Title: Preparation contextuality in its myriad forms

Date: Jul 19, 2005 02:00 PM

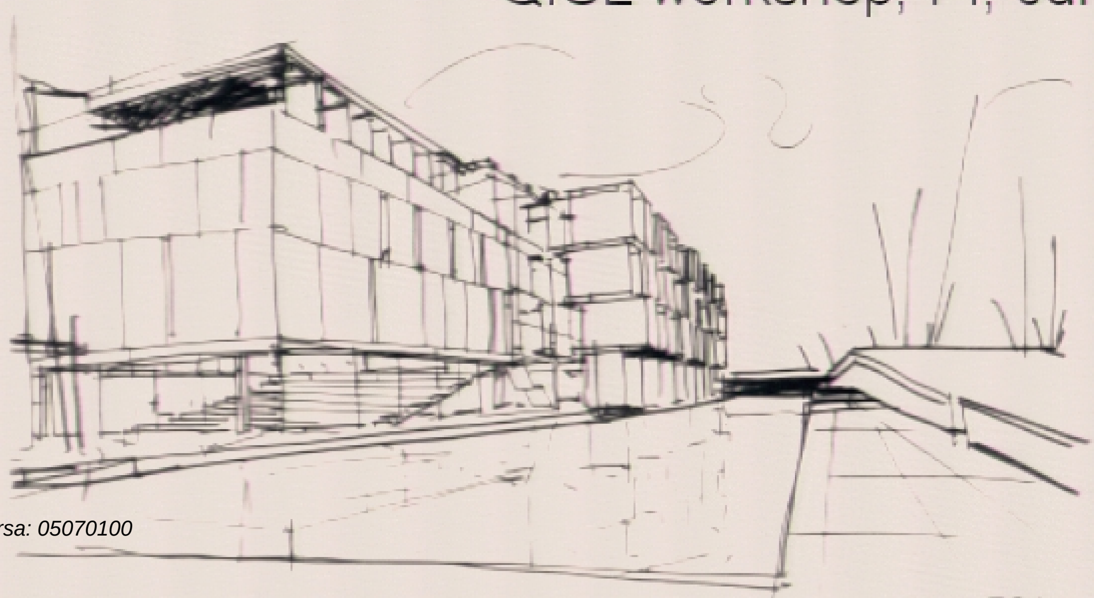
URL: http://pirsa.org/05070100

Abstract:

Preparation Contextuality in its myriad forms

Robert Spekkens
Perimeter Institute for Theoretical Physics,
Waterloo, Canada

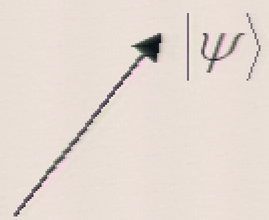
QICL workshop, PI, July 19, 2005

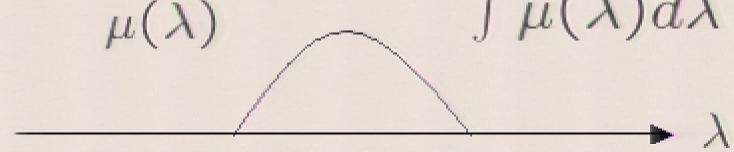


PERIMETER INSTITUTE
FOR THEORETICAL PHYSICS

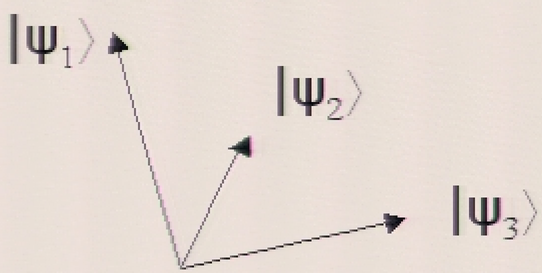
Traditional Measurement Contextuality

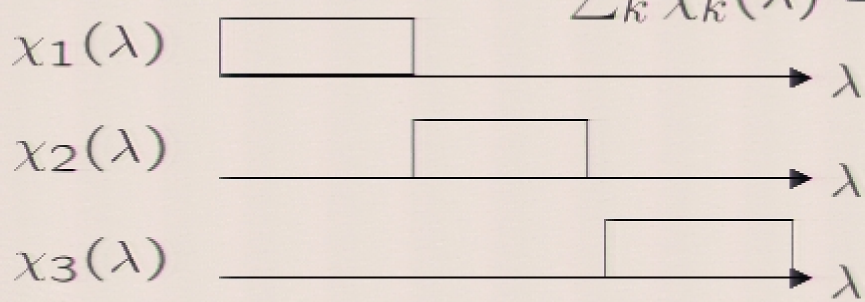
The idea of a deterministic hidden variable theory is that





$$0 \leq \mu(\lambda)d\lambda \leq 1$$
$$\int \mu(\lambda)d\lambda = 1$$





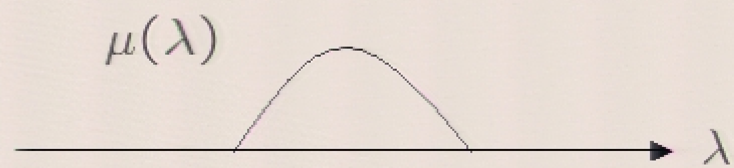
$$\chi_k : \Omega \rightarrow \{0, 1\}$$
$$\sum_k \chi_k(\lambda) = 1$$

$$|\langle\psi|\psi_k\rangle|^2 = \int \mu(\lambda)\chi_k(\lambda)d\lambda$$

The idea of a deterministic hidden variable theory is that

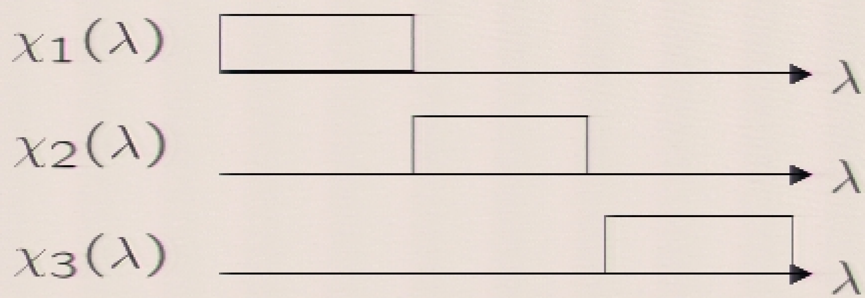
$$\rho$$
$$\langle\psi|\rho|\psi\rangle \geq 0, \forall\psi$$
$$\mathrm{Tr}(\rho) = 1$$

$\Longleftrightarrow$



$$\{P_k\}$$
$$\langle\psi|P_k|\psi\rangle \geq 0, \forall\psi$$
$$\sum P_k = I$$
$$P_k P_{k'} = 0 \text{ for } k \neq k'$$

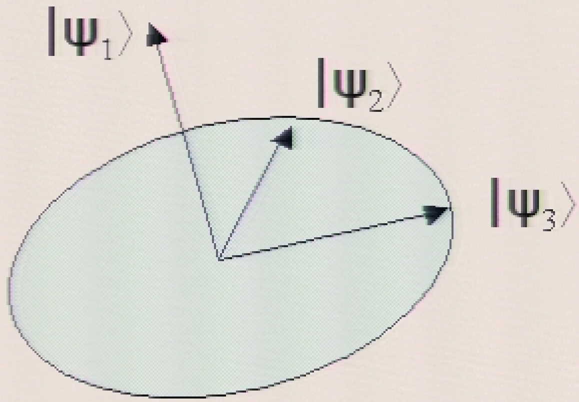
$\Longleftrightarrow$

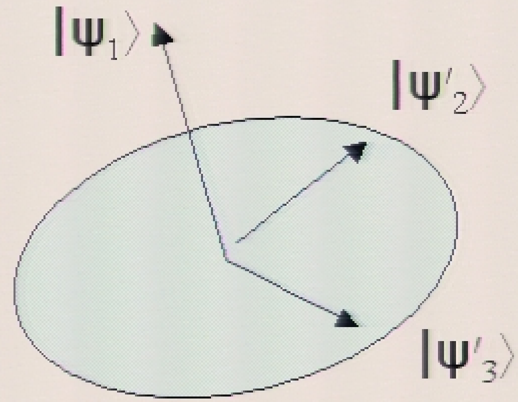


$$\mathrm{Tr}(\rho P_k) = \int \mu(\lambda)\chi_k(\lambda)d\lambda$$

There are many ways of measuring a non-rank-1 PVM

Ex:





M = measure PVM
$\{|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|, |\psi_3\rangle\langle\psi_3|\}$
then coarse-grain 2 and 3

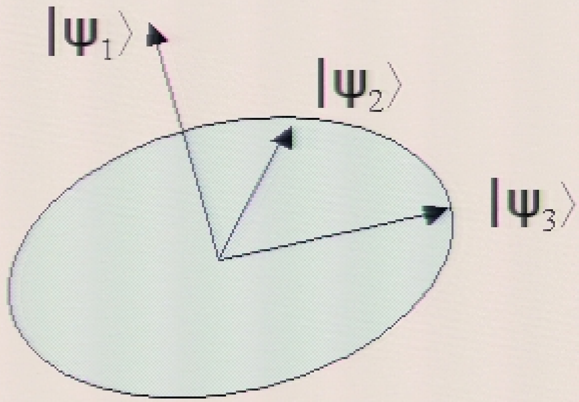
M′ = measure PVM
$\{|\psi_1\rangle\langle\psi_1|, |\psi'_2\rangle\langle\psi'_2|, |\psi'_3\rangle\langle\psi'_3|\}$
then coarse-grain 2 and 3

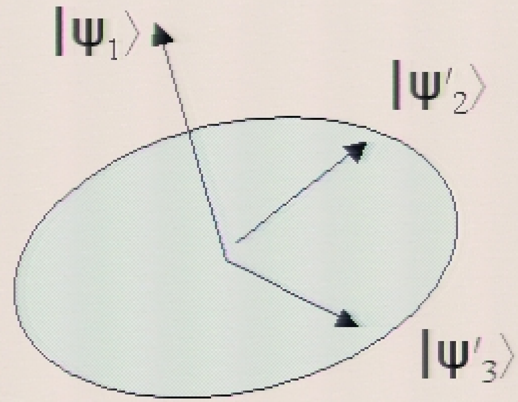
$$\{|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|\}$$
$$= \{|\psi_1\rangle\langle\psi_1|, I - |\psi_1\rangle\langle\psi_1|\}$$

$$\{|\psi_1\rangle\langle\psi_1|, |\psi'_2\rangle\langle\psi_2| + |\psi'_3\rangle\langle\psi'_3|\}$$
$$= \{|\psi_1\rangle\langle\psi_1|, I - |\psi_1\rangle\langle\psi_1|\}$$

There are many ways of measuring a non-rank-1 PVM

Ex:





M = measure PVM

$\{|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|, |\psi_3\rangle\langle\psi_3|\}$

then coarse-grain 2 and 3

M′ = measure PVM

$\{|\psi_1\rangle\langle\psi_1|, |\psi_2'\rangle\langle\psi_2'|, |\psi_3'\rangle\langle\psi_3'|\}$

then coarse-grain 2 and 3

$$\{|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|\}$$
$$= \{|\psi_1\rangle\langle\psi_1|, I - |\psi_1\rangle\langle\psi_1|\}$$

$$\{|\psi_1\rangle\langle\psi_1|, |\psi_2'\rangle\langle\psi_2| + |\psi_3'\rangle\langle\psi_3'|\}$$
$$= \{|\psi_1\rangle\langle\psi_1|, I - |\psi_1\rangle\langle\psi_1|\}$$

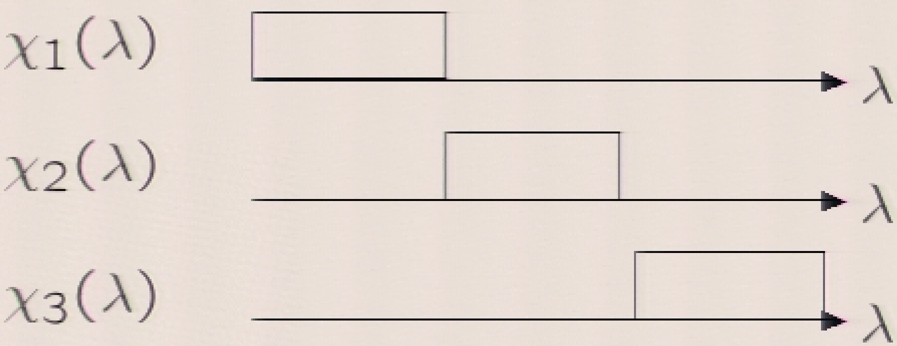
Traditional Measurement Noncontextuality

if M≃M′ then $\chi_{\mathrm{M},k}(\lambda) = \chi_{\mathrm{M}',k}(\lambda)$

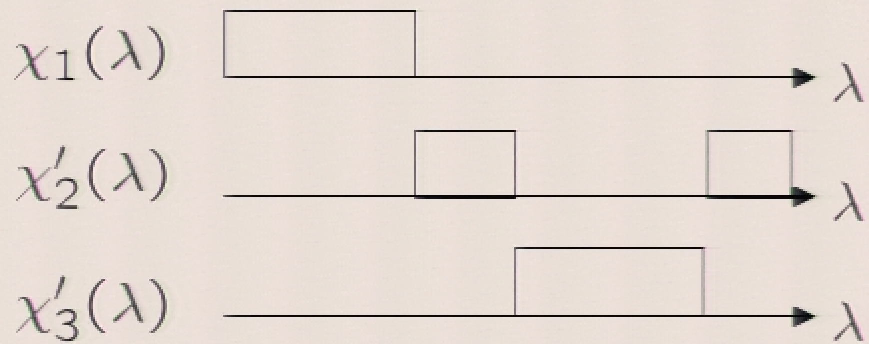
Rep'd by same PVM → Rep'd by same indicator functions

The hope is to represent this as follows:

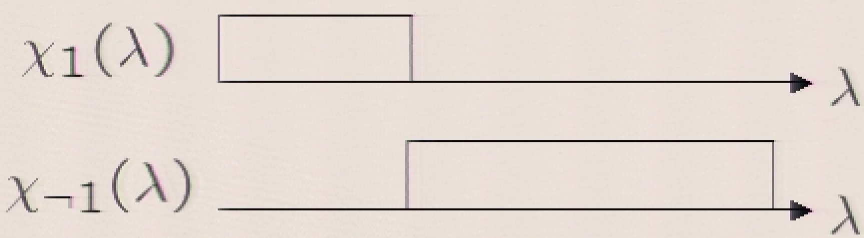
$$\{|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|, |\psi_3\rangle\langle\psi_3|\}$$



$$\{|\psi_1\rangle\langle\psi_1|, |\psi_2'\rangle\langle\psi_2'|, |\psi_3'\rangle\langle\psi_3'|\}$$



$$\{|\psi_1\rangle\langle\psi_1|, I - |\psi_1\rangle\langle\psi_1|\}$$



Every P is associated with the same $\chi(\lambda)$ regardless of how it is measured

It was shown by Bell (1966) and Kochen and Specker (1967) that a noncontextual hidden variable model of quantum theory for Hilbert spaces of dimensionality 3 or greater is impossible. That is, quantum theory is contextual

This is the Bell-Kochen-Specker theorem

Key fact for proof: there is a multiplicity of decompositions of a rank-2 projector into rank-1 projectors.

$$
\begin{aligned}
I - |\psi_1\rangle\langle\psi_1| &= |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3| \\
&= |\psi_2'\rangle\langle\psi_2'| + |\psi_3'\rangle\langle\psi_3'|
\end{aligned}
$$

Recall a similar fact for preparations: there is a multiplicity of convex decompositions of a mixed state into pure states
a.k.a the ambiguity of mixtures

$$
\begin{aligned}
\frac{1}{2}I &= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \\
&= \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| \\
&= \frac{1}{2}|+i\rangle\langle +i| + \frac{1}{2}|-i\rangle\langle -i|
\end{aligned}
$$

$$
\begin{aligned}
|\pm\rangle &= |0\rangle \pm |1\rangle \\
|\pm i\rangle &= |0\rangle \pm i|1\rangle
\end{aligned}
$$

Can we derive a no-go theorem from this? Yes.

Preparation Contextuality

$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$

P = implement P_j
with probability p_j

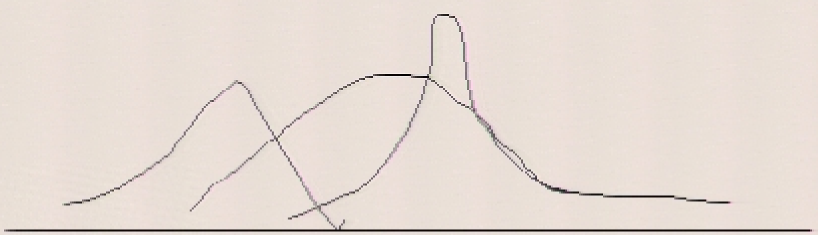
$\rho = \sum_i q_i |\chi_i\rangle\langle\chi_i|$

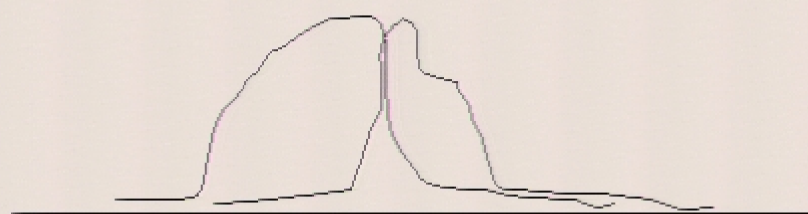
P′ = implement P'_j
with probability q_j

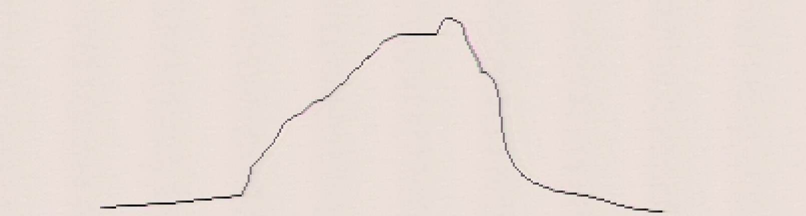
Preparation Noncontextuality

if $\mathsf{P} \simeq \mathsf{P}'$ then $\mu_{\mathsf{P}}(\lambda) = \mu_{\mathsf{P}'}(\lambda)$

Rep'd by same density operator → Rep'd by same distribution







Proof of preparation contextuality

(a preparation noncontextual hidden variable model is impossible)

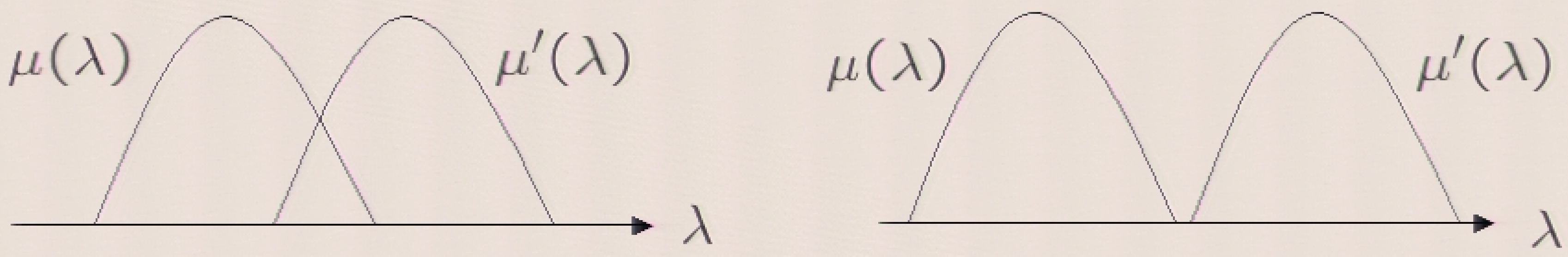
Important features of hidden variable models

$$\text{Let } \mathsf{P} \leftrightarrow \mu(\lambda)$$
$$\mathsf{P}' \leftrightarrow \mu'(\lambda)$$

Representing distinguishability:

If P and P′ are distinguishable with certainty

$$\text{then } \mu(\lambda)\,\mu'(\lambda) = 0$$



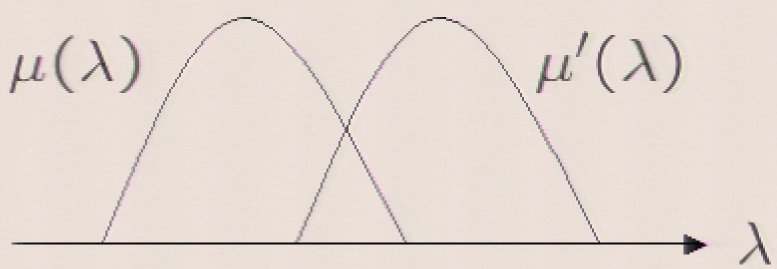
Important features of hidden variable models

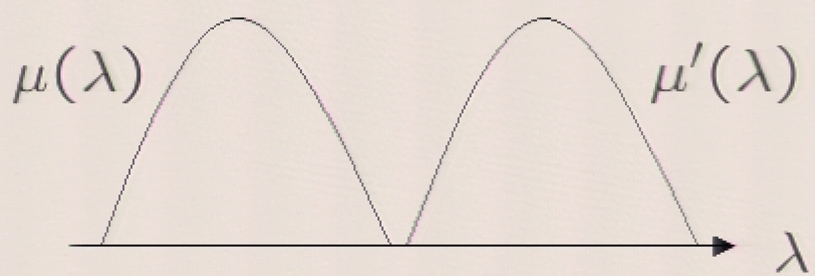
$$\text{Let P} \leftrightarrow \mu(\lambda)$$
$$\text{P}' \leftrightarrow \mu'(\lambda)$$

Representing distinguishability:

If P and P$'$ are distinguishable with certainty

then $\mu(\lambda)\,\mu'(\lambda) = 0$





Representing convex combination:

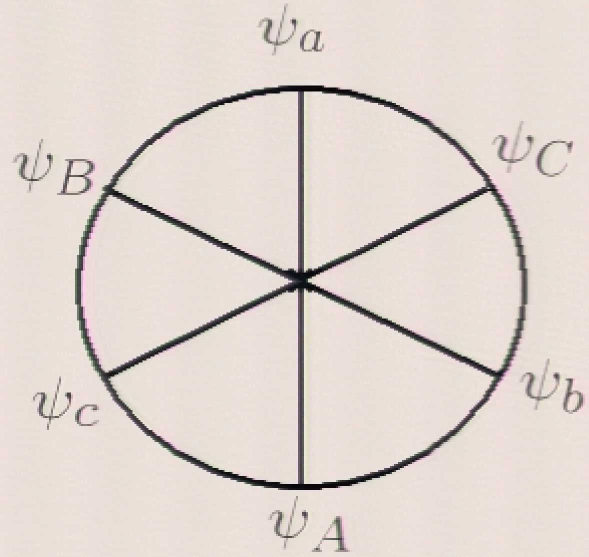
If P$''$ = P with prob. p and P$'$ with prob. $1-p$

Then $\mu''(\lambda) = p\,\mu(\lambda) + (1-p)\,\mu'(\lambda)$

Proof of preparation contextuality in 2d

$$
\begin{aligned}
\mathsf{P}_a &\leftrightarrow \psi_a = (1,0) \\
\mathsf{P}_A &\leftrightarrow \psi_A = (0,1) \\
\mathsf{P}_b &\leftrightarrow \psi_b = (1/2, \sqrt{3}/2) \\
\mathsf{P}_B &\leftrightarrow \psi_B = (\sqrt{3}/2, -1/2) \\
\mathsf{P}_c &\leftrightarrow \psi_c = (1/2, -\sqrt{3}/2) \\
\mathsf{P}_C &\leftrightarrow \psi_C = (\sqrt{3}/2, 1/2)
\end{aligned}
$$

$$
\psi_a \perp \psi_A \quad \psi_b \perp \psi_B \quad \psi_c \perp \psi_C
$$



Proof of preparation contextuality in 2d

$$\mathsf{P}_a \leftrightarrow \sigma_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathsf{P}_A \leftrightarrow \sigma_A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathsf{P}_b \leftrightarrow \sigma_b = \begin{pmatrix} \frac{1}{4} & \frac{1}{4}\sqrt{3} \\ \frac{1}{4}\sqrt{3} & \frac{3}{4} \end{pmatrix}$$

$$\mathsf{P}_B \leftrightarrow \sigma_B = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{1}{4} \end{pmatrix}$$

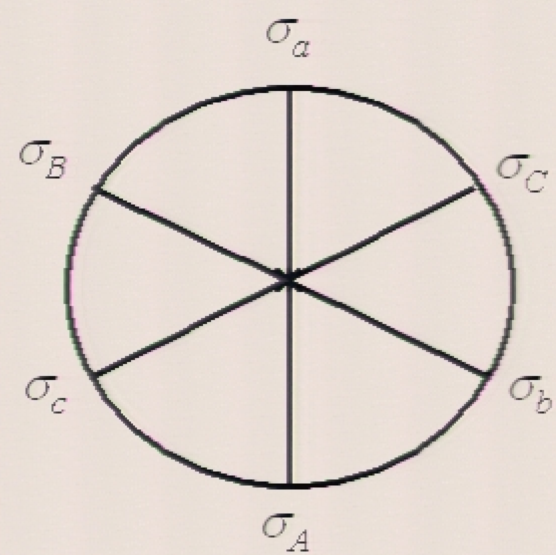
$$\mathsf{P}_c \leftrightarrow \sigma_c = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{3}{4} \end{pmatrix}$$

$$\mathsf{P}_C \leftrightarrow \sigma_C = \begin{pmatrix} \frac{3}{4} & \frac{1}{4}\sqrt{3} \\ \frac{1}{4}\sqrt{3} & \frac{1}{4} \end{pmatrix}$$

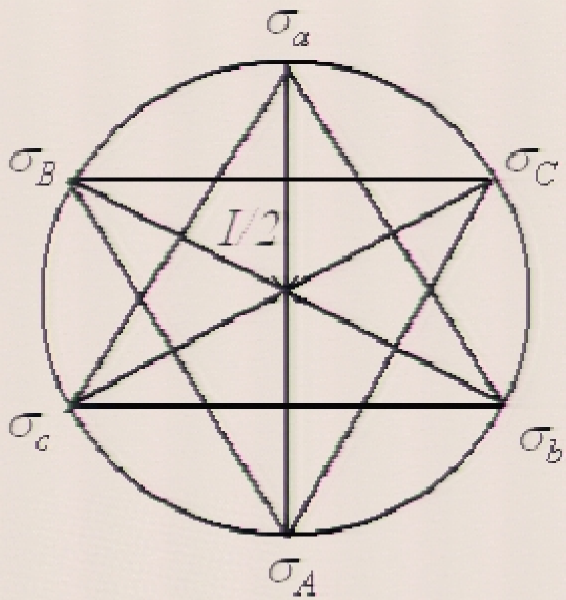
$$\sigma_a \sigma_A = 0$$

$$\sigma_b \sigma_B = 0$$

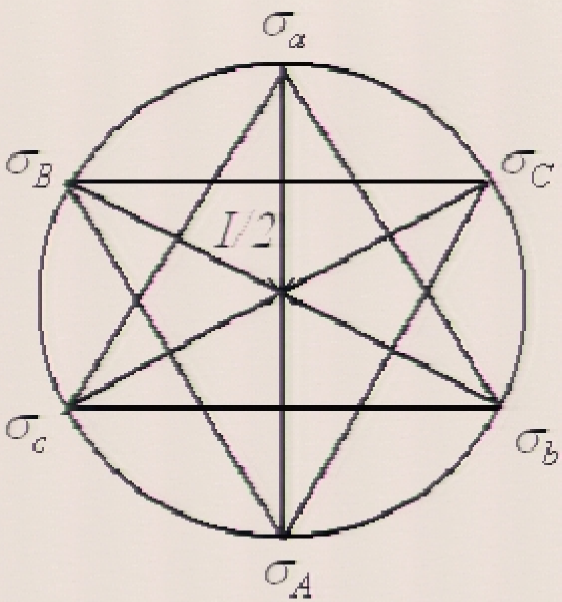
$$\sigma_c \sigma_C = 0$$



$$
\begin{aligned}
\mathsf{P}_{aA} &\equiv \mathsf{P}_a \text{ and } \mathsf{P}_A \text{ with prob. } 1/2 \text{ each}\\
\mathsf{P}_{bB} &\equiv \mathsf{P}_b \text{ and } \mathsf{P}_B \text{ with prob. } 1/2 \text{ each}\\
\mathsf{P}_{cC} &\equiv \mathsf{P}_c \text{ and } \mathsf{P}_C \text{ with prob. } 1/2 \text{ each}\\
\mathsf{P}_{abc} &\equiv \mathsf{P}_a,\ \mathsf{P}_b \text{ and } \mathsf{P}_c \text{ with prob. } 1/3 \text{ each}\\
\mathsf{P}_{ABC} &\equiv \mathsf{P}_A,\ \mathsf{P}_B \text{ and } \mathsf{P}_C \text{ with prob. } 1/3 \text{ each}
\end{aligned}
$$



$$
\begin{aligned}
I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\
&= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
&= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\
&= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\
&= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.
\end{aligned}
$$



$$
\begin{aligned}
I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\
&= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
&= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\
&= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\
&= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C .
\end{aligned}
$$

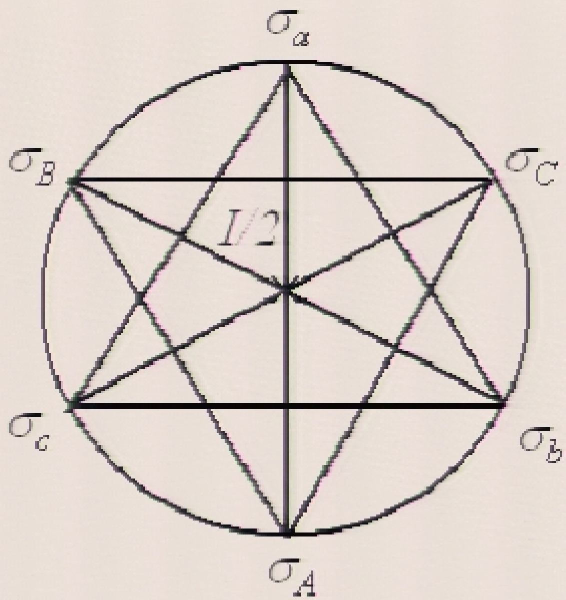
$$
\mathsf{P}_{aA} \simeq \mathsf{P}_{bB} \simeq \mathsf{P}_{cC} \\
\simeq \mathsf{P}_{abc} \simeq \mathsf{P}_{ABC}
$$

By preparation noncontextuality

$$
\mu_{aA}(\lambda) = \mu_{bB}(\lambda) = \mu_{cC}(\lambda) \\
= \mu_{abc}(\lambda) = \mu_{ABC}(\lambda) \\
\equiv \nu(\lambda)
$$

$$
\begin{aligned}
\mathsf{P}_{aA} &\equiv \mathsf{P}_a \text{ and } \mathsf{P}_A \text{ with prob. } 1/2 \text{ each} \\
\mathsf{P}_{bB} &\equiv \mathsf{P}_b \text{ and } \mathsf{P}_B \text{ with prob. } 1/2 \text{ each} \\
\mathsf{P}_{cC} &\equiv \mathsf{P}_c \text{ and } \mathsf{P}_C \text{ with prob. } 1/2 \text{ each} \\
\mathsf{P}_{abc} &\equiv \mathsf{P}_a,\ \mathsf{P}_b \text{ and } \mathsf{P}_c \text{ with prob. } 1/3 \text{ each} \\
\mathsf{P}_{ABC} &\equiv \mathsf{P}_A,\ \mathsf{P}_B \text{ and } \mathsf{P}_C \text{ with prob. } 1/3 \text{ each}
\end{aligned}
$$

$$
\longrightarrow \quad
\begin{aligned}
\mu_{aA}(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\
\mu_{bB}(\lambda) &= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\
\mu_{cC}(\lambda) &= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\
\mu_{abc}(\lambda) &= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\
\mu_{ABC}(\lambda) &= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda)
\end{aligned}
$$



$$
\begin{aligned}
I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\
&= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
&= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\
&= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\
&= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{P}_{aA} \simeq \mathsf{P}_{bB} \simeq \mathsf{P}_{cC} \\
&\simeq \mathsf{P}_{abc} \simeq \mathsf{P}_{ABC}
\end{aligned}
$$

Our task is to find
$\mu_a(\lambda)$, $\mu_A(\lambda)$, $\mu_b(\lambda)$,
$\mu_B(\lambda)$, $\mu_c(\lambda)$, $\mu_C(\lambda)$,
and $\nu(\lambda)$ such that

$$
\begin{aligned}
\mu_a(\lambda)\,\mu_A(\lambda) &= 0 \\
\mu_b(\lambda)\,\mu_B(\lambda) &= 0 \\
\mu_c(\lambda)\,\mu_C(\lambda) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\
&= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\
&= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\
&= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\
&= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda).
\end{aligned}
$$

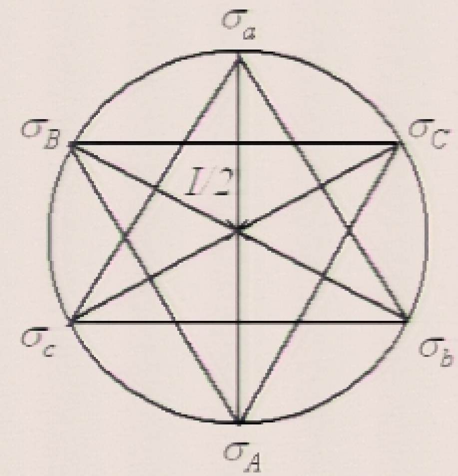
Our task is to find $\mu_a(\lambda)$, $\mu_A(\lambda)$, $\mu_b(\lambda)$, $\mu_B(\lambda)$, $\mu_c(\lambda)$, $\mu_C(\lambda)$, and $\nu(\lambda)$ such that

$$\mu_a(\lambda)\,\mu_A(\lambda) = 0$$
$$\mu_b(\lambda)\,\mu_B(\lambda) = 0$$
$$\mu_c(\lambda)\,\mu_C(\lambda) = 0$$

$$\begin{aligned}\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda)\\ &= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda)\\ &= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda)\\ &= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda)\\ &= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda).\end{aligned}$$

i.e., paralleling the quantum structure:

$$\sigma_a\sigma_A = 0$$
$$\sigma_b\sigma_B = 0$$
$$\sigma_c\sigma_C = 0$$



$$\begin{aligned}I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A\\ &= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B\\ &= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C\\ &= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c\\ &= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.\end{aligned}$$

Our task is to find

$\mu_a(\lambda)$, $\mu_A(\lambda)$, $\mu_b(\lambda)$,
$\mu_B(\lambda)$, $\mu_c(\lambda)$, $\mu_C(\lambda)$,
and $\nu(\lambda)$ such that

$$\begin{aligned}\mu_a(\lambda)\,\mu_A(\lambda) &= 0\\ \mu_b(\lambda)\,\mu_B(\lambda) &= 0\\ \mu_c(\lambda)\,\mu_C(\lambda) &= 0\end{aligned}$$

$$\begin{aligned}\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda)\\ &= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda)\\ &= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda)\\ &= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda)\\ &= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda)\end{aligned}$$

At a given λ, Suppose

$$\begin{aligned}\mu_a(\lambda) &= 0\\ \mu_b(\lambda) &= 0\\ \mu_c(\lambda) &= 0\end{aligned}$$

Then we obtain
the all-zero solution

Our task is to find

$\mu_a(\lambda)$, $\mu_A(\lambda)$, $\mu_b(\lambda)$, $\mu_B(\lambda)$, $\mu_c(\lambda)$, $\mu_C(\lambda)$, and $\nu(\lambda)$ such that

$$\begin{aligned}\mu_a(\lambda)\,\mu_A(\lambda) &= 0\\ \mu_b(\lambda)\,\mu_B(\lambda) &= 0\\ \mu_c(\lambda)\,\mu_C(\lambda) &= 0\end{aligned}$$

$$\begin{aligned}\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda)\\ &= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda)\\ &= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda)\\ &= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda)\\ &= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda)\end{aligned}$$

At a given λ, Suppose

$$\begin{aligned}\mu_a(\lambda) &= 0\\ \mu_b(\lambda) &= 0\\ \mu_c(\lambda) &= 0\end{aligned}$$

Then we obtain the all-zero solution

Suppose

$$\begin{aligned}\mu_a(\lambda) &= 0\\ \mu_b(\lambda) &= 0\\ \mu_C(\lambda) &= 0\end{aligned}$$

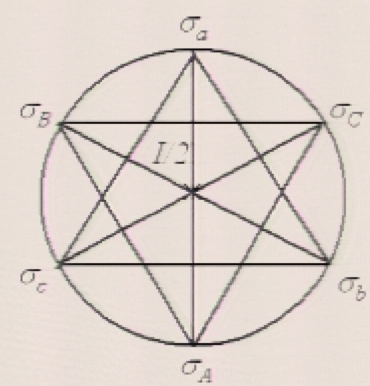
Then

$$\begin{aligned}\nu(\lambda) &= \frac{1}{3}\mu_c(\lambda)\\ &= \frac{1}{2}\mu_c(\lambda).\end{aligned}$$

Thus $\mu_c(\lambda) = 0$

Again yielding the all-zero solution

By symmetry, all other cases are similar



For all λ, we have the all-zero solution

CONTRADICTION

A proof starting from a Gleason-like theorem

Consider a function on projectors

$P \mapsto \omega(P)$, satisfying:

1) $0 \leq \omega(P) \leq 1$ for all P

2) $\omega(I) = 1$

3) $\omega(\sum_k P_k) = \sum_k \omega(P_k)$

Gleason's theorem: For $\dim(\mathcal{H}) \geq 3$,

$$\omega(P) = \mathrm{Tr}(\rho P)$$

where ρ is a density operator

($\rho \geq 0$, $\mathrm{Tr}(\rho) = 1$).

Consider a function on density operators $\rho \mapsto f(\rho)$, satisfying:

1) $0 \leq f(\rho) \leq 1$ for all ρ

2) $f(\sum_k w_k \rho_k) = \sum_k w_k f(\rho_k)$ where $0 \leq w_k \leq 1$ and $\sum_k w_k = 1$.

The "reverse" Gleason's theorem:

$$f(\rho) = Tr\,(E\rho)$$

for some effect E (i.e. $0 \leq E \leq I$).

This is the dual of the generalized Gleason's theorem

Busch, Phys. Rev. Lett. **91**, 120403 (2003).

Caves, Fuchs, Manne, and Renes, Found. Phys. **34**, 193 (2004).

Also similar to a theorem in

L. Hardy, e-print quant-ph/9906123.

Suppose $\rho \leftrightarrow \mu_\rho(\lambda)$ (assume preparation noncontextuality)

$\mu_\rho(\lambda) \geq 0$

If $\rho = \sum_k w_k \rho_k$ then $\mu_\rho(\lambda) = \sum_k w_k \mu_{\rho_k}(\lambda)$

At a given value of λ, the μ considered as a function of ρ satisfy the conditions of the reverse Gleason's theorem.

Thus:

$$\mu_\rho(\lambda) = \mathrm{Tr}(\rho E_\lambda) \quad \text{for some effect } E_\lambda$$

Recall: If $\rho_1\rho_2 = 0$, then $\mu_{\rho_1}(\lambda)\mu_{\rho_2}(\lambda) = 0$

Let $\{\rho_k = |\psi_k\rangle\langle\psi_k|\}$ be an orthogonal basis

Consider a λ such that $\mu_{\rho_{k\neq 1}}(\lambda) = 0$.

Then, $E_\lambda = |\psi_1\rangle\langle\psi_1|$

But for $\{\rho'_k = |\psi'_k\rangle\langle\psi'_k|\}$ mutually noncollinear

we have $\mu_{\rho'_k}(\lambda) = \mathrm{Tr}(\rho'_k|\psi_1\rangle\langle\psi_1|) \neq 0$ for all k.

CONTRADICTION

Consider a function on density operators
$\rho \mapsto f(\rho)$, satisfying:
1) $0 \leq f(\rho) \leq 1$ for all ρ
2) $f(\sum_k w_k \rho_k) = \sum_k w_k f(\rho_k)$ where $0 \leq w_k \leq 1$ and $\sum_k w_k = 1$.

The "reverse" Gleason's theorem:

$$f(\rho) = Tr\,(E\rho)$$

for some effect E (i.e. $0 \leq E \leq I$).

This is the dual of the generalized Gleason's theorem
Busch, Phys. Rev. Lett. **91**, 120403 (2003).
Caves, Fuchs, Manne, and Renes, Found. Phys. **34**, 193 (2004).
Also similar to a theorem in
L. Hardy, e-print quant-ph/9906123.

Suppose $\rho \leftrightarrow \mu_\rho(\lambda)$ (assume preparation noncontextuality)

$\mu_\rho(\lambda) \geq 0$

If $\rho = \sum_k w_k \rho_k$ then $\mu_\rho(\lambda) = \sum_k w_k \mu_{\rho_k}(\lambda)$

At a given value of λ, the μ considered as a function of ρ satisfy the conditions of the reverse Gleason's theorem.

Thus:

$\mu_\rho(\lambda) = \mathrm{Tr}(\rho E_\lambda)$ for some effect E_λ

Recall: If $\rho_1 \rho_2 = 0$, then $\mu_{\rho_1}(\lambda)\mu_{\rho_2}(\lambda) = 0$

Let $\{\rho_k = |\psi_k\rangle\langle\psi_k|\}$ be an orthogonal basis

Consider a λ such that $\mu_{\rho_{k\neq 1}}(\lambda) = 0$.

Then, $E_\lambda = |\psi_1\rangle\langle\psi_1|$

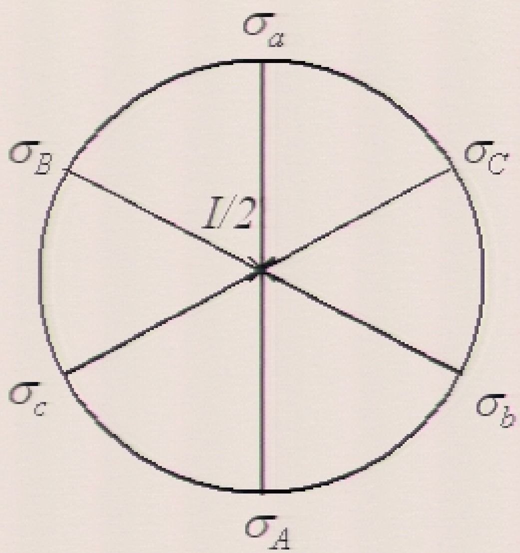
But for $\{\rho'_k = |\psi'_k\rangle\langle\psi'_k|\}$ mutually noncollinear

we have $\mu_{\rho'_k}(\lambda) = \mathrm{Tr}(\rho'_k |\psi_1\rangle\langle\psi_1|) \neq 0$ for all k.

CONTRADICTION

A statistical proof

(joint work with Terry Rudolph)



$$
\begin{aligned}
I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\
&= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
&= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C
\end{aligned}
$$

By preparation NC

$$
\begin{aligned}
\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\
&= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\
&= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda)
\end{aligned}
$$

$$
\begin{aligned}
w^{HV}(b|a) &= w_1 + w_2 \\
w^{HV}(C|b) &= w_2 + w_6 \\
w^{HV}(A|C) &= w_6 + w_8
\end{aligned}
$$

Thus

$$
w^{HV}(b|a) - w^{HV}(C|b) + w^{HV}(A|C) = w_1 + w_8 \geq 0
$$

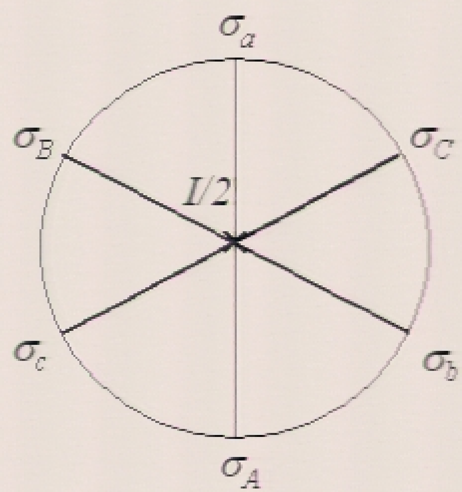
$$
\begin{aligned}
w^{HV}(b|a) &= w_1 + w_2 \\
w^{HV}(C|b) &= w_2 + w_6 \\
w^{HV}(A|C) &= w_6 + w_8
\end{aligned}
$$

Thus

$$w^{HV}(b|a) - w^{HV}(C|b) + w^{HV}(A|C) = w_1 + w_8 \geq 0$$

However, the quantum probabilities are

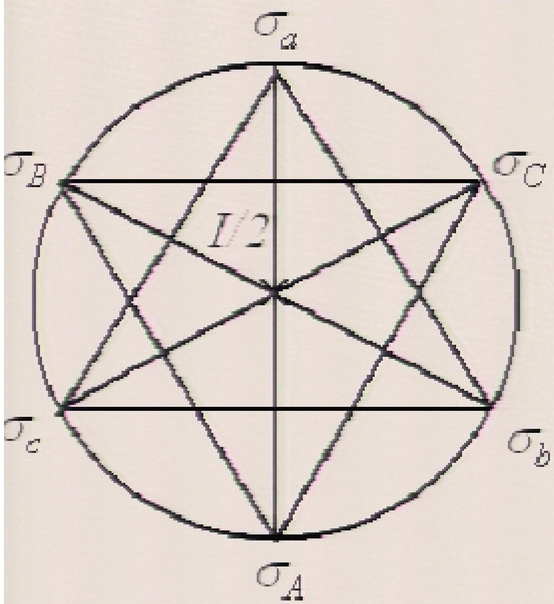
$$
\begin{aligned}
w^{Q}(b|a) &= 1/4 \\
w^{Q}(C|b) &= 3/4 \\
w^{Q}(A|C) &= 1/4
\end{aligned}
$$



$$w^{Q}(b|a) - w^{Q}(C|b) + w^{Q}(A|C) = -1/4 \leq 0$$

Proofs of nonlocality from proofs of contextuality

Consider the first proof given



$$
\begin{aligned}
I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\
&= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
&= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\
&= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\
&= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C .
\end{aligned}
$$

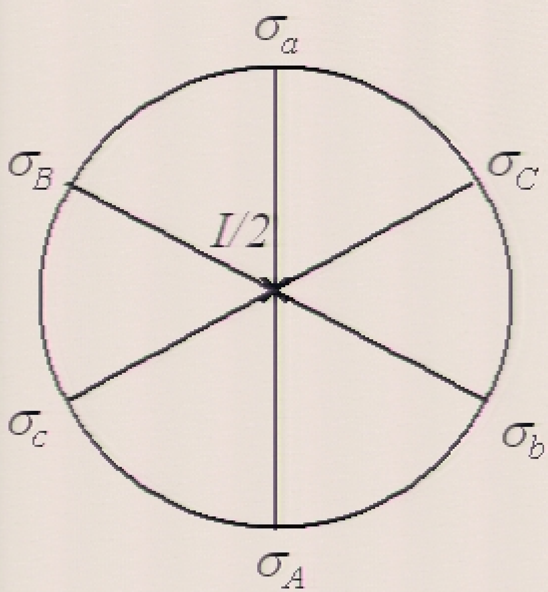
By preparation noncontextuality

$$
\begin{aligned}
\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\
&= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\
&= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\
&= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\
&= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda).
\end{aligned}
$$

PNC for *I*/2 can be justified by locality

But PNC for σ_x cannot be justified by locality

Consider the statistical proof



$$
\begin{aligned}
I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\
&= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
&= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C
\end{aligned}
$$

By preparation noncontextuality

$$
\begin{aligned}
\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\
&= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\
&= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda)
\end{aligned}
$$

The proof only required PNC for *I/2*

this can be justified by locality!

Locality $\rightarrow$ PNC $\rightarrow$ CONTRADICTION

Also,

Any bipartite Bell-type proof of nonlocality $\rightarrow$ proof of preparation contextuality

(proof due to Jon Barrett)

<u>Other results</u>

Noncontextuality can be defined for any operational theory:
If two experimental procedures are operationally indistinguishable, they should be represented in the same way in the hidden variable model

One can also generalize the notion of noncontextuality to unsharp measurements and transformations and to more general sorts of contexts

Every proof of preparation contextuality based on the ambiguity of mixtures yields a proof of contextuality for unsharp measurements

There is a connection between Horn's problem and contextuality for unsharp measurements

Open questions

For quantum logicians

- Does preparation contextuality have a natural expression within the convex set approach to quantum logic (see e.g. Mielnik's work)
-Does contextuality for unsharp measurements or transformations have a natural expression in quantum logic?
- What sorts of theories, besides quantum mechanics, are contextual (using the operational definition)

For quantum information theorists

- Is contextuality critical for any of the information-theoretic advantages of quantum theory?
Random access codes? (see Ernesto Galvão's thesis)
Exponential speed-up?

For quantum foundations types

- Is there a simple physical principle which implies contextuality?

For more on preparation contextuality, see:
RWS, Phys. Rev. A **71**, 052108 (2005) or quant-ph/0406166.