Title: Symmetry and quantum logic

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Abstract:

Symmetry and Quantum Logic

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Outline

1. Test spaces and orthoalgebras (Background)
2. Fully symmetric test spaces (mainly definitions and examples)
3. Constructing fully symmetric test spaces
4. Planar test spaces

Questions

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Definition: An **orthoalgebra** [2] is a structure $(L, \oplus, 0, 1)$ consisting of a set L, two distinguished elements 0 and 1, and a commutative, associative, cancellative partial operation $\oplus$ such that, for all $a \in L$,

- $a \oplus 0 = a$;
- $\exists\, a' \in L$ with $a \oplus a' = 1$;
- $a \oplus a$ exists only if $a = 0$.

An orthoalgebra L can be partially ordered by setting

$$a \leq b \quad \Leftrightarrow \quad \exists c \in L \text{ with } b = a \oplus c.$$

The mapping $a \mapsto a'$ is an orthocomplementation with respect to $\leq$, and $a \oplus b$ is defined iff $a \perp b$, i.e., $a \leq b'$. If $a \perp b$, then $a \oplus b$ is a *minimal* (but not necessarily least) upper bound for $a, b \in L$.

Example: Any orthomodular poset – in particular, any Boolean algebra – can be regarded as an orthoalgebra with $a \oplus b = a \vee b$ provided $a \leq b'$. The given order then coincides with the one defined above.

Facts [2]: *Let $(L, \oplus, 0, 1)$ be an orthoalgebra. The following are equivalent:*
(a) *$(L, \leq, ')$ is an OMP*
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Condition (c) is called **ortho-coherence.** Hence:

OMP $\Leftrightarrow$ ortho-coherent OA
OML $\Leftrightarrow$ lattice-ordered OA

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An orthoalgebra L can be partially ordered by setting

$$a \leq b \quad \Leftrightarrow \quad \exists c \in L \text{ with } b = a \oplus c.$$

The mapping $a \mapsto a'$ is an orthocomplementation with respect to $\leq$, and $a \oplus b$ is defined iff $a \perp b$, i.e., $a \leq b'$. If $a \perp b$, then $a \oplus b$ is a *minimal* (but not necessarily least) upper bound for $a, b \in L$.

Example: Any orthomodular poset – in particular, any Boolean algebra – can be regarded as an orthoalgebra with $a \oplus b = a \vee b$ provided $a \leq b'$. The given order then coincides with the one defined above.

Facts [2]: *Let* $(L, \oplus, 0, 1)$ *be an orthoalgebra. The following are equivalent:*

(a) $(L, \leq, ')$ *is an OMP*
(b) where defined, $a \oplus b$ *is the least upper bound of* $a, b \in L$;
(c) a, b, c *pairwise orthogonal* $\Rightarrow (a \oplus b) \oplus c$ *exists.*

1. Test Spaces and Orthoalgebras

Definition: An **orthoalgebra** [2] is a structure $(L, \oplus, 0, 1)$ consisting of a set L, two distinguished elements 0 and 1, and a commutative, associative, cancellative partial operation $\oplus$ such that, for all $a \in L$,

- $a \oplus 0 = a$;
- $\exists\, a' \in L$ with $a \oplus a' = 1$;
- $a \oplus a$ exists only if $a = 0$.

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Condition (c) is called **ortho-coherence**. Hence:

OMP $\Leftrightarrow$ ortho-coherent OA
OML $\Leftrightarrow$ lattice-ordered OA

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Example 2: Quantum Test Spaces Let $\mathbf{H}$ be a Hilbert space; let X be $\mathbf{H}$'s unit sphere and $\mathfrak{F}$, the set of frames (unordered orthonormal bases) for $\mathbf{H}$. The pair $(X, \mathfrak{F})$, the **frame manual** for $\mathbf{H}$, is a (very basic) model for the set of quantum statistical experiments. Gleason's theorem tells us that so long as $\dim(\mathbf{H}) > 2$, every state on $(X, \mathfrak{F})$ arises from a density operator on $\mathbf{H}$ via the "Born rule" $\omega(x) = \mathrm{Tr}\,(\rho P_x)$.

We can also consider the projective unit sphere PX, i.e., the set of one-dimensional subspaces of $\mathbf{H}$, and the collection $P\mathfrak{F}$ of *projective frames*, i.e., maximal pairwise orthogonal subsets of PX. I'll call the test space $(PX, P\mathfrak{F})$ the **projective frame manual** of $\mathbf{H}$.

Notice that both of these examples are uniform, and have rank equal to $\dim(\mathbf{H})$.

Logics of Test Spaces

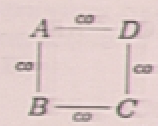
An **event** of a test space $(X, \mathfrak{A})$ is a subset of a test. We write $\mathcal{E} = \mathcal{E}(X, \mathfrak{A})$ for the set of all events. Events $A, B \in \mathcal{E}$ are

- **compatible** iff their union is an event;
- **orthogonal** ($A \perp B$) iff they are disjoint and compatible;
- **Complementary** ($A \,\mathrm{co}\, B$) iff they partition a test;
- **Perspective** ($A \sim B$) iff $\exists C \in \mathcal{E}$ with $A \,\mathrm{co}\, C \,\mathrm{co}\, B$.

$(X, \mathfrak{A})$ is **algebraic**, or a **manual**, iff perspective events have *the same* complementary events – equivalently, if

$$A \,\mathrm{co}\, B \,\mathrm{co}\, C \,\mathrm{co}\, D \;\Rightarrow\; A \,\mathrm{co}\, D$$

for all events $A, B, C, D \in \mathcal{E}$:



For a proof of the following, see [1] or [3]:

Theorem 1: *If $(X, \mathfrak{A})$ is algebraic, then perspectivity is an equivalence relation on $\mathcal{E}$. In this case, the quotient set*

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carries a well-defined partial operation given by $[A] \oplus [B] := [A \cup B]$ whenever $A \perp B$ in $\mathcal{E}$, making Π an orthoalgebra.

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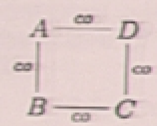

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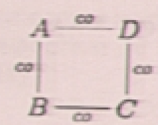

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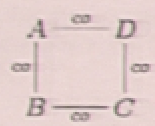

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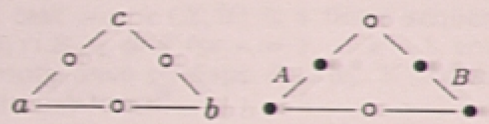
Example: Consider three 3-outcome tests pasted together in a loop, as in figure (a). This is algebraic. The events A and B indicated in figure (b) are perspective, hence, $[A] = [B] = p \in \Pi$. The outcomes a and b are orthogonal, and as $a \in A$, $b \in B$, $[a], [b] \leq p$; however, $[a] \oplus [b] = [\{a, b\}] \not\leq p$. Hence, Π is an orthoalgebra, but not an OMP.



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(a) If $X = E$ and $\mathfrak{A} = \{E\}$, then $\Pi \simeq \mathcal{P}(E)$.

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A test space is **ortho-coherent** iff every pairwise orthogonal triple of events is compatible, i.e., unions to an event. A **square** in a test space is a set of four outcomes $\{x, y, u, v\}$ with $x \perp y \perp u \perp v \perp x$, and with no other pairs orthogonal.

The following result, usually called the *loop lemma*, goes back to Dick Greechie's doctoral dissertation of 1966. The present formulation is a special case of one due to Foulis, Greechie and Ruttimann [1].

Theorem 2: *Let $(X, \mathfrak{A})$ be an algebraic test space of finite rank. Then $\Pi(X, \mathfrak{A})$ is lattice-ordered – hence, an OML – iff $\mathfrak{A}$ is ortho-coherent and square-free.*

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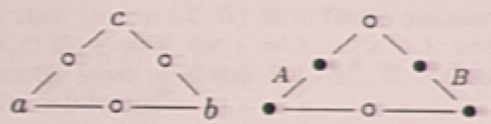

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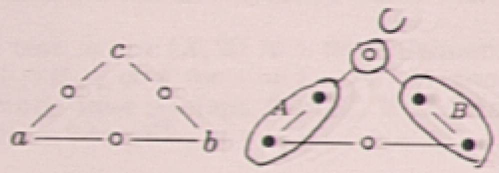

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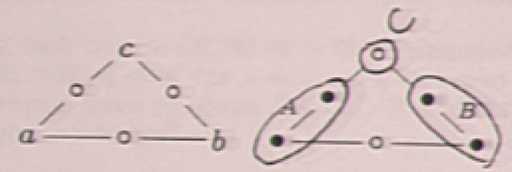

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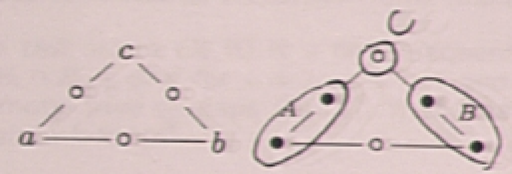

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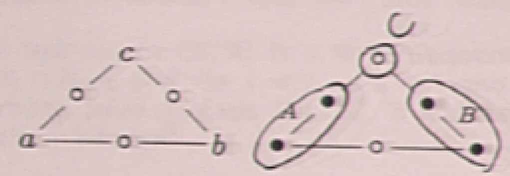

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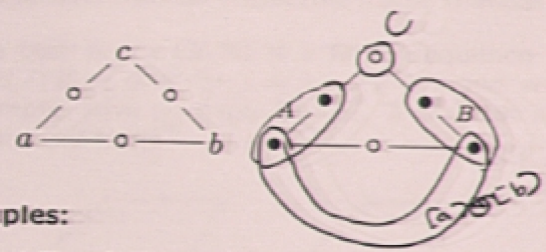

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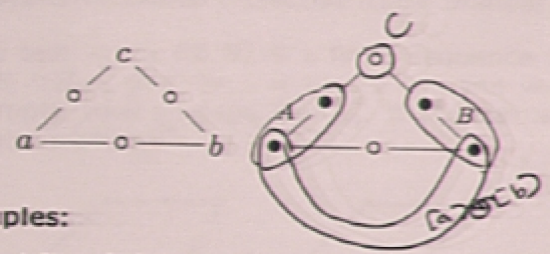

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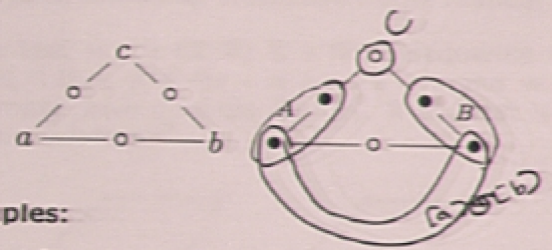

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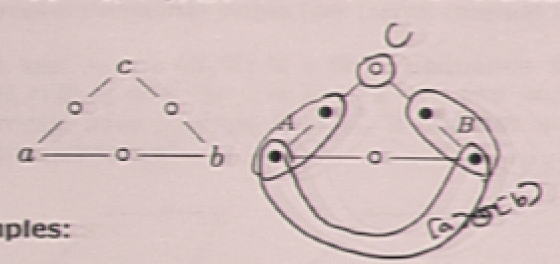

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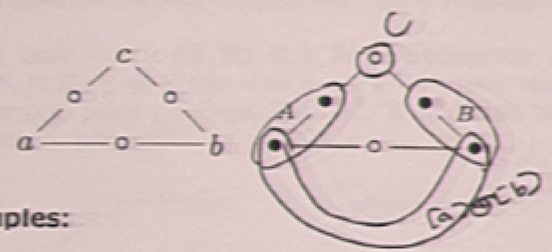

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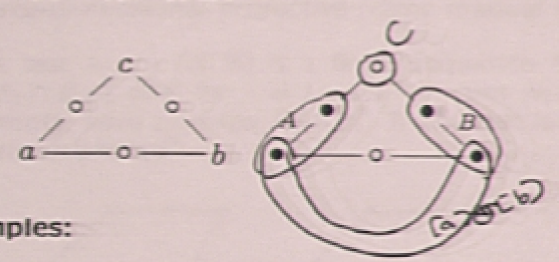

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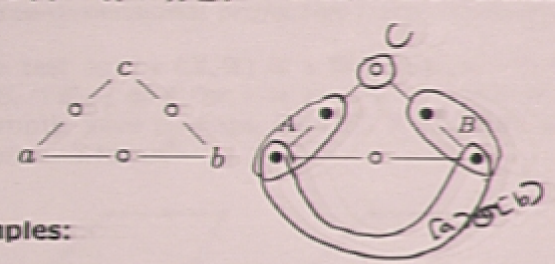

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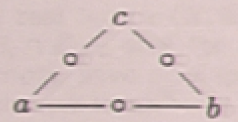

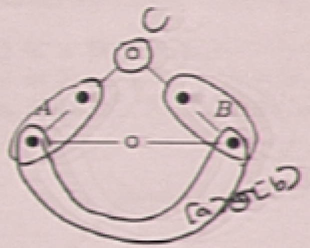

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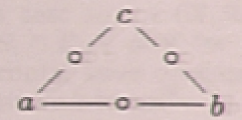
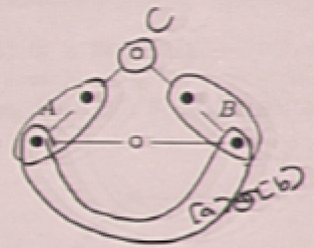
Example: Consider three 3-outcome tests pasted together in a loop, as in figure (a). This is algebraic. The events A and B indicated in figure (b) are perspective, hence, $[A] = [B] = p \in \Pi$. The outcomes a and b are orthogonal, and as $a \in A$, $b \in B$, $[a], [b] \leq p$; however, $[a] \oplus [b] = [\{a, b\}] \not\leq p$. Hence, Π is an orthoalgebra, but not an OMP.





Further Examples:

(a) If $X = E$ and $\mathfrak{A} = \{E\}$, then $\Pi \simeq \mathcal{P}(E)$.

(b) If $(X, \mathfrak{F})$ is the quantum test space associated with a Hilbert space $\mathbf{H}$, then tests are orthonormal bases, events are orthonormal sets, and two events are perspective iff they have the same closed span. It follows that $\Pi \simeq L(\mathbf{H})$.

The Loop Lemma

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The following result, usually called the *loop lemma*, goes back to Dick Greechie's doctoral dissertation of 1966. The present formulation is a special case of one due to Foulis, Greechie and Ruttimann [1].

Theorem 2: *Let $(X, \mathfrak{A})$ be an algebraic test space of finite rank. Then $\Pi(X, \mathfrak{A})$ is lattice-ordered – hence, an OML – iff $\mathfrak{A}$ is ortho-coherent and square-free.*

Definition: A test space $(X, \mathfrak{A})$ is **Greechie** iff (i) $|E| \geq 3$ for all $E \in \mathfrak{A}$, and (ii) $|E \cap F| \leq 1$ for all $E \neq F \in \mathfrak{A}$.

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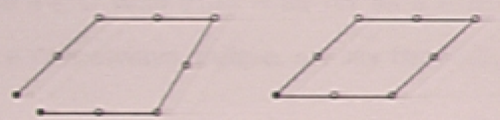

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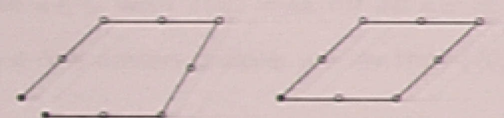

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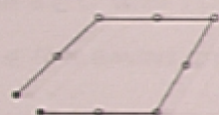

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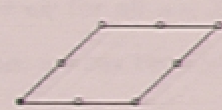

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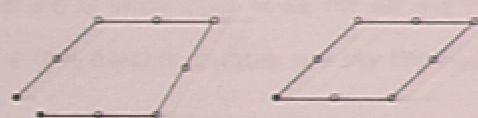

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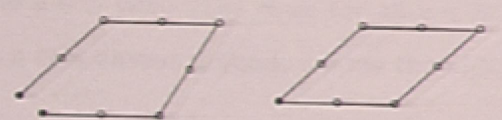

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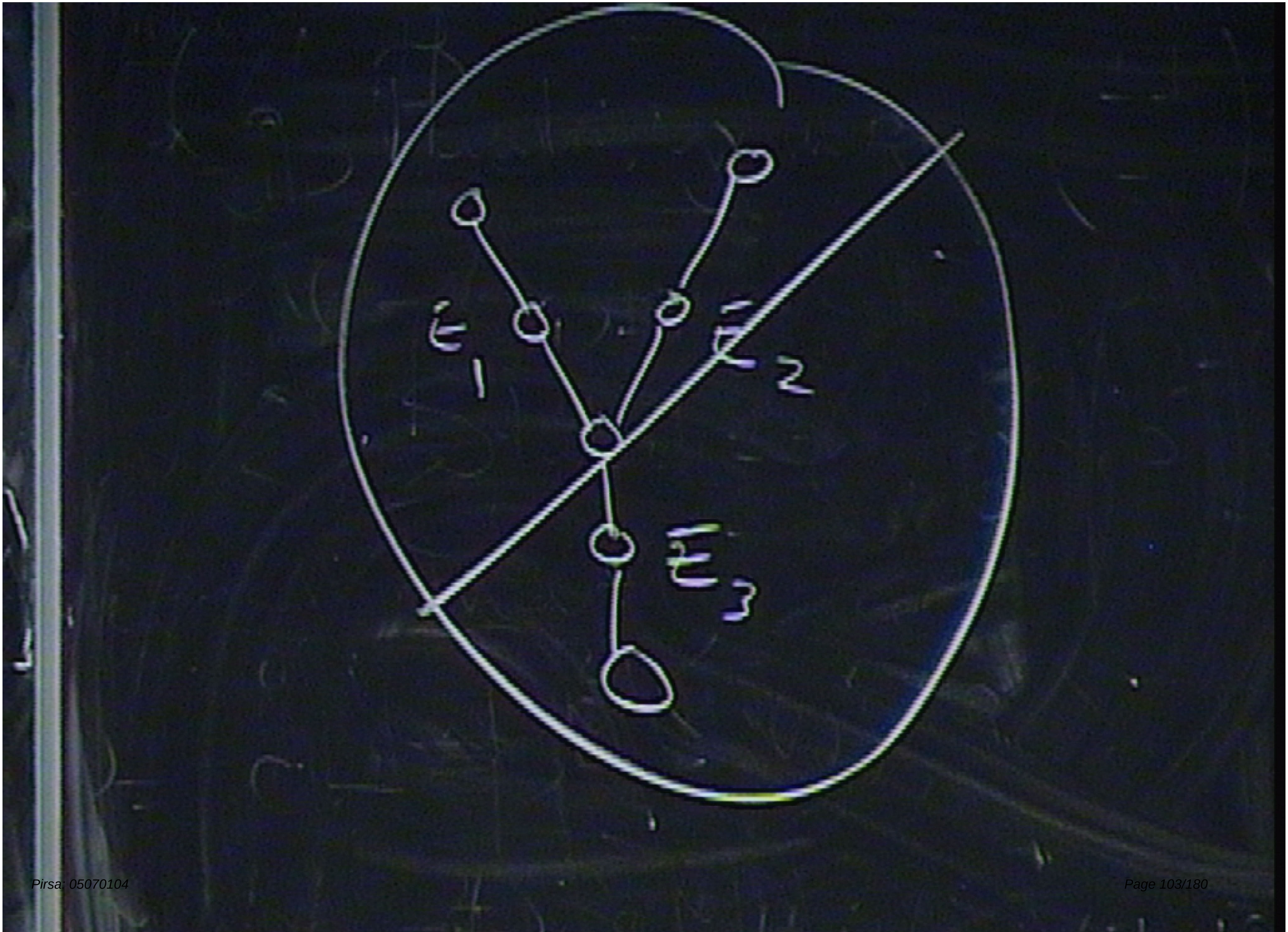
$\mathcal{E}_1$
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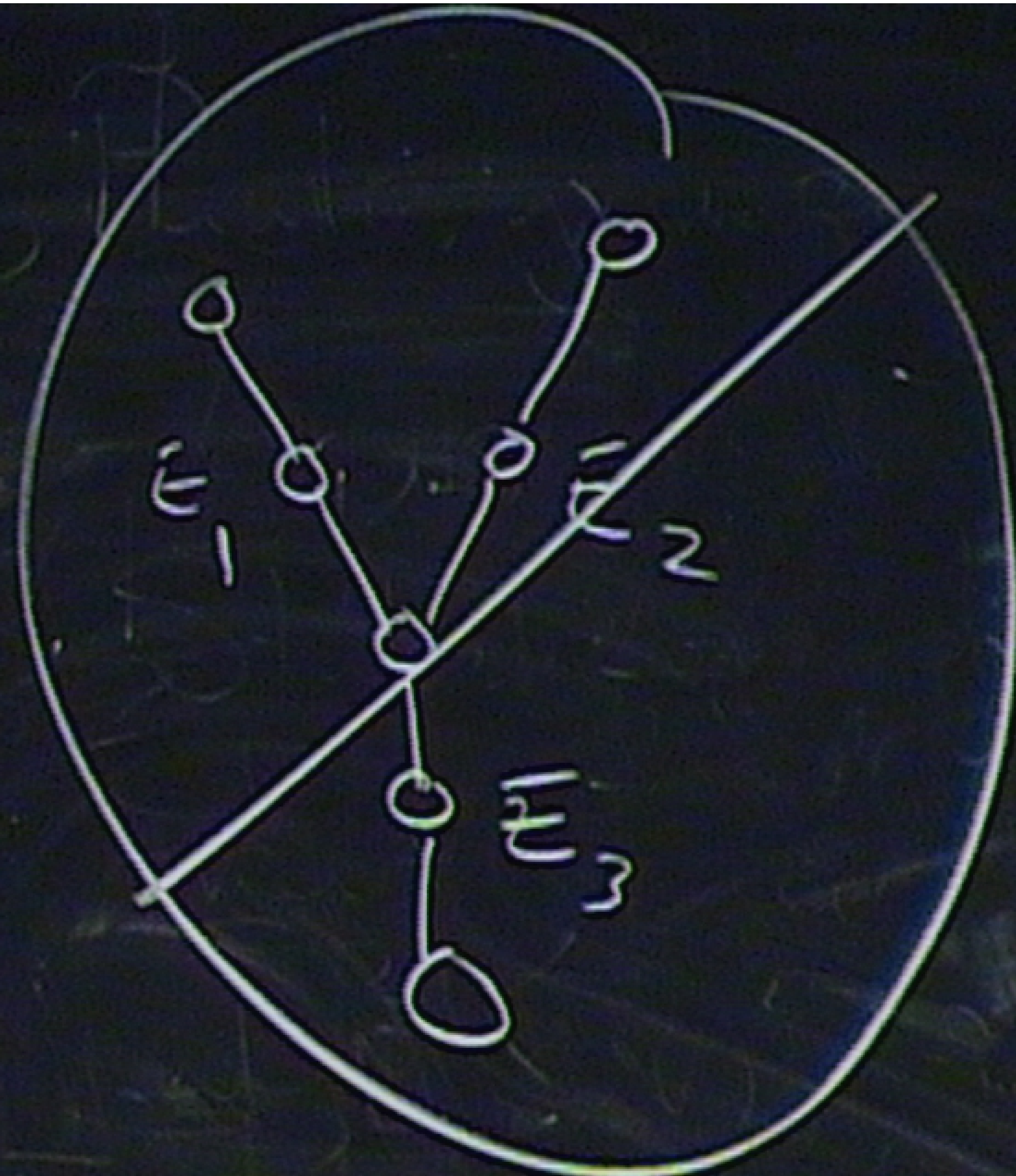

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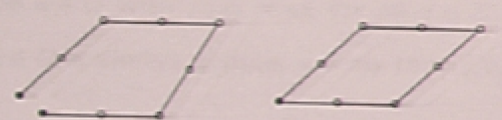

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2. Fully Symmetric Test Spaces

Let G be a group. A G-**test space** is a test space $(X, \mathfrak{A})$ equipped with an action of G on X such that, for every test $E \in \mathfrak{A}$ and every $\alpha \in G$, $\alpha(E) \in \mathfrak{A}$ as well.

Among the various transitivity conditions one might impose on a G-test space, the following seems particularly interesting:

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If the element $a \in G$ is always unique, we say that $(X, \mathfrak{A})$ is **strongly symmetric.**

Example 0: A classical test space $\{E\}$ is fully (indeed, strongly) symmetric under the symmetric group S_E.

Example 1: The frame manual of a Hilbert space is strongly symmetric with respect to that space's unitary group, since any bijection between two frames uniquely determines a unitary operator on $\mathbf{H}$. On the other hand, the projective frame manual is fully, but not strongly, symmetric, since any diagonal unitary will fix every point of a frame.

A fully, or even strongly, symmetric algebraic test space can still be rather far from $L(\mathbf{H})$. To illustrate this, here are some further examples.

Example 2: Uniform Partitions Let S be a finite set of size $|X| = nk$; let X denote the set of k-element subsets of S, and let $\mathfrak{A}$ consist of all partitions of S into n k-element blocks. Then $(X, \mathfrak{A})$ is a fully symmetric algebraic test space of rank n. Note that this space typically has four-loops, so its logic isn't an OML.

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2. Fully Symmetric Test Spaces

Let G be a group. A **G-test space** is a test space $(X, \mathfrak{A})$ equipped with an action of G on X such that, for every test $E \in \mathfrak{A}$ and every $\alpha \in G$, $\alpha(E) \in \mathfrak{A}$ as well.

Among the various transitivity conditions one might impose on a G-test space, the following seems particularly interesting:

Definition: A G-test space $(X, \mathfrak{A})$ is **fully symmetric** iff

(i) Any two tests have the same cardinality, and

(ii) for bijection $f : E \to F$ between two tests $E, F \in \mathfrak{A}$, there exists some $a \in G$ with $f(x) = ax$ for all $x \in E$.

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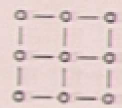
Example 3: Grids Let $X = n \times n$ consist of the points, and $\mathfrak{A}$, of the rows and columns, of the $n \times n$ grid: this is algebraic, orthocoherent, but (obviously) not square-deficient. It's fully symmetric under the subgroup of $S(n^2)$ generated by $S_n \times S_n$, together with the bijection (transposition) that exchanges the two factors.



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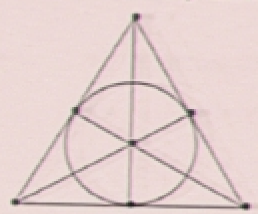
If any one line is finite, with $n+1$ points, then all lines contain $n+1$ points. In this case, we say the plane has *order* n. The projective plane of order 2 is the famous *Fano plane*, pictured below.



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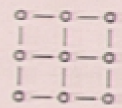

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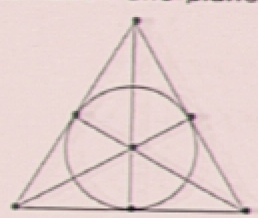

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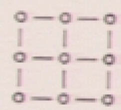

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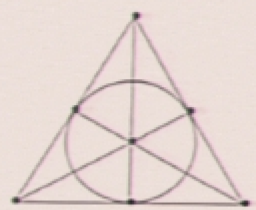

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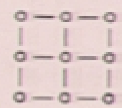

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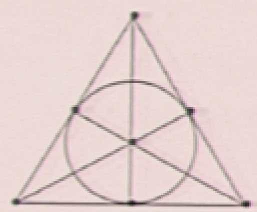

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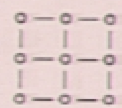

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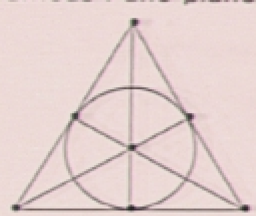

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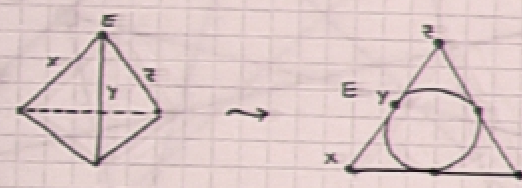
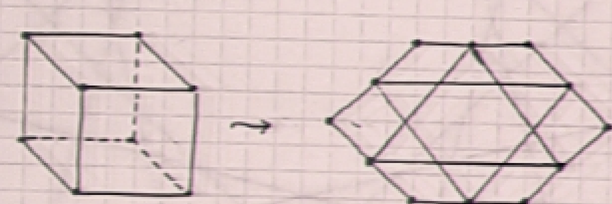


3 edges per face → 3 loop

b)



(NB: $\triangle = \bigcirc$)

4 edges per face → 4 loop: OMP, not OML

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No loops of order < 5

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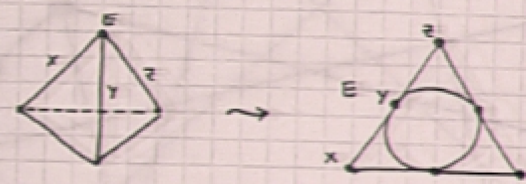

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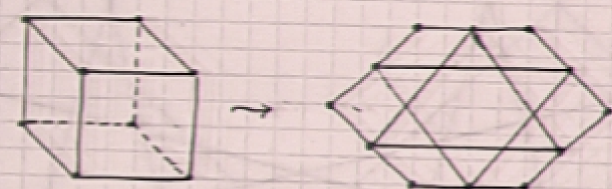

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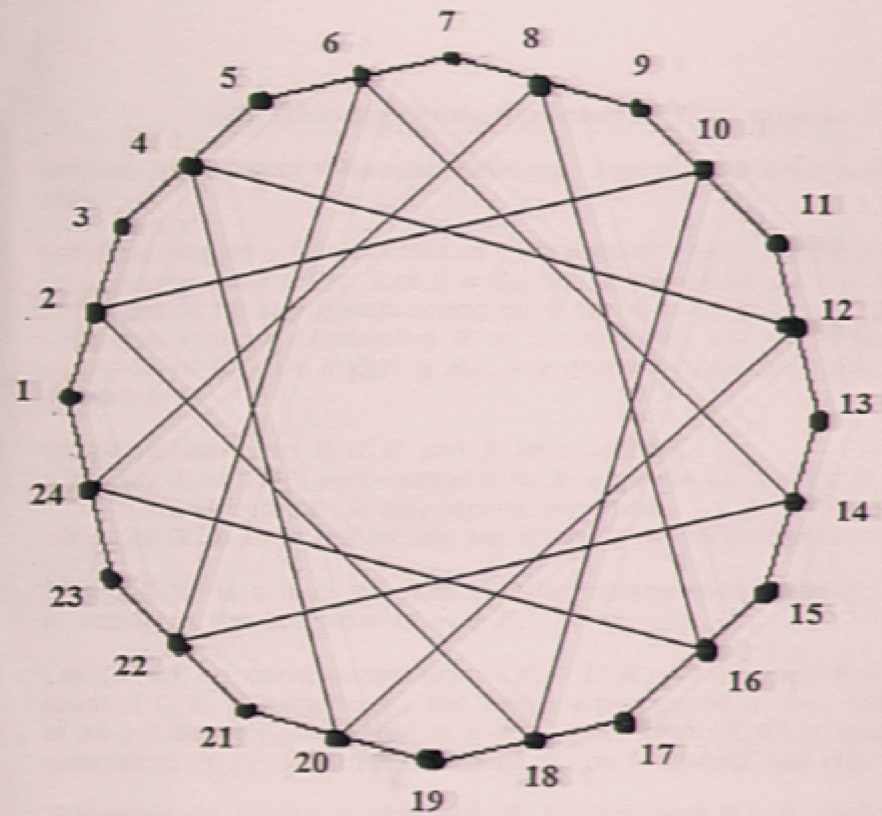

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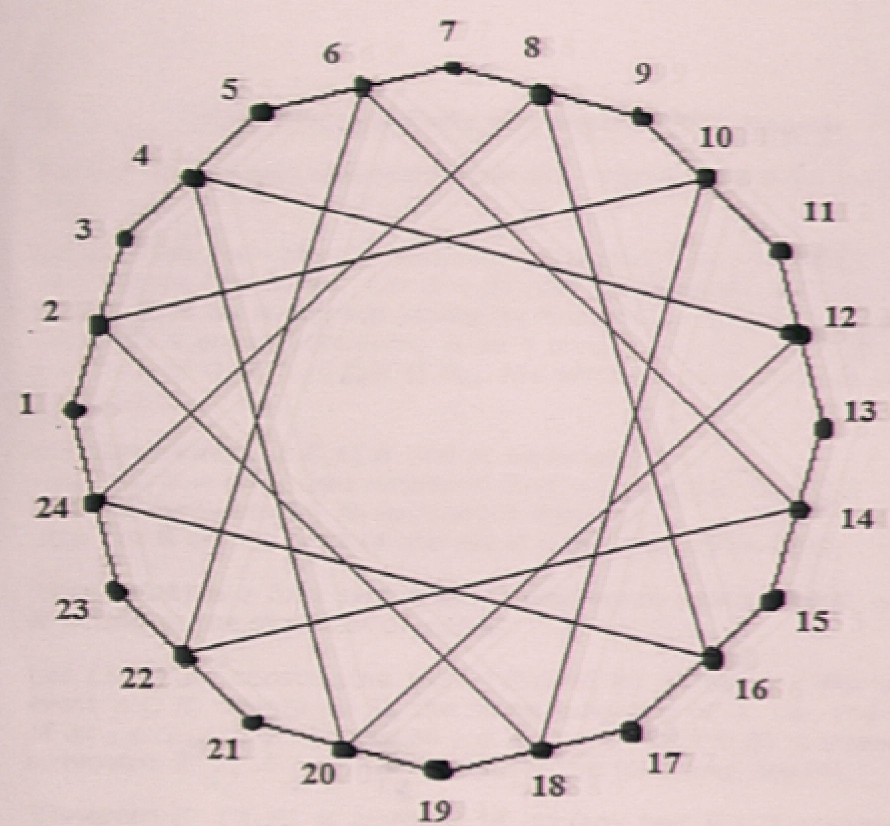

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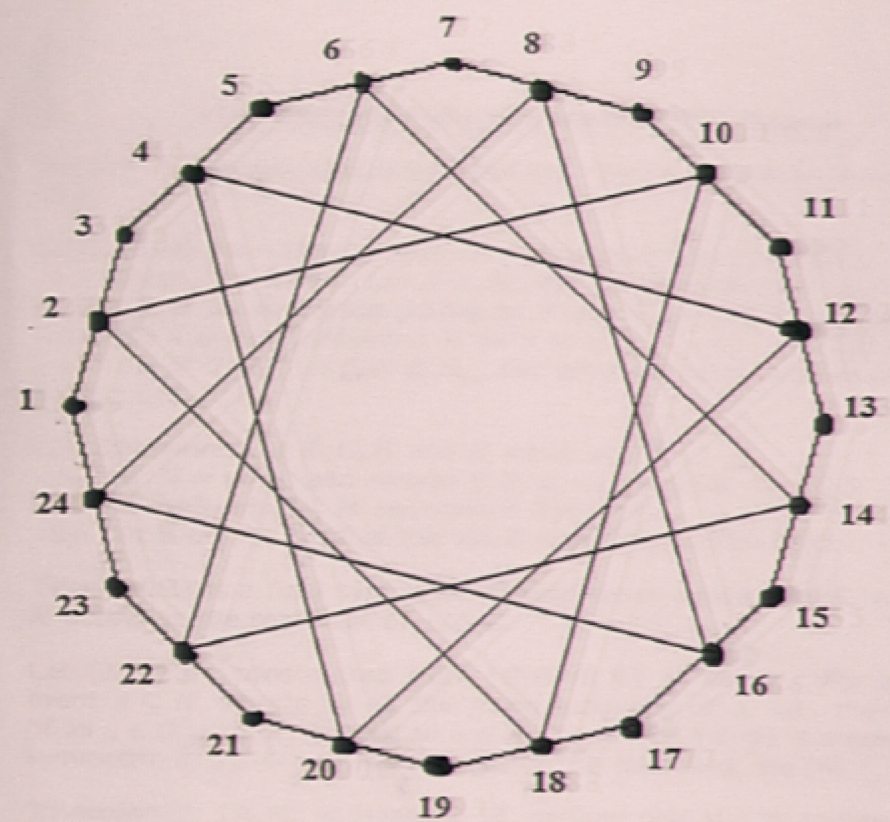

3. Constructing Fully Symmetric Test Spaces

We can reconstruct fully symmetric test spaces from their symmetries.

Let E be any set – think of this as a "standard" test – and let $x_o \in E$ be a chosen base point. Let $S = S_E$, the group of all permutations of E. Let H be any group acting on E as its full symmetry group. Let G be a group containing H as a subgroup, and let K be any subgroup of G with $H \cap K \subseteq H_{x_o}$, the group of permutations in H fixing $x \in E$.

Construction: Let E, G, H and K be as above.

(a) Set $X = G/K$, and embed E in X via $sx \mapsto sK$ for $s \in S$ – a well-defined, H-equivariant injection.

(b) Let $\mathfrak{A}$ be the orbit of the set E under the action of G.

Then $(X, \mathfrak{A})$ is a fully symmetric G-test space containing E, with K acting as the stabilizer of $x_o \in E$.

Let $(X, \mathfrak{A})$ be constructed from (E, G, H, K) as above. For any event $A \subseteq E$, denote by F_A the *fixing* subgroup of A, i.e., the set of all $g \in G$ with $gx = x$ for all $x \in A$. Note that $(X, \mathfrak{A})$ is strongly symmetric iff $F_E = \{e\}$. For a proof of the following, see [4]:

Theorem 2: *$(X, \mathfrak{A})$ is algebraic iff, for (any test $E \in \mathfrak{A}$ and) every $A \subseteq E \in \mathfrak{A}$, $F_A F_{E \setminus A} = F_{E \setminus A} F_A$.*

Remark: The foregoing construction can be topologized. If $(X, \mathfrak{A})$ is a symmetric G-test space where G is a compact group, then one can identify X with G/K in its quotient topology (here K is the stabilizer of a point in X, as above). This gives us in turn a natural topology on $\mathcal{E}$, and hence, a quotient topology on $\Pi = \mathcal{E}/\sim$. This all *just* hangs together to yield the result that Π is a compact – hence, complete – atomistic topological orthoalgebra. For details, see [3] and [4].

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Let E be any set – think of this as a "standard" test – and let $x_o \in E$ be a chosen base point. Let $S = S_E$, the group of all permutations of E. Let H be any group acting on E as its full symmetry group. Let G be a group containing H as a subgroup, and let K be any subgroup of G with $H \cap K \subseteq H_{x_o}$, the group of permutations in H fixing $x_o \in E$.

Construction: Let E, G, H and K be as above.

(a) Set $X = G/K$, and embed E in X via $sx_o \mapsto sK$ for $s \in H$ – a well-defined, H-equivariant injection.

(b) Let $\mathfrak{A}$ be the orbit of the set E under the action of G.

Then $(X, \mathfrak{A})$ is a fully symmetric G-test space containing E, with K acting as the stabilizer of $x_o \in E$.

Let $(X, \mathfrak{A})$ be constructed from (E, G, H, K) as above. For any event $A \subseteq E$, denote by F_A the *fixing* subgroup of A, i.e., the set of all $g \in G$ with $gx = x$ for all $x \in A$. Note that $(X, \mathfrak{A})$ is strongly symmetric iff $F_E = \{e\}$. For a proof of the following, see [4]:

Theorem 2: *$(X, \mathfrak{A})$ is algebraic iff, for (any test $E \in \mathfrak{A}$ and) every $A \subseteq E \in \mathfrak{A}$, $F_A F_{E \setminus A} = F_{E \setminus A} F_A$.*

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Pivoting. Suppose that ax_o
$\{x_o, y, ax_o\}$ containing x_o and
$f : E_o \to F$ sending $x_o \to$
element $k_1 \in K$, we h
$k_1 \sigma = a k_2$ for some $k_2 \in$

k_1

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Lemma 2: $ax_o \perp x_o$ *iff* $a \in K\sigma K$

Corollary 1: $ax_o \perp bx_o$ *iff* $b^{-1}a \in K$

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Lemma 1: *$(X, \mathfrak{A})$ is Greechie iff $K \cap \sigma K \sigma^{-1} = \{e\}$.*

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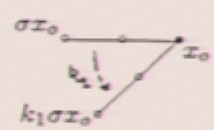

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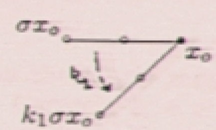

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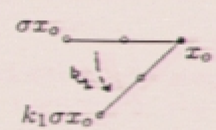

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