

Title: Preparation contextuality in its myriad forms

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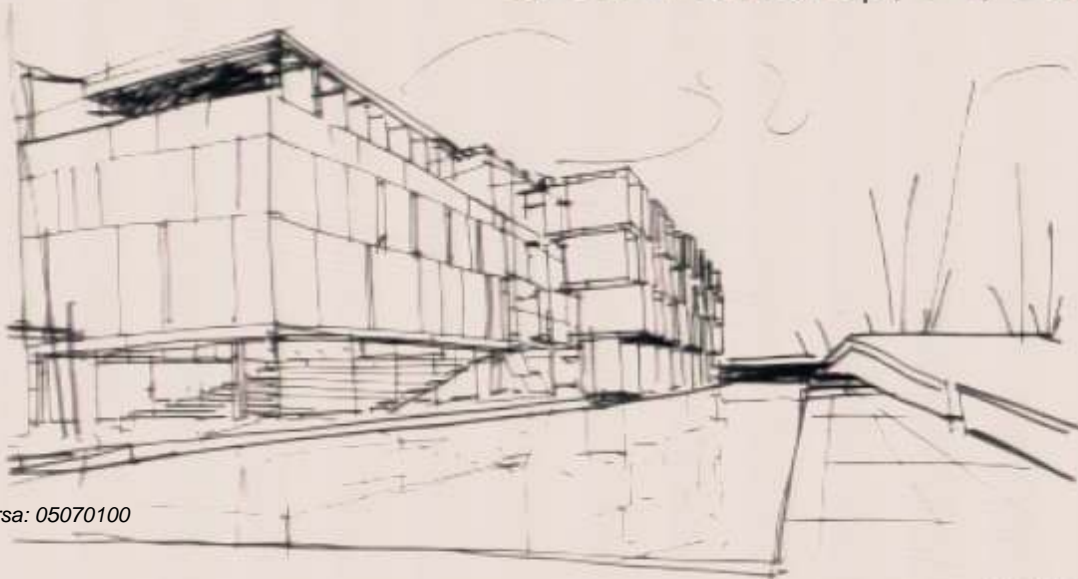
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Abstract:

Preparation Contextuality in its myriad forms

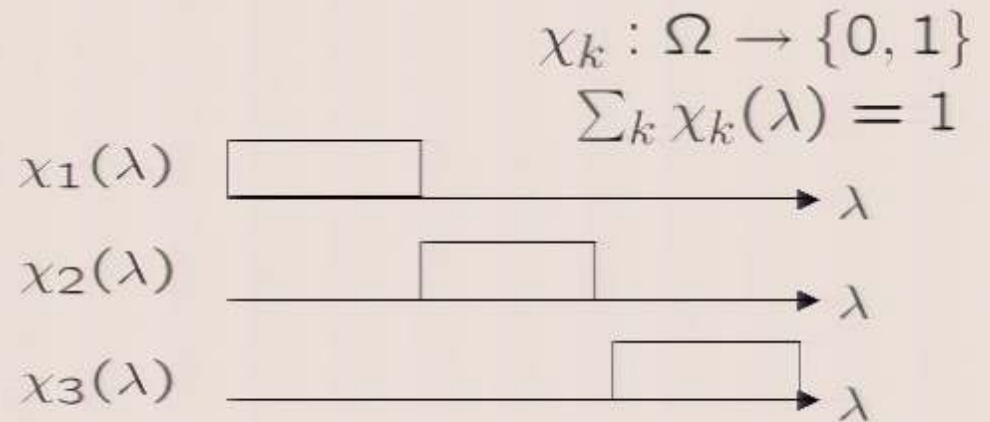
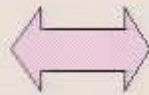
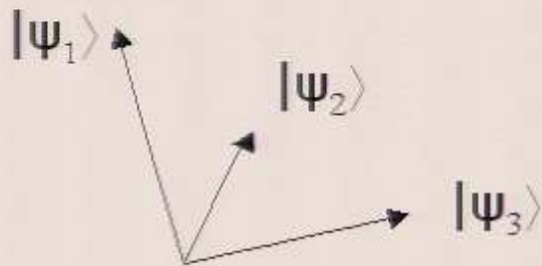
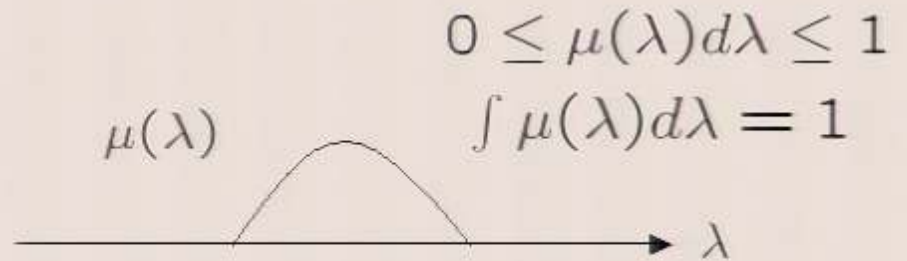
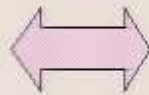
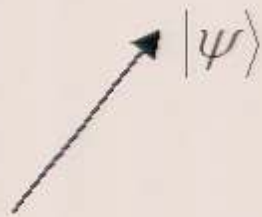
Robert Spekkens
Perimeter Institute for Theoretical Physics,
Waterloo, Canada

QICL workshop, PI, July 19, 2005



Traditional Measurement Contextuality

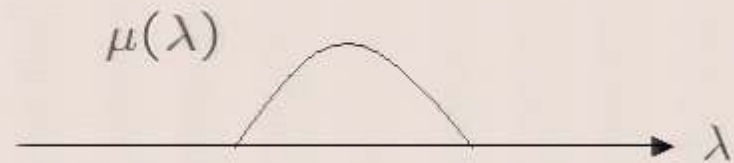
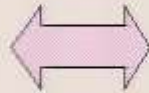
The idea of a **deterministic hidden variable theory** is that



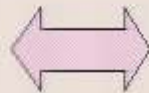
$$|\langle \psi | \psi_k \rangle|^2 = \int \mu(\lambda) \chi_k(\lambda) d\lambda$$

The idea of a **deterministic hidden variable theory** is that

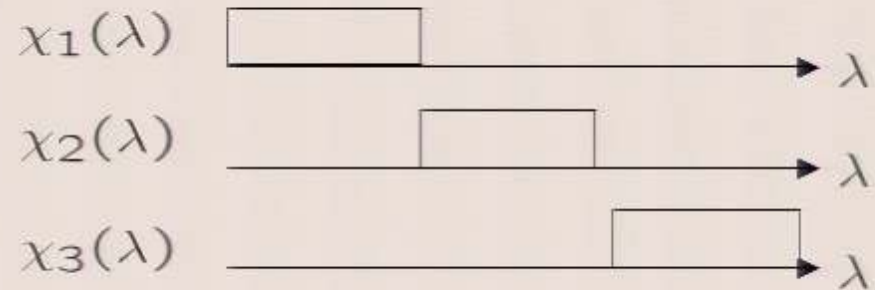
$$\begin{aligned} \rho \\ \langle \psi | \rho | \psi \rangle \geq 0, \forall \psi \\ \text{Tr}(\rho) = 1 \end{aligned}$$



$$\begin{aligned} \{P_k\} \\ \langle \psi | P_k | \psi \rangle \geq 0, \forall \psi \\ \sum P_k = I \end{aligned}$$



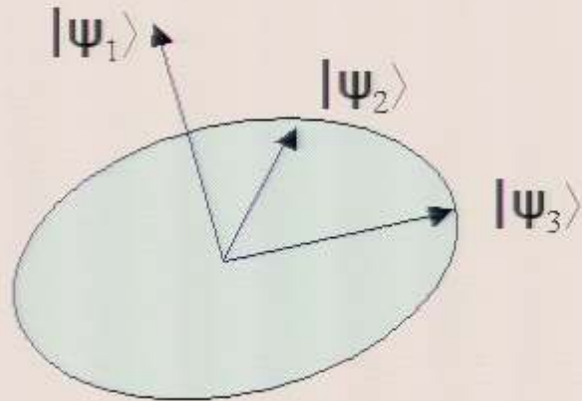
$$P_k P_{k'} = 0 \text{ for } k \neq k'$$



$$\text{Tr}(\rho P_k) = \int \mu(\lambda) \chi_k(\lambda) d\lambda$$

There are many ways of measuring a non-rank-1 PVM

Ex:

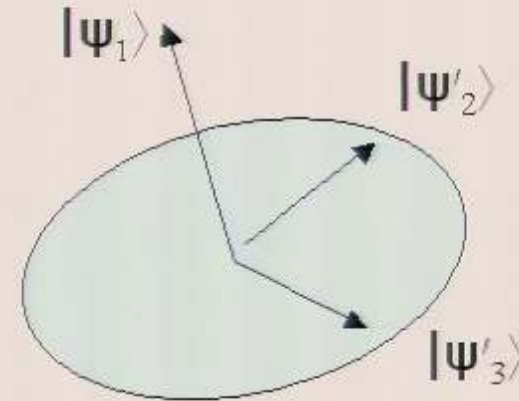


$M = \text{measure PVM}$

$$\{|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|, |\psi_3\rangle\langle\psi_3|\}$$

then coarse-grain 2 and 3

$$\begin{aligned} & \{|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|\} \\ &= \{|\psi_1\rangle\langle\psi_1|, I - |\psi_1\rangle\langle\psi_1|\} \end{aligned}$$



$M' = \text{measure PVM}$

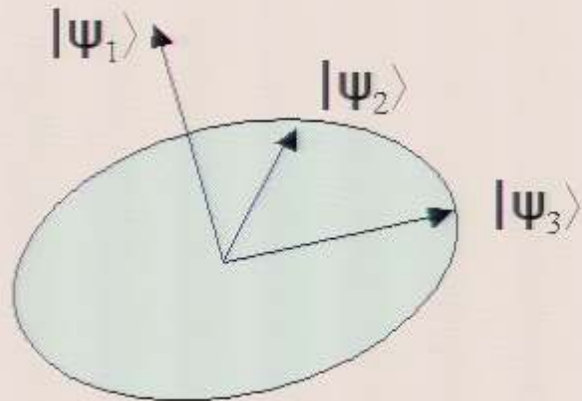
$$\{|\psi_1\rangle\langle\psi_1|, |\psi'_2\rangle\langle\psi'_2|, |\psi'_3\rangle\langle\psi'_3|\}$$

then coarse-grain 2 and 3

$$\begin{aligned} & \{|\psi_1\rangle\langle\psi_1|, |\psi'_2\rangle\langle\psi'_2| + |\psi'_3\rangle\langle\psi'_3|\} \\ &= \{|\psi_1\rangle\langle\psi_1|, I - |\psi_1\rangle\langle\psi_1|\} \end{aligned}$$

There are many ways of measuring a non-rank-1 PVM

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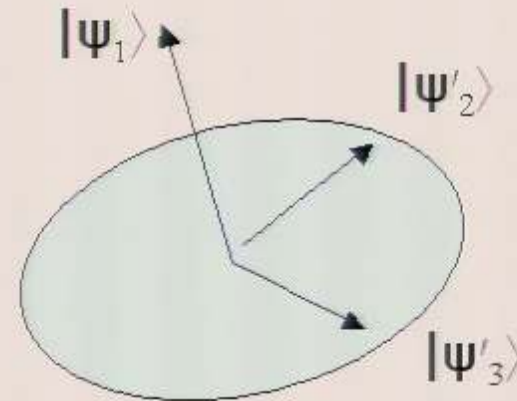


$M = \text{measure PVM}$

$$\{|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|, |\psi_3\rangle\langle\psi_3|\}$$

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$M' = \text{measure PVM}$

$$\{|\psi_1\rangle\langle\psi_1|, |\psi'_2\rangle\langle\psi'_2|, |\psi'_3\rangle\langle\psi'_3|\}$$

then coarse-grain 2 and 3

$$\begin{aligned} & \{|\psi_1\rangle\langle\psi_1|, |\psi'_2\rangle\langle\psi'_2| + |\psi'_3\rangle\langle\psi'_3|\} \\ &= \{|\psi_1\rangle\langle\psi_1|, I - |\psi_1\rangle\langle\psi_1|\} \end{aligned}$$

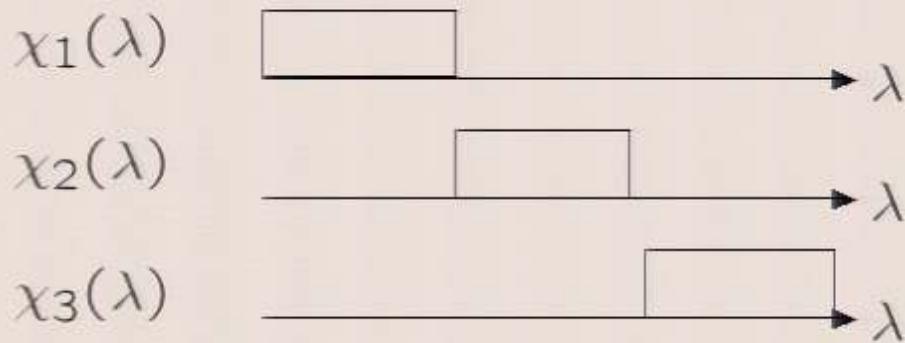
Traditional Measurement Noncontextuality

$$\text{if } M \simeq M' \text{ then } \chi_{M,k}(\lambda) = \chi_{M',k}(\lambda)$$

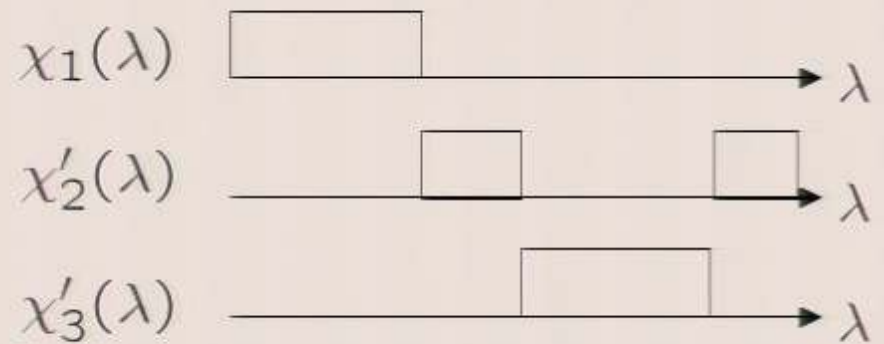
Rep'd by same PVM \rightarrow Rep'd by same indicator functions

The hope is to represent this as follows:

$$\{|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|, |\psi_3\rangle\langle\psi_3|\}$$



$$\{|\psi_1\rangle\langle\psi_1|, |\psi'_2\rangle\langle\psi'_2|, |\psi'_3\rangle\langle\psi'_3|\}$$



$$\{|\psi_1\rangle\langle\psi_1|, I - |\psi_1\rangle\langle\psi_1|\}$$



Every P is associated with the same $\chi(\lambda)$ regardless of how it is measured

It was shown by Bell (1966) and Kochen and Specker (1967) that a noncontextual hidden variable model of quantum theory for Hilbert spaces of dimensionality 3 or greater is **impossible**. That is, **quantum theory is contextual**

This is the **Bell-Kochen-Specker theorem**

Key fact for proof: there is a multiplicity of decompositions of a rank-2 projector into rank-1 projectors.

$$\begin{aligned} I - |\psi_1\rangle\langle\psi_1| &= |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3| \\ &= |\psi'_2\rangle\langle\psi'_2| + |\psi'_3\rangle\langle\psi'_3| \end{aligned}$$

Recall a similar fact for preparations: there is a multiplicity of convex decompositions of a mixed state into pure states
a.k.a the ambiguity of mixtures

$$\begin{aligned} \frac{1}{2}I &= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| & |\pm\rangle &= |0\rangle \pm |1\rangle \\ &= \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| & |\pm i\rangle &= |0\rangle \pm i|1\rangle \\ &= \frac{1}{2}|+i\rangle\langle +i| + \frac{1}{2}|-i\rangle\langle -i| \end{aligned}$$

Can we derive a no-go theorem from this? Yes.

Preparation Contextuality

$$\rho = \sum_j p_j |\psi_j\rangle \langle \psi_j|$$

P = implement P_j
with probability p_j

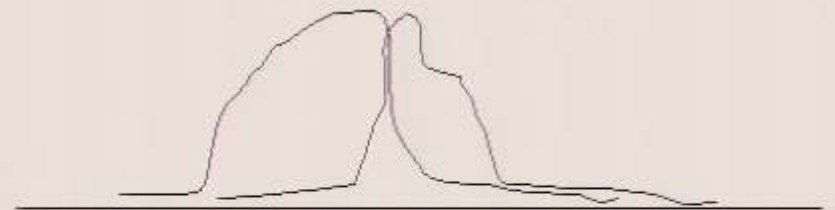
$$\rho = \sum_i q_i |\chi_i\rangle \langle \chi_i|$$

P' = implement P'_j
with probability q_j

Preparation Noncontextuality

if $P \simeq P'$ then $\mu_P(\lambda) = \mu_{P'}(\lambda)$

Rep'd by same density operator \rightarrow Rep'd by same distribution



Proof of preparation contextuality

(a preparation noncontextual hidden variable model is impossible)

Important features of hidden variable models

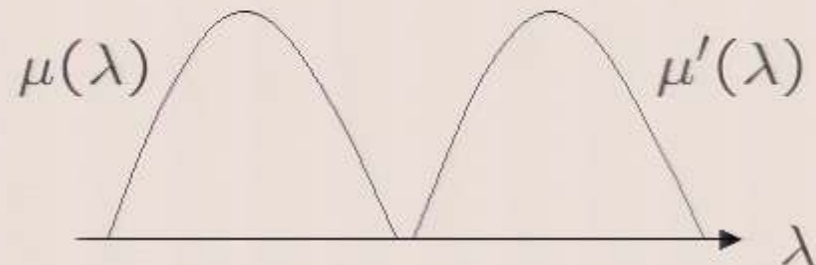
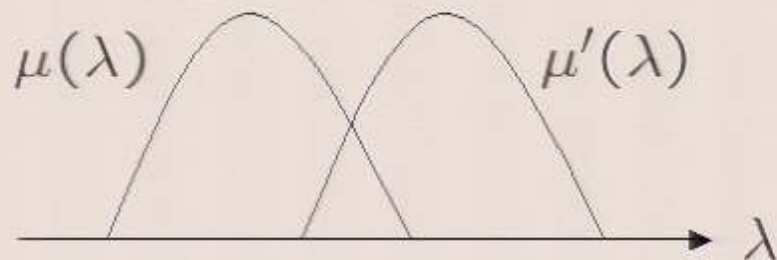
Let $P \leftrightarrow \mu(\lambda)$

$P' \leftrightarrow \mu'(\lambda)$

Representing distinguishability:

If P and P' are distinguishable with certainty

then $\mu(\lambda) \mu'(\lambda) = 0$



Important features of hidden variable models

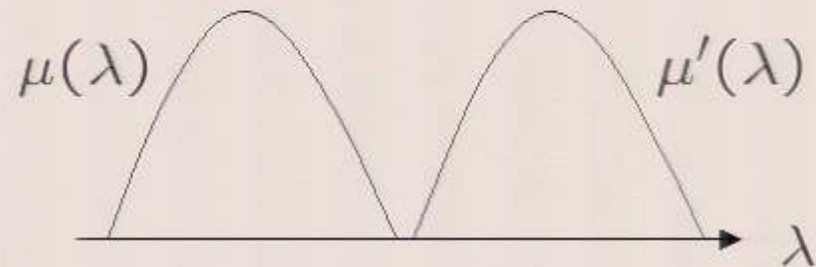
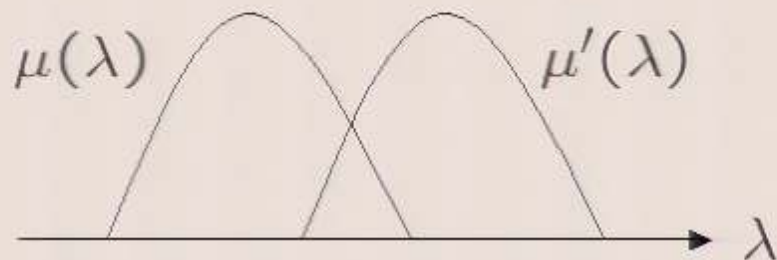
Let $P \leftrightarrow \mu(\lambda)$

$P' \leftrightarrow \mu'(\lambda)$

Representing distinguishability:

If P and P' are distinguishable with certainty

then $\mu(\lambda) \mu'(\lambda) = 0$



Representing convex combination:

If $P'' = P$ with prob. p and P' with prob. $1 - p$

Then $\mu''(\lambda) = p \mu(\lambda) + (1 - p) \mu'(\lambda)$

Proof of preparation contextuality in 2d

$$P_a \leftrightarrow \psi_a = (1, 0)$$

$$P_A \leftrightarrow \psi_A = (0, 1)$$

$$P_b \leftrightarrow \psi_b = (1/2, \sqrt{3}/2)$$

$$P_B \leftrightarrow \psi_B = (\sqrt{3}/2, -1/2)$$

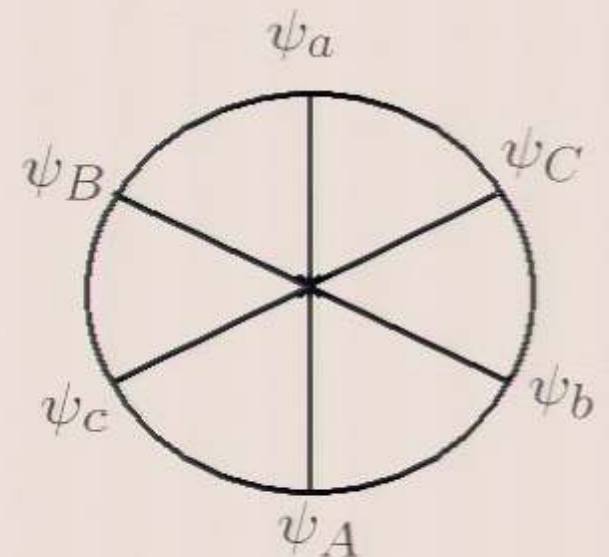
$$P_c \leftrightarrow \psi_c = (1/2, -\sqrt{3}/2)$$

$$P_C \leftrightarrow \psi_C = (\sqrt{3}/2, 1/2)$$

$$\psi_a \perp \psi_A$$

$$\psi_b \perp \psi_B$$

$$\psi_c \perp \psi_C$$



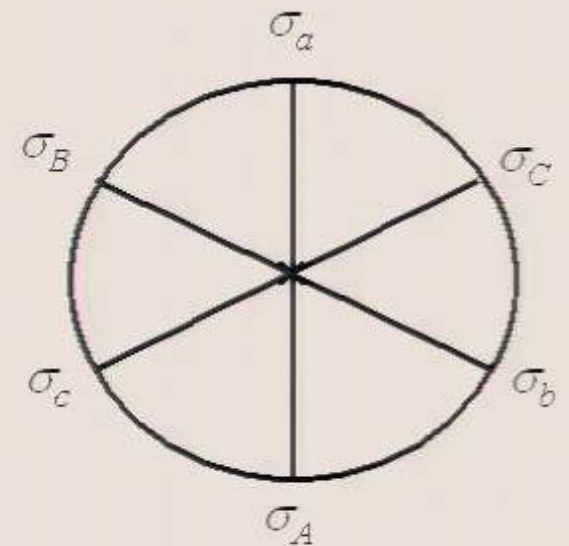
Proof of preparation contextuality in 2d

$$\begin{array}{lcl}
 P_a & \leftrightarrow & \sigma_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
 P_A & \leftrightarrow & \sigma_A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
 P_b & \leftrightarrow & \sigma_b = \begin{pmatrix} \frac{1}{4} & \frac{1}{4}\sqrt{3} \\ \frac{1}{4}\sqrt{3} & \frac{3}{4} \end{pmatrix} \\
 P_B & \leftrightarrow & \sigma_B = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{1}{4} \end{pmatrix} \\
 P_c & \leftrightarrow & \sigma_c = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{3}{4} \end{pmatrix} \\
 P_C & \leftrightarrow & \sigma_C = \begin{pmatrix} \frac{3}{4} & \frac{1}{4}\sqrt{3} \\ \frac{1}{4}\sqrt{3} & \frac{1}{4} \end{pmatrix}
 \end{array}$$

$$\sigma_a \sigma_A = 0$$

$$\sigma_b \sigma_B = 0$$

$$\sigma_c \sigma_C = 0$$



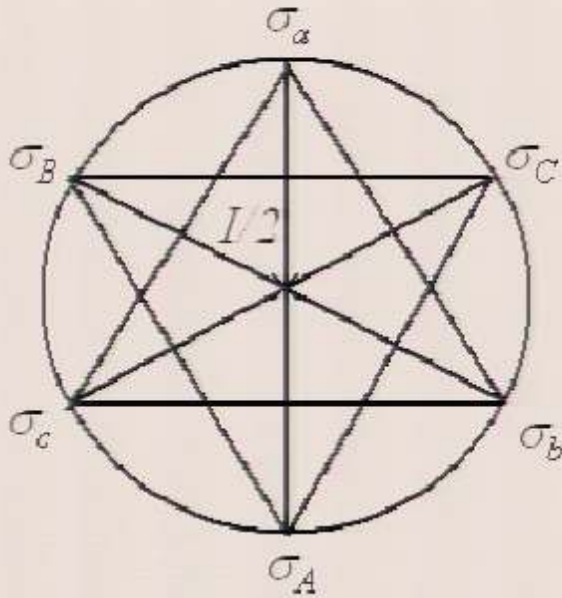
$P_{aA} \equiv P_a$ and P_A with prob. $1/2$ each

$P_{bB} \equiv P_b$ and P_B with prob. $1/2$ each

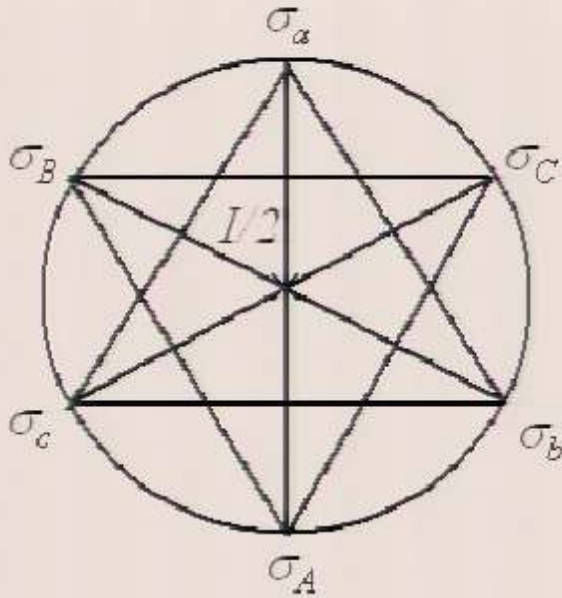
$P_{cC} \equiv P_c$ and P_C with prob. $1/2$ each

$P_{abc} \equiv P_a, P_b$ and P_c with prob. $1/3$ each

$P_{ABC} \equiv P_A, P_B$ and P_C with prob. $1/3$ each



$$\begin{aligned}
 I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\
 &= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
 &= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\
 &= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\
 &= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.
 \end{aligned}$$



$$\begin{aligned}
 I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\
 &= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
 &= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\
 &= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\
 &= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.
 \end{aligned}$$

$$\begin{aligned}
 P_{aA} &\simeq P_{bB} \simeq P_{cC} \\
 &\simeq P_{abc} \simeq P_{ABC}
 \end{aligned}$$

By **preparation noncontextuality**

$$\begin{aligned}
 \mu_{aA}(\lambda) &= \mu_{bB}(\lambda) = \mu_{cC}(\lambda) \\
 &= \mu_{abc}(\lambda) = \mu_{ABC}(\lambda) \\
 &\equiv \nu(\lambda)
 \end{aligned}$$

$P_{aA} \equiv P_a$ and P_A with prob. 1/2 each

$P_{bB} \equiv P_b$ and P_B with prob. 1/2 each

$P_{cC} \equiv P_c$ and P_C with prob. 1/2 each

$P_{abc} \equiv P_a, P_b$ and P_c with prob. 1/3 each

$P_{ABC} \equiv P_A, P_B$ and P_C with prob. 1/3 each



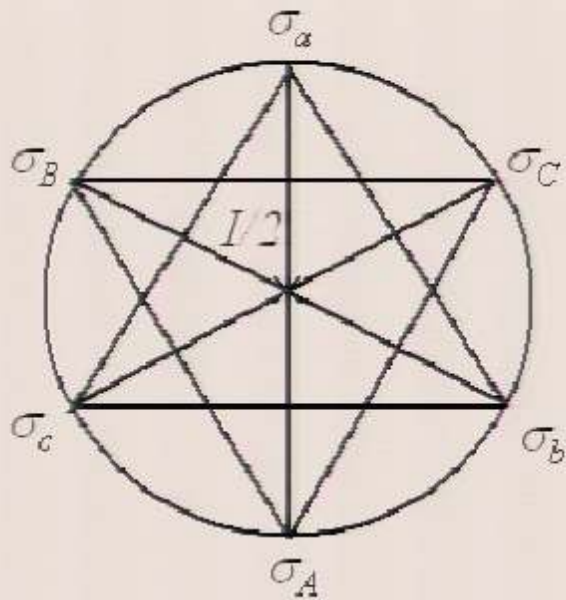
$$\mu_{aA}(\lambda) = \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda)$$

$$\mu_{bB}(\lambda) = \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda)$$

$$\mu_{cC}(\lambda) = \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda)$$

$$\mu_{abc}(\lambda) = \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda)$$

$$\mu_{ABC}(\lambda) = \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda)$$



$$\begin{aligned}
 I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\
 &= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
 &= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\
 &= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\
 &= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.
 \end{aligned}$$

$$\begin{aligned}
 P_{aA} &\simeq P_{bB} \simeq P_{cC} \\
 &\simeq P_{abc} \simeq P_{ABC}
 \end{aligned}$$

Our task is to find

$\mu_a(\lambda)$, $\mu_A(\lambda)$, $\mu_b(\lambda)$,
 $\mu_B(\lambda)$, $\mu_c(\lambda)$, $\mu_C(\lambda)$,
and $\nu(\lambda)$ such that

$$\mu_a(\lambda) \mu_A(\lambda) = 0$$

$$\mu_b(\lambda) \mu_B(\lambda) = 0$$

$$\mu_c(\lambda) \mu_C(\lambda) = 0$$

$$\begin{aligned}\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\ &= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\ &= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\ &= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\ &= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda).\end{aligned}$$

Our task is to find
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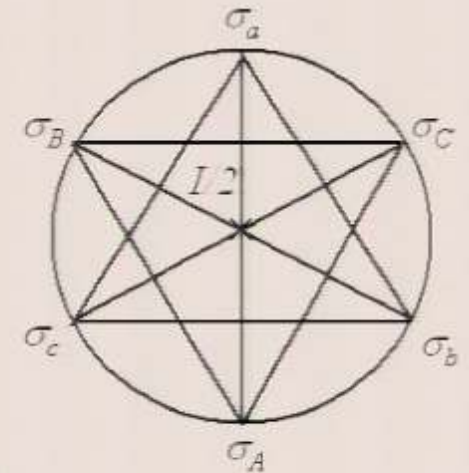
$$\begin{aligned} \nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\ &= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\ &= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\ &= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\ &= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda). \end{aligned}$$

i.e., paralleling the
 quantum structure:

$$\sigma_a \sigma_A = 0$$

$$\sigma_b \sigma_B = 0$$

$$\sigma_c \sigma_C = 0$$



$$\begin{aligned} I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\ &= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\ &= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\ &= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\ &= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C. \end{aligned}$$

Our task is to find

$\mu_a(\lambda)$, $\mu_A(\lambda)$, $\mu_b(\lambda)$,
 $\mu_B(\lambda)$, $\mu_c(\lambda)$, $\mu_C(\lambda)$,
and $\nu(\lambda)$ such that

$$\mu_a(\lambda) \mu_A(\lambda) = 0$$

$$\mu_b(\lambda) \mu_B(\lambda) = 0$$

$$\mu_c(\lambda) \mu_C(\lambda) = 0$$

$$\begin{aligned}\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\ &= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\ &= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\ &= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\ &= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda)\end{aligned}$$

At a given λ , Suppose

$$\mu_a(\lambda) = 0$$

$$\mu_b(\lambda) = 0$$

$$\mu_c(\lambda) = 0$$

Then we obtain
the all-zero solution

Our task is to find $\mu_a(\lambda)$, $\mu_A(\lambda)$, $\mu_b(\lambda)$, $\mu_B(\lambda)$, $\mu_c(\lambda)$, $\mu_C(\lambda)$, and $\nu(\lambda)$ such that

$$\begin{aligned}\mu_a(\lambda) \mu_A(\lambda) &= 0 \\ \mu_b(\lambda) \mu_B(\lambda) &= 0 \\ \mu_c(\lambda) \mu_C(\lambda) &= 0\end{aligned}$$

$$\begin{aligned}\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\ &= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\ &= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\ &= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\ &= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda)\end{aligned}$$

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Then we obtain the all-zero solution

Suppose

$$\begin{aligned}\mu_a(\lambda) &= 0 \\ \mu_b(\lambda) &= 0 \\ \mu_C(\lambda) &= 0\end{aligned}$$

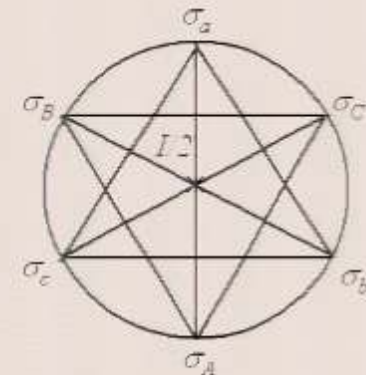
Then

$$\begin{aligned}\nu(\lambda) &= \frac{1}{3}\mu_c(\lambda) \\ &= \frac{1}{2}\mu_c(\lambda).\end{aligned}$$

Thus $\mu_c(\lambda) = 0$

Again yielding the all-zero solution

By symmetry, all other cases are similar



For all λ , we have the all-zero solution

CONTRADICTION

A proof starting from a Gleason-like theorem

Consider a function on projectors

$P \mapsto \omega(P)$, satisfying:

- 1) $0 \leq \omega(P) \leq 1$ for all P
- 2) $\omega(I) = 1$
- 3) $\omega(\sum_k P_k) = \sum_k \omega(P_k)$

Gleason's theorem: For $\dim(\mathcal{H}) \geq 3$,

$$\omega(P) = \text{Tr}(\rho P)$$

where ρ is a density operator

($\rho \geq 0$, $\text{Tr}(\rho) = 1$).

Consider a function on density operators

$\rho \mapsto f(\rho)$, satisfying:

1) $0 \leq f(\rho) \leq 1$ for all ρ

2) $f(\sum_k w_k \rho_k) = \sum_k w_k f(\rho_k)$ where $0 \leq w_k \leq 1$
and $\sum_k w_k = 1$.

The "reverse" Gleason's theorem:

$$f(\rho) = \text{Tr}(E\rho)$$

for some effect E (i.e. $0 \leq E \leq I$).

This is the dual of the generalized Gleason's theorem

Busch, Phys. Rev. Lett. **91**, 120403 (2003).

Caves, Fuchs, Manne, and Renes, Found. Phys. **34**, 193 (2004).

Also similar to a theorem in

L. Hardy, e-print quant-ph/9906123.

Suppose $\rho \leftrightarrow \mu_\rho(\lambda)$ (assume preparation noncontextuality)

$$\mu_\rho(\lambda) \geq 0$$

$$\text{If } \rho = \sum_k w_k \rho_k \text{ then } \mu_\rho(\lambda) = \sum_k w_k \mu_{\rho_k}(\lambda)$$

At a given value of λ , the μ considered as a function of ρ satisfy the conditions of the reverse Gleason's theorem.

Thus:

$$\mu_\rho(\lambda) = \text{Tr}(\rho E_\lambda) \text{ for some effect } E_\lambda$$

Recall: If $\rho_1 \rho_2 = 0$, then $\mu_{\rho_1}(\lambda) \mu_{\rho_2}(\lambda) = 0$

Let $\{\rho_k = |\psi_k\rangle\langle\psi_k|\}$ be an orthogonal basis

Consider a λ such that $\mu_{\rho_{k \neq 1}}(\lambda) = 0$.

Then, $E_\lambda = |\psi_1\rangle\langle\psi_1|$

But for $\{\rho'_k = |\psi'_k\rangle\langle\psi'_k|\}$ mutually noncollinear

we have $\mu_{\rho'_k}(\lambda) = \text{Tr}(\rho'_k |\psi_1\rangle\langle\psi_1|) \neq 0$ for all k .

Consider a function on density operators

$\rho \mapsto f(\rho)$, satisfying:

1) $0 \leq f(\rho) \leq 1$ for all ρ

2) $f(\sum_k w_k \rho_k) = \sum_k w_k f(\rho_k)$ where $0 \leq w_k \leq 1$
and $\sum_k w_k = 1$.

The "reverse" Gleason's theorem:

$$f(\rho) = \text{Tr}(E\rho)$$

for some effect E (i.e. $0 \leq E \leq I$).

This is the dual of the generalized Gleason's theorem

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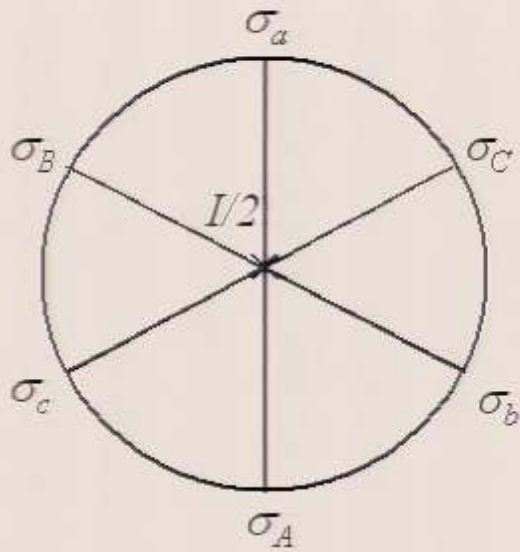
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A statistical proof

(joint work with Terry Rudolph)



$$\begin{aligned}
 I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\
 &= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
 &= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C
 \end{aligned}$$

By preparation NC

$$\begin{aligned}
 \nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\
 &= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\
 &= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda)
 \end{aligned}$$

$$\begin{aligned}w^{HV}(b|a) &= w_1 + w_2 \\w^{HV}(C|b) &= w_2 + w_6 \\w^{HV}(A|C) &= w_6 + w_8\end{aligned}$$

Thus

$$w^{HV}(b|a) - w^{HV}(C|b) + w^{HV}(A|C) = w_1 + w_8 \geq 0$$

$$\begin{aligned}
 w^{HV}(b|a) &= w_1 + w_2 \\
 w^{HV}(C|b) &= w_2 + w_6 \\
 w^{HV}(A|C) &= w_6 + w_8
 \end{aligned}$$

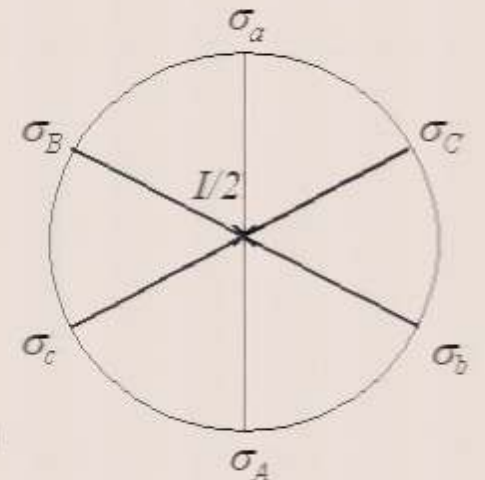
Thus

$$w^{HV}(b|a) - w^{HV}(C|b) + w^{HV}(A|C) = w_1 + w_8 \geq 0$$

However, the quantum probabilities are

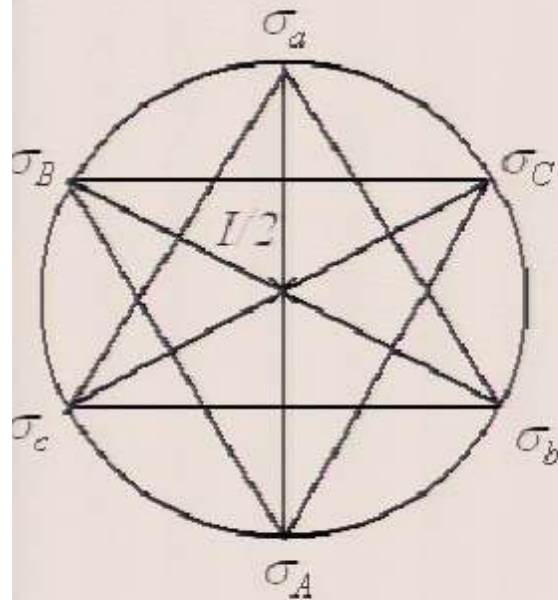
$$\begin{aligned}
 w^Q(b|a) &= 1/4 \\
 w^Q(C|b) &= 3/4 \\
 w^Q(A|C) &= 1/4
 \end{aligned}$$

$$w^Q(b|a) - w^Q(C|b) + w^Q(A|C) = -1/4 \leq 0$$



Proofs of nonlocality from proofs of contextuality

Consider the first proof given



$$\begin{aligned}
 I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\
 &= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
 &= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\
 &= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\
 &= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.
 \end{aligned}$$

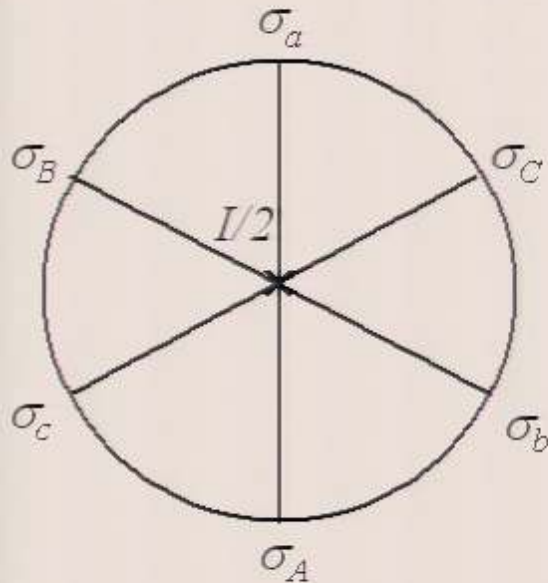
By **preparation noncontextuality**

$$\begin{aligned}
 \nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\
 &= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\
 &= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\
 &= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\
 &= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda).
 \end{aligned}$$

PNC for $I/2$ can be justified by locality

But PNC for σ_x cannot be justified by locality

Consider the statistical proof



$$\begin{aligned} I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\ &= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\ &= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \end{aligned}$$

By **preparation noncontextuality**

$$\begin{aligned} \nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\ &= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\ &= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \end{aligned}$$

The proof only
required PNC for $I/2$

this can be justified
by locality!

Locality \rightarrow PNC \rightarrow **CONTRADICTION**

Also,

Any bipartite Bell-type
proof of nonlocality



proof of preparation
contextuality

(proof due to Jon Barrett)

Other results

Noncontextuality can be defined for **any operational theory**:

If two experimental procedures are operationally indistinguishable, they should be represented in the same way in the hidden variable model

One can also generalize the notion of noncontextuality to **unsharp measurements** and **transformations** and to **more general sorts of contexts**

Every proof of preparation contextuality based on the ambiguity of mixtures yields a proof of contextuality for unsharp measurements

There is a connection between **Horn's problem** and contextuality for unsharp measurements

Open questions

For quantum logicians

- Does preparation contextuality have a natural expression within the convex set approach to quantum logic (see e.g. Mielnik's work)
- Does contextuality for unsharp measurements or transformations have a natural expression in quantum logic?
- What sorts of theories, besides quantum mechanics, are contextual (using the operational definition)

For quantum information theorists

- Is contextuality critical for any of the information-theoretic advantages of quantum theory?
Random access codes? (see Ernesto Galvão's thesis)
Exponential speed-up?

For quantum foundations types

- Is there a simple physical principle which implies contextuality?

For more on preparation contextuality, see:

RWS, Phys. Rev. A **71**, 052108 (2005) or [quant-ph/0406166](https://arxiv.org/abs/quant-ph/0406166).