$, $|\uparrow \text{out}\rangle + |\downarrow \text{in}\rangle$, $|\uparrow \text{out}\rangle - |\downarrow \text{in}\rangle$

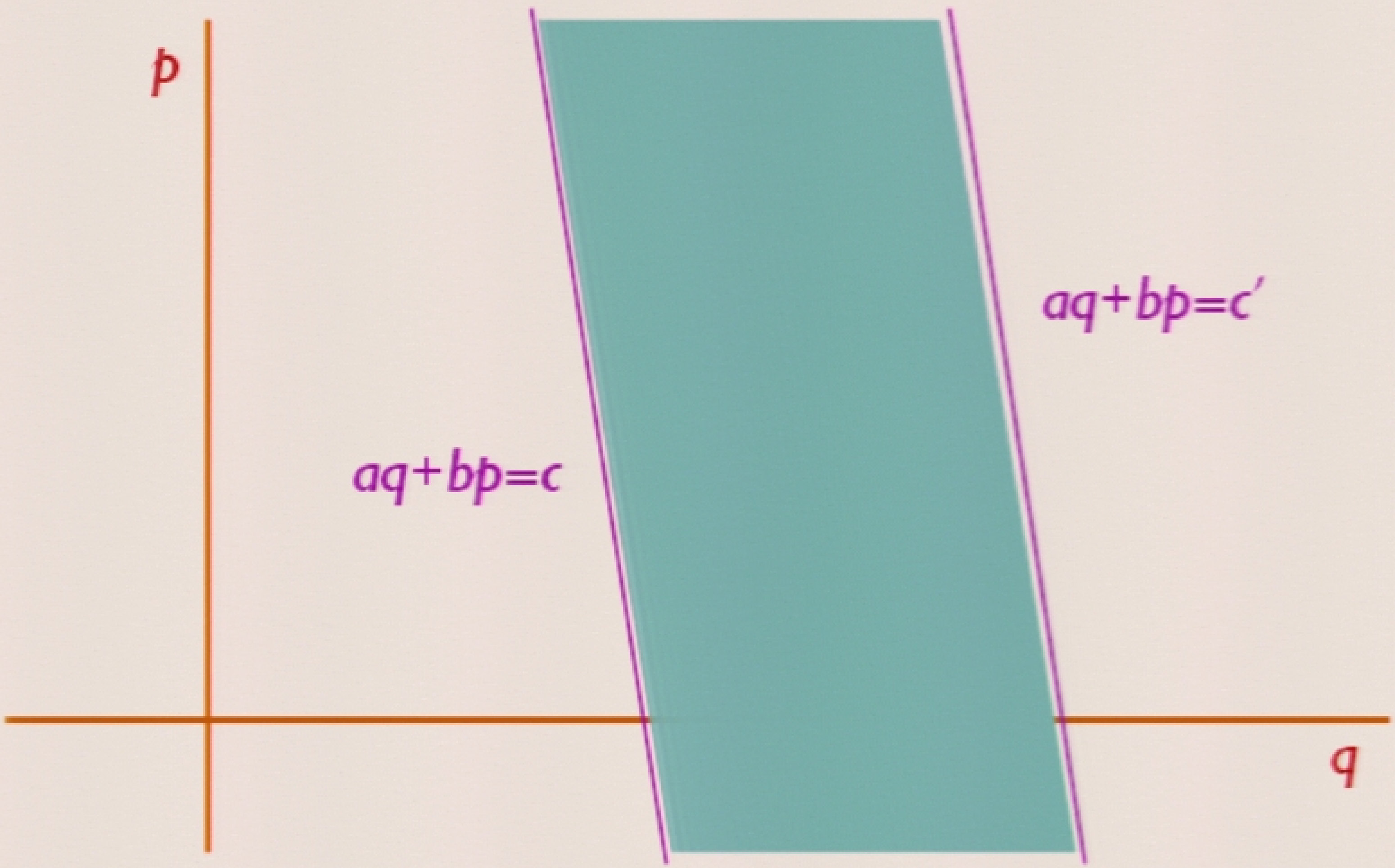
The Wigner Function



$$\rho \longleftrightarrow W$$

Translational covariance: $\rho' = T_{(a,b)}\, \rho\, T^{\dagger}_{(a,b)} \longrightarrow W'(q,p) = W(q-a,\, p-b)$

The Wigner function and probabilities



$\int W(q,p)\ dq\ dp$ = probability that the *operator* $aQ+bP$

shaded area

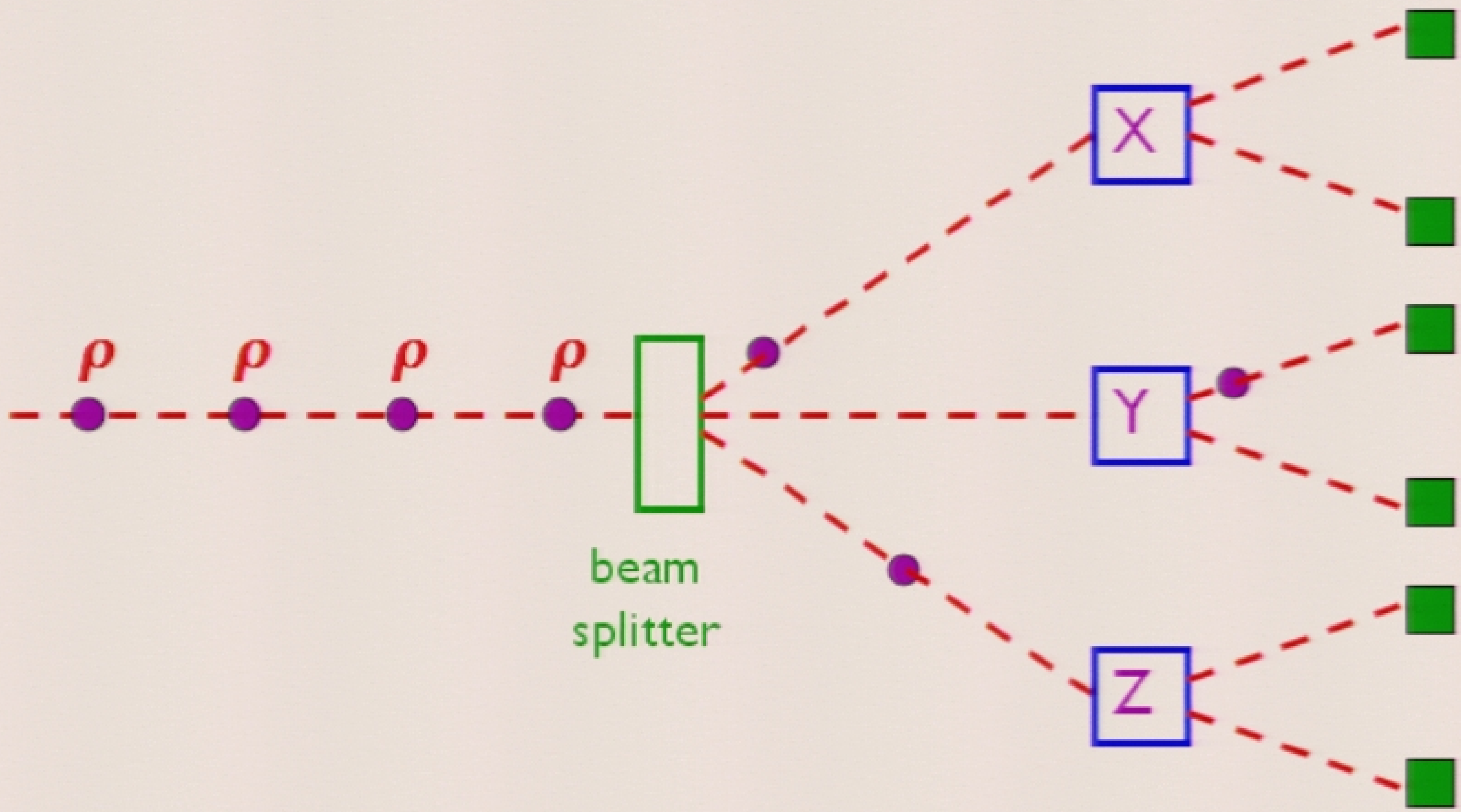
takes a value between c and c'.

State Determination for a continuous degree of freedom.

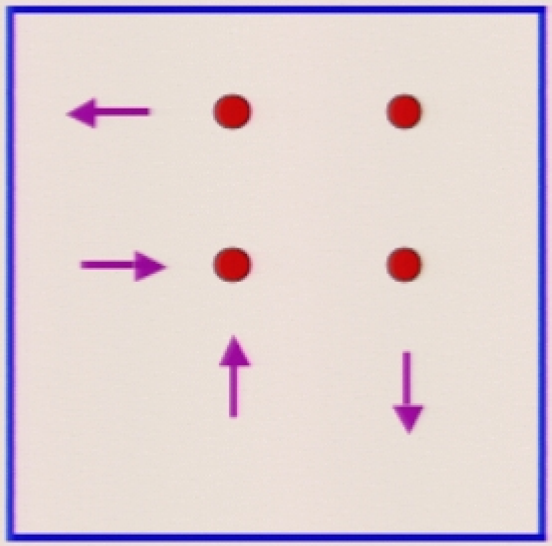
- Measure the probability distribution for each $aQ + bP$. Get the integral of the Wigner function over each direction.
- Reconstruct the Wigner function from these integrals.
- Obtain the density matrix from the Wigner function.

This is tomography.

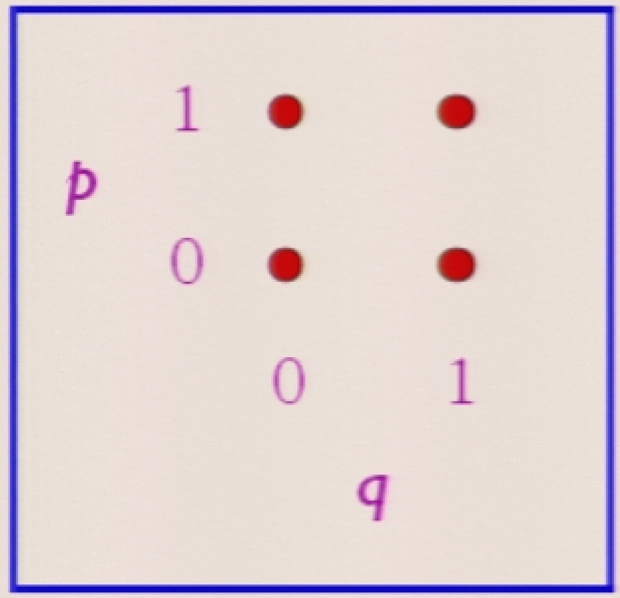
Now back to a single qubit.
Do *these* measurements arise from a phase space?



Phase Space for a Single Qubit



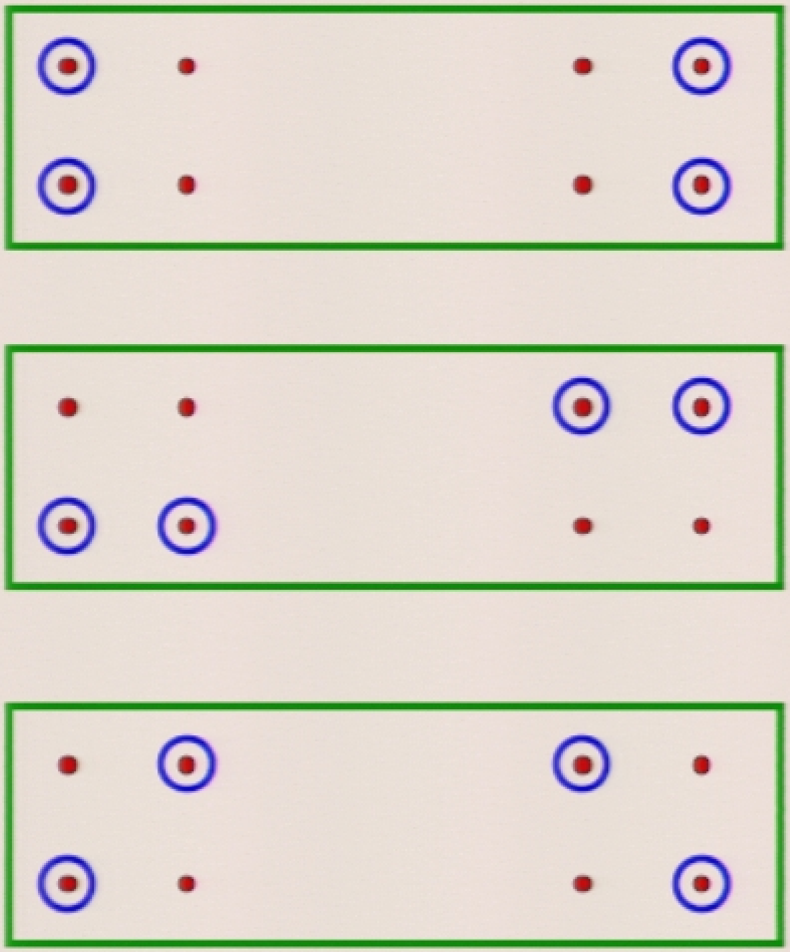
OR



q and p take values in $\{0,1\}$, with arithmetic mod 2.

The Wigner function representing the state of a qubit will be a real function on this discrete phase space.

The three "striations" of the one-qubit phase space



Each striation will be associated with an orthogonal basis.

Each line will be associated with a *vector* in that basis.

The *Wigner function* is defined so that its sum over any line is the probability of the associated vector.

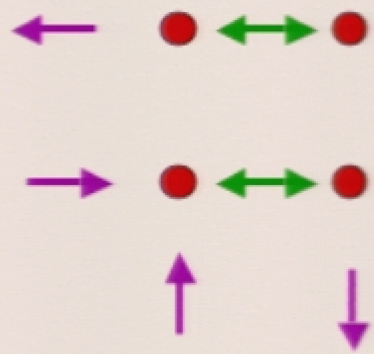
The next few slides are aimed at assigning, in a natural way, a state vector to each line of phase space.

Once this is done, the definition of the Wigner function will be determined.

Translation Operators (for defining translational covariance)

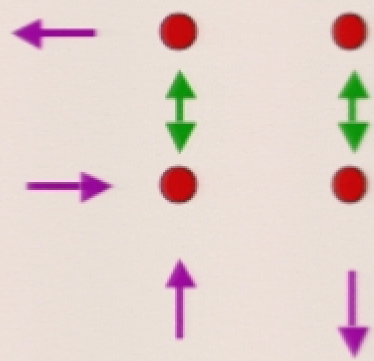
$$T_{(1,0)} = X:$$

$$X|\uparrow\rangle = |\downarrow\rangle$$
$$X|\downarrow\rangle = |\uparrow\rangle$$



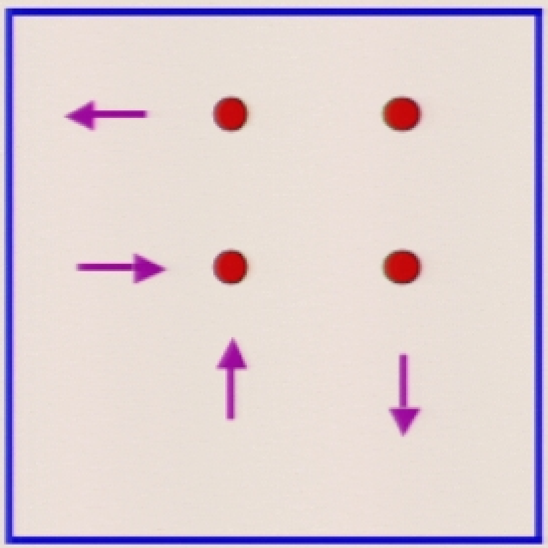
$$T_{(0,1)} = Z:$$

$$Z|\uparrow\rangle = |\uparrow\rangle$$
$$Z|\downarrow\rangle = -|\downarrow\rangle$$



What basis goes with the diagonal striation?

These lines are invariant under a vertical translation followed by a horizontal translation.



So we associate them with the eigenvectors of $T_{(1,1)} = XZ$. But $XZ = -iY$.

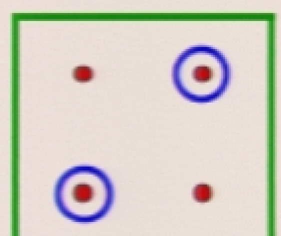
So this striation goes with a measurement of the *y component* of spin.

Some freedom in the definition of the Wigner function

Which eigenvector of Y goes with which line?

There are two possible assignments of vectors to lines, so two possible definitions of the Wigner function.

In the following slide, we arbitrarily choose to associate the $+1$ eigenstate of Y with the line



A Wigner Function for a Single Qubit

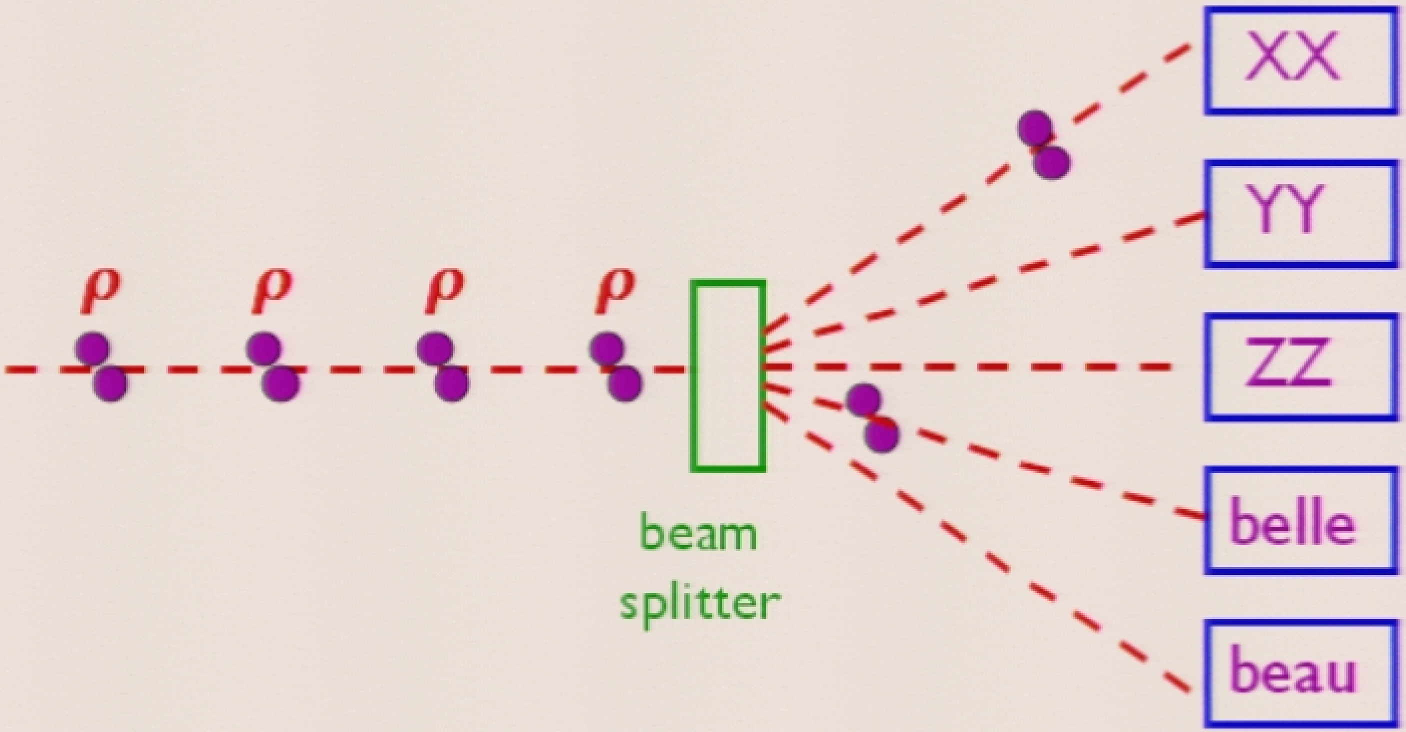
The up state:

	↑	↓
←	1/2	0
→	1/2	0

The negative eigenstate of $X + Y + Z$:

	↑	↓
←	0.394	0.394
→	−0.183	0.394

Now back to two qubits.
Do *these* measurements arise from a phase space?



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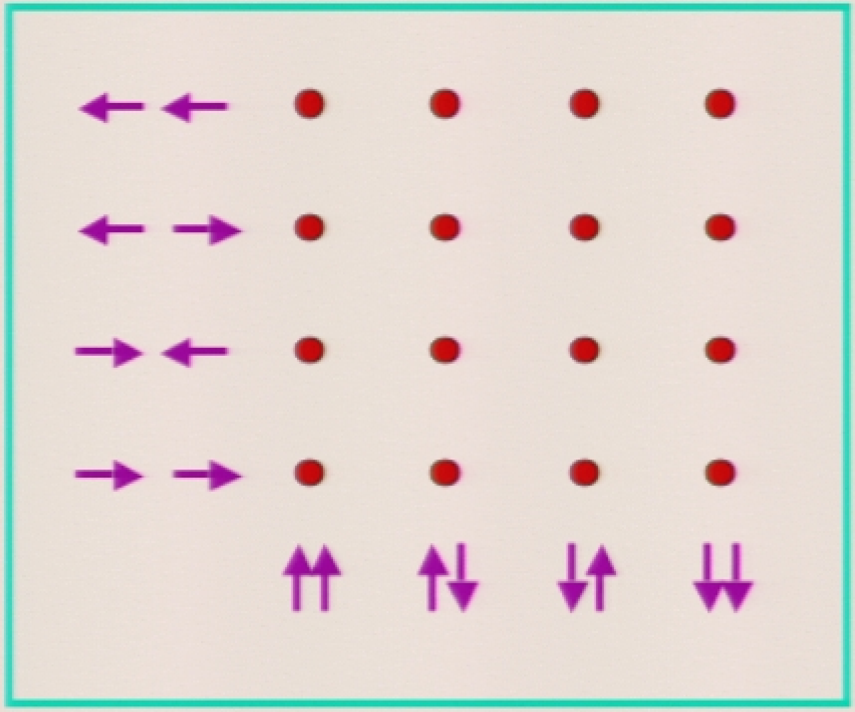
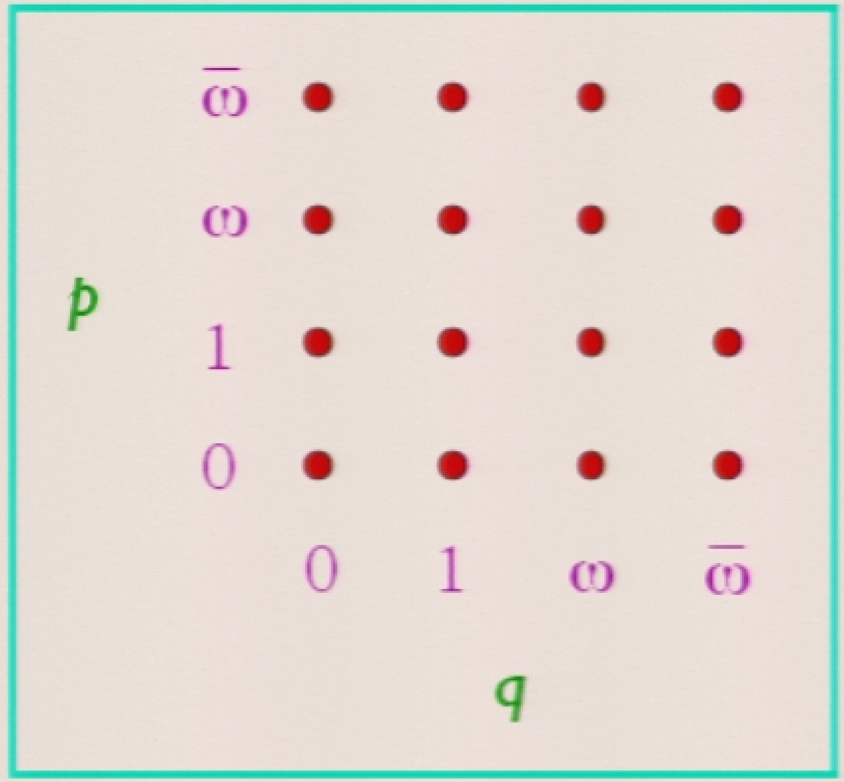
Phase Space for Two Qubits





The "numbers" $\{0, 1, \omega, \bar{\omega}\}$ constitute the 4-element field.

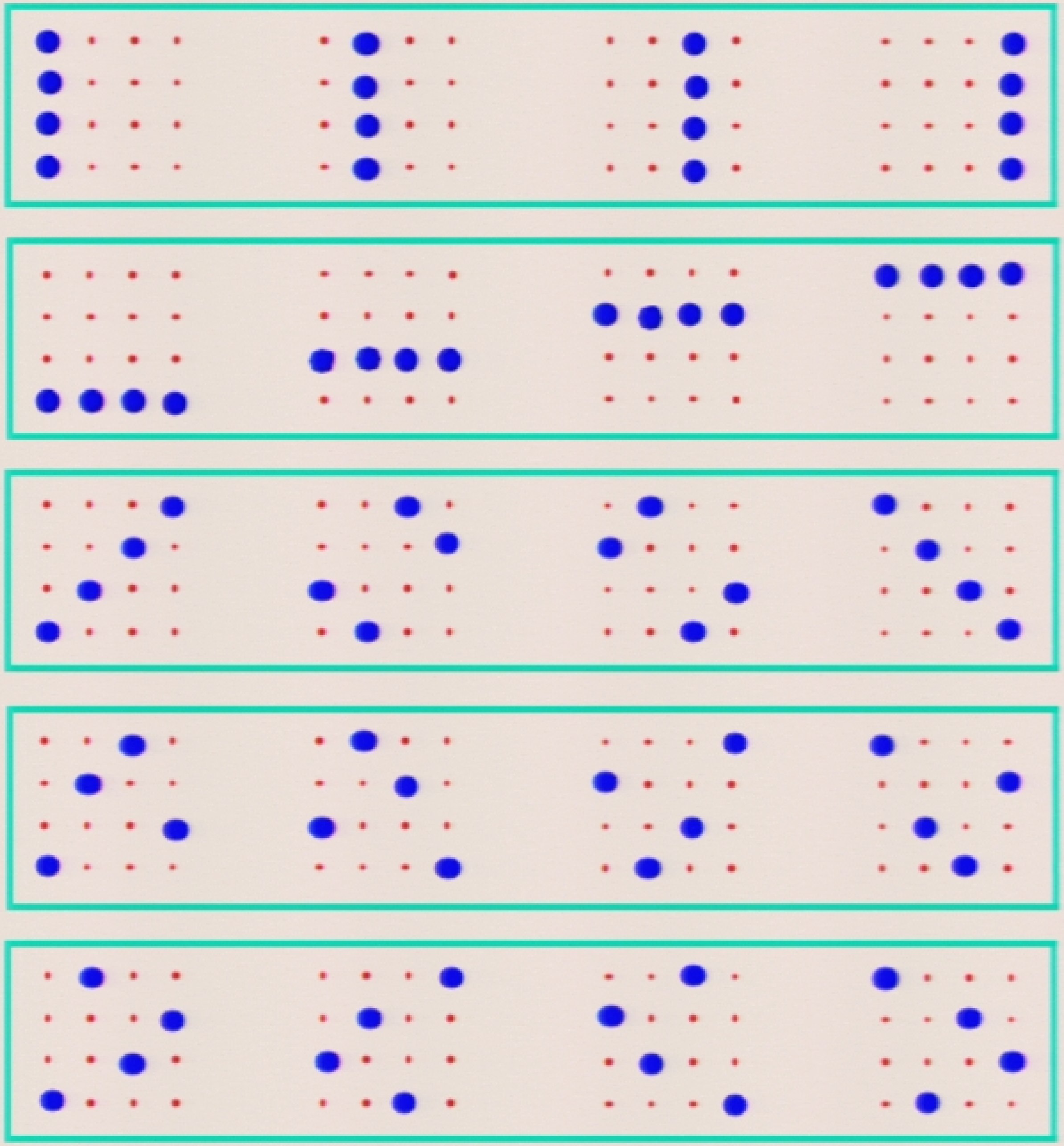
The Four-Element Field

+	0	1	ω	$\bar{\omega}$
0	0	1	ω	$\bar{\omega}$
1	1	0	$\bar{\omega}$	ω
ω	ω	$\bar{\omega}$	0	1
$\bar{\omega}$	$\bar{\omega}$	ω	1	0

×	0	1	ω	$\bar{\omega}$
0	0	0	0	0
1	0	1	ω	$\bar{\omega}$
ω	0	ω	$\bar{\omega}$	1
$\bar{\omega}$	0	$\bar{\omega}$	1	ω

This is the only commutative, associative, and distributive arithmetic on 4 elements such that addition and multiplication are both invertible.

The Five Striations of the Two-Qubit Phase Space

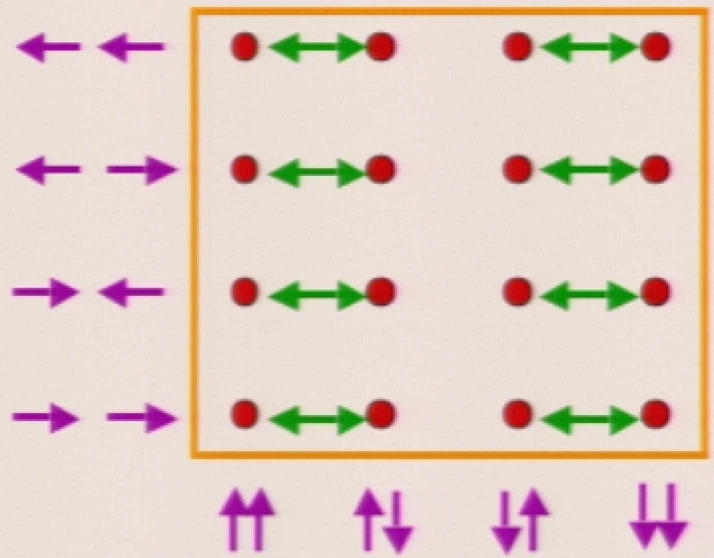


Translation Operators for Two Qubits

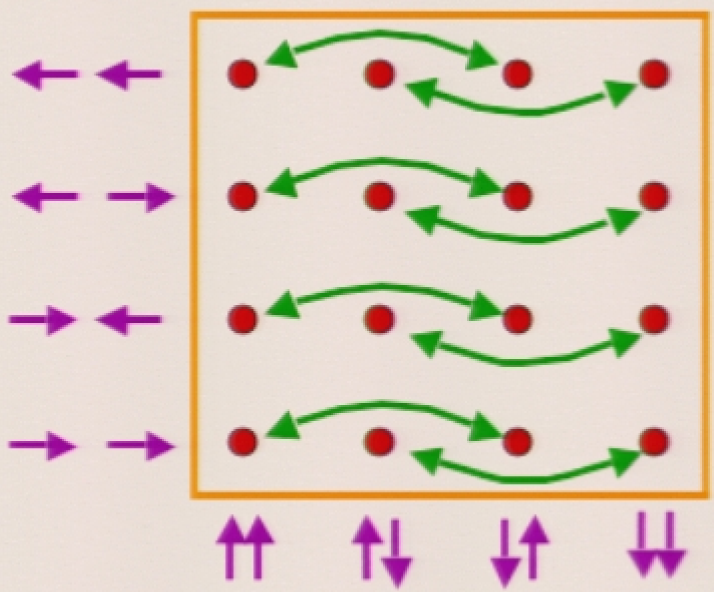
$T_{(1,0)} = I \otimes X :$

$X|\uparrow\rangle = |\downarrow\rangle$

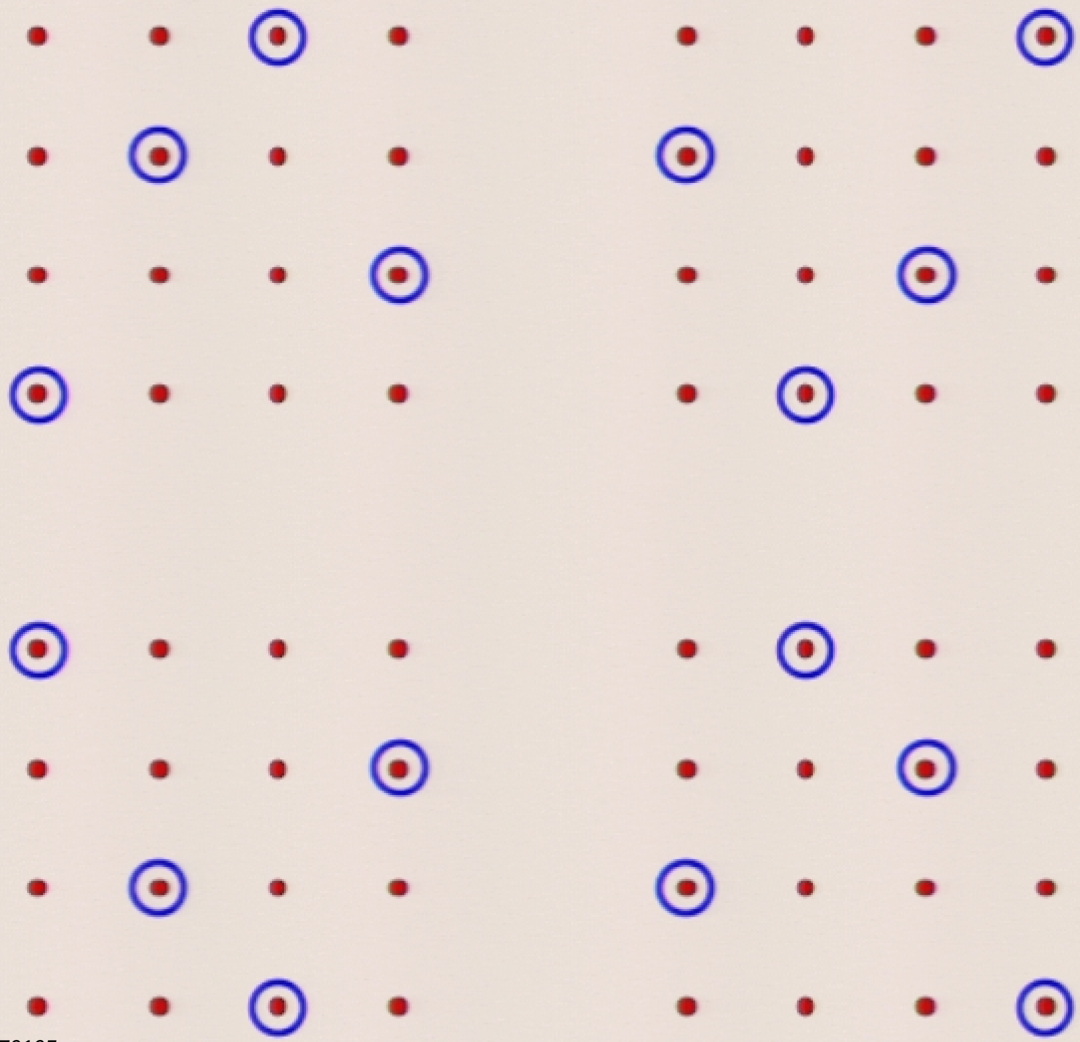
$X|\downarrow\rangle = |\uparrow\rangle$



$T_{(\omega,0)} = X \otimes I :$



What measurement goes with *this* striation (for example)?

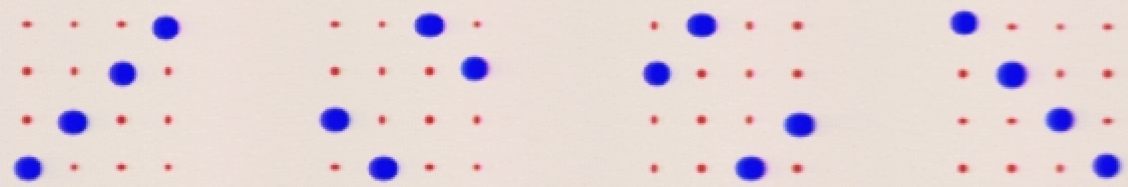


Each of these lines is invariant under three translations: $T_{(1,\,\omega)}$, $T_{(\omega,\,\bar{\omega})}$, $T_{(\bar{\omega},\,1)}$

So, find simultaneous eigenvectors of these operators.

These eigenvectors turn out to be the "belle" basis.

Freedom in the definition of the two-qubit Wigner function



Which *vector* in the YY basis goes with each line in this striation?

There are 4 choices for the first line. The others are then determined by the translation operators.

Similarly for the two weird striations.

So altogether there are $4 \times 4 \times 4 = 64$ different ways to define a Wigner function for two qubits.

A Wigner Function for Two Qubits (based on a specific assignment of state vectors to lines)

State: ↑↑

←←	1/4	0	0	0
←→	1/4	0	0	0
→←	1/4	0	0	0
→→	1/4	0	0	0
	↑↑	↑↓	↓↑	↓↓

State: ↑↓ − ↓↑

←←	0	0	0	0
←→	0	1/4	1/4	0
→←	0	1/4	1/4	0
→→	0	0	0	0
	↑↑	↑↓	↓↑	↓↓

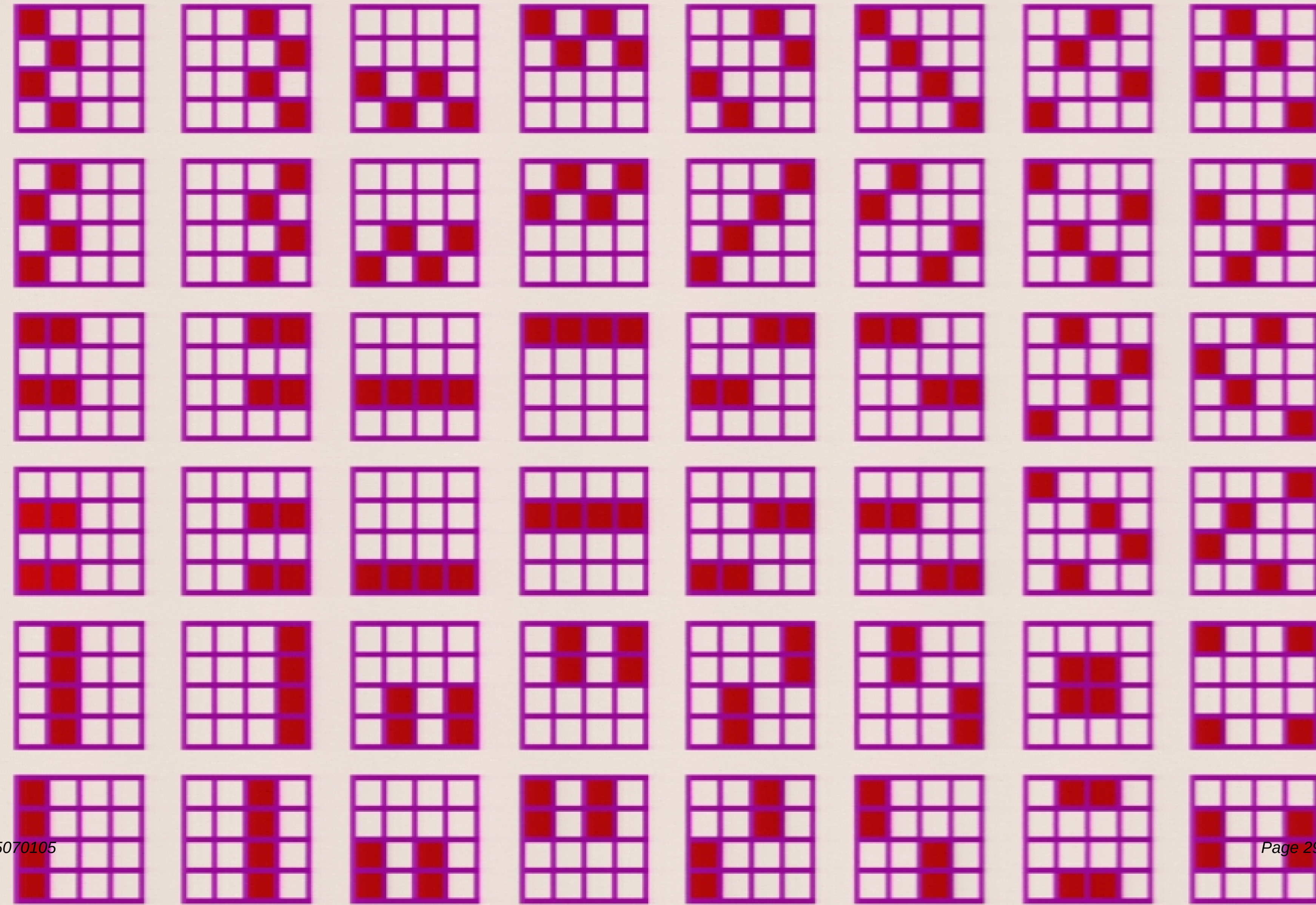
Computing overlaps using the discrete Wigner function

Let ρ and ρ' be two quantum states (density matrices), and let W and W' be their respective Wigner functions. Then,

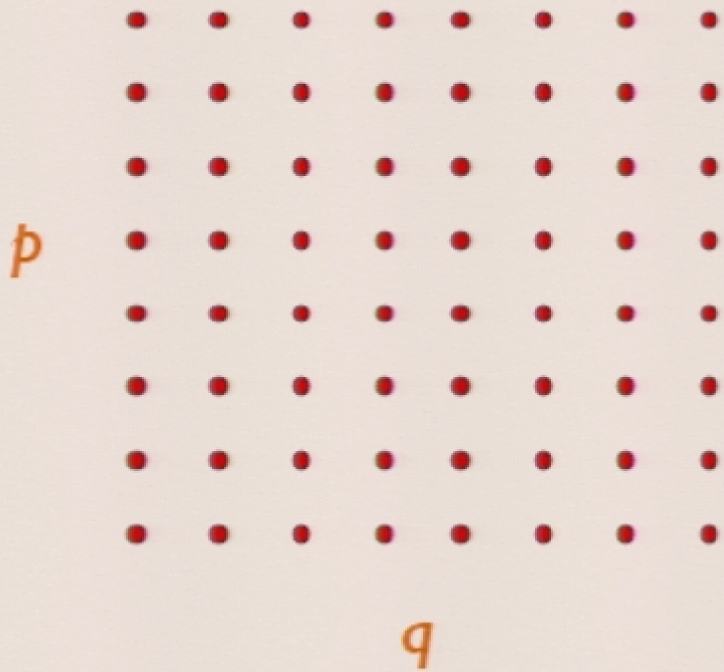
$$\mathrm{Tr}(\rho\rho') = d\sum_{\alpha} W_{\alpha} W'_{\alpha}$$

where d is the dimension of the state space (d=4 for two qubits) and the sum is over all points α of phase space.

All states with only 4 nonzero entries in the Wigner function



Generalizing to All Finite Fields



q and *p* take values in the finite field that has $d=r^n$ elements, where *r* is prime.

$$q = \sum_{i=1}^{n} q_i e_i \qquad p = \sum_{i=1}^{n} p_i f_i$$

Here $\{e_i\}$ and $\{f_i\}$ are *bases* for the field, chosen appropriately, and q_i and p_i are in $\{0,1,\ldots,r-1\}$.

Define translation operators: $T_{(q,p)} = X^{q_1}Z^{p_1} \otimes \cdots \otimes X^{q_n}Z^{p_n}$.

Then we automatically get $d+1$ mutually unbiased bases associated with the $d+1$ striations of phase space.

The discrete Wigner function and classicality

(Cormick, Galvão, Gottesman, Paz and Pittenger, 2005)

Consider the set of pure states having non-negative *W* for *all* possible definitions of *W*.

- These states have a description that grows polynomially (in fact linearly) in the number of particles.
- The unitary transformations that preserve this set of states can be simulated efficiently on a classical computer.

Main Conclusions

- We can express quantum states of discrete systems as real functions on a discrete phase space.
- Complete sets of mutually unbiased bases for the state space emerge naturally from this construction. These can be used for tomography.
- Positivity of W has quantum computational implications.

Open problems

- Classification of the possible definitions of *W*.
- For any dimension *d*, does there exist a definition of *W* such that the *W* of any product state is itself a product?
  Yes, if *d* is a power of an *odd* prime. (Pittenger and Rubin)
- Relation to toy models of quantum mechanics.
- Different routes to the continuum?
  Different "classical" limits?