Title: Operational Quantum Logic

Date: Jul 17, 2005 04:30 PM

URL: http://pirsa.org/05070092

Abstract: Introductory lecture summary:

Operational Quantum Logic I: Effect Algebras, States, and Basic Convexity
Â• Effect algebras, effect test-spaces, PAS's (partial abelian semigroups).
Â• Morphisms, states, dynamics. Classes of effect algebras whose state-set has nice properties.
Â• Operational derivation of effect algberas, summarized.
Â• "Theories"--- Effect-state systems.
Â• Tensor product (defined, existence result stated).
Â• Some notions of sharpness in EA's, examples that separate them, conditional equivalences that are interesting.
Â• Convex cones/sets, ordered linear space basics. Partially ordered abelian groups.
Operational Quantum Logic II: Convexity, Representations, and Operations
Â• Convex cones and convex sets. Extremality. Krein-Milman. Caratheodory. Affine maps.
Â• Positive maps. Automorphisms. Dual space, Dual cone. Adjoint map. Faces. Exposed faces. Lattices of faces.
Â• Interval EA's, representations on partially ordered abelian groups, unigroups. Analogues of Naimark's theorem, open problems.
Â• Convex EA's. Observables, "generalized" observables. Representation theorem for convex EA's. Relation of observables to effects formulation.
Â• State representation theorem for finite-d homogeneous self-dual cones (statement).
Â• Homogeneous cones as slices of positive semidefinite cones (statement).
Â• Axioms concerning the face lattice.

Effect Algebras

Motivate by: POVM elements
fuzzy sets special case
general operational?

Definition: Set E, partial binary operation $\oplus$, distinguished element $\mathbb{1} \in E$ satisfying:

(Notation: $a \perp b := a \oplus b$ exists)

1.) Strong commutativity: $a \perp b \Rightarrow (b \perp a$ and $a \oplus b = b \oplus a)$.

2.) Strong associativity: $b \oplus c$ exists and $a \oplus (b \oplus c)$ exists $\Longrightarrow$ $(a \oplus b)$ exists and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

3.) Orthosupplementation: $\forall a \in E, \exists! b : a \oplus b = \mathbb{1}$.

(Notation: $\exists! :=$ "there exists a unique";
call the unique b corresponding to a, a'
aka the "orthosupplement of a".
$0 := \mathbb{1}'$).

R1
4.) $a \perp \mathbb{1} \Rightarrow a = 0$.

R2

Examples on board

Def. of "Sub-effect algebra": F is a sub-EA of E iff:
$F \subseteq E$,
R3 $\mathbb{1} \in F$, $a \in F \Rightarrow a' \in F$, & $a, b \in F$ & $a \perp b \Rightarrow a \oplus b \in F$.
(subset containing $\mathbb{1}$, & closed under $'$ and $\oplus$).

(1)

Motivate by: POVM elements, fuzzy sets special case, general operational?

Effect Algebras

Definition: Set E, partial binary operation $\oplus$, distinguished element $\mathbb{1} \in E$ satisfying:
(Notation: $a \perp b := a \oplus b$ exists)

1.) Strong commutativity: $a \perp b \Rightarrow (b \perp a$ and $a \oplus b = b \oplus a)$.

2.) Strong associativity: $b \oplus c$ exists and $a \oplus (b \oplus c)$ exists $\Rightarrow (a \oplus b)$ exists and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

3.) Orthosupplementation: $\forall a \in E$, $\exists! b : a \oplus b = \mathbb{1}$.
(Notation: $\exists! :=$ "there exists a unique"; call the unique b corresponding to a, a', aka the "orthosupplement of a".
$0 := \mathbb{1}'$).

4.) $a \perp \mathbb{1} \Rightarrow a = 0$.

(R1) (R2)

Examples on board

Def. of "Sub-effect algebra": F is a sub-EA of E iff:
$F \subseteq E$,
(R3) $\mathbb{1} \in F$, $a \in F \Rightarrow a' \in F$, & $a, b \in F$ & $a \perp b \Rightarrow a \oplus b \in F$.
(subset containing $\mathbb{1}$, & closed under $'$ and $\oplus$).

①

Effect Algebras

Motivate by: POVM elements, fuzzy sets special case, general operational?

Definition: Set E, partial binary operation $\oplus$, distinguished element $\mathbb{1} \in E$ satisfying:

(Notation: $a \perp b := a \oplus b$ exists)

1.) Strong commutativity: $a \perp b \Rightarrow (b \perp a$ and $a \oplus b = b \oplus a)$.

2.) Strong associativity: $b \oplus c$ exists and $a \oplus (b \oplus c)$ exists $\Longrightarrow (a \oplus b)$ exists and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

3.) Orthosupplementation: $\forall a \in E,\ \exists! b : a \oplus b = \mathbb{1}$.

(Notation: $\exists! :=$ "there exists a unique"; call the unique b corresponding to a, a' aka the "orthosupplement of a". $0 := \mathbb{1}'$).

4.) $a \perp \mathbb{1} \Rightarrow a = 0$.

Examples on board

(R1) (R2) Def. of "Sub-effect algebra": F is a sub-EA of E iff:

$F \subseteq E$,

(R3) $\mathbb{1} \in F$, $a \in F \Rightarrow a' \in F$, & $a, b \in F$ & $a \perp b \Rightarrow a \oplus b \in F$.

(subset containing $\mathbb{1}$, & closed under $'$ and $\oplus$).

(1)

Motivate by: POVM elements
fuzzy sets special case
general operational?

Effect Algebras

distinguished element $\mathbb{1} \in E$

<u>Definition:</u> Set E, partial binary operation $\oplus$, $\wedge$satisfying:

(Notation: $a \perp b := a \oplus b$ exists)

1.) Strong commutativity: $a \perp b \Rightarrow (b \perp a$ and $a \oplus b = b \oplus a)$.

2.) Strong associativity: $b \oplus c$ exists and $a \oplus (b \oplus c)$ exists

$\Longrightarrow (a \oplus b)$ exists and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

3.) Orthosupplementation: $\forall a \in E,\ \exists! b : a \oplus b = \mathbb{1}$.

(Notation: $\exists! :=$ "there exists a unique";
call the unique b corresponding to a, $'a'$
aka the "orthosupplement of a".
$0 := \mathbb{1}'$).

4.) $a \perp \mathbb{1} \Rightarrow a = 0$.

(R1)
(R2)

Examples on board

Def. of "Sub-effect algebra": F is a sub-EA of E iff:

$F \subseteq E$,

(R3) $\mathbb{1} \in F$, $a \in F \Rightarrow a' \in F$, & $a, b \in F$ & $a \perp b \Rightarrow a \oplus b \in F$.

(subset containing $\mathbb{1}$, & closed under $'$ and $\oplus$).

(1)

Effect Algebras

Motivate by: POVM elements, fuzzy sets special case, general operational?

Definition: Set E, partial binary operation $\oplus$, distinguished element $\mathbb{1} \in E$ satisfying:

(Notation: $a \perp b := a \oplus b$ exists)

1.) Strong commutativity: $a \perp b \Rightarrow (b \perp a \text{ and } a \oplus b = b \oplus a)$.

2.) Strong associativity: $b \oplus c$ exists and $a \oplus (b \oplus c)$ exists

$\Longrightarrow (a \oplus b)$ exists and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

3.) Orthosupplementation: $\forall a \in E, \exists ! b : a \oplus b = \mathbb{1}$.

(Notation: $\exists ! :=$ "there exists a unique"; {call the unique b corresponding to a, a', aka the "orthosupplement of a".} $0 := \mathbb{1}'$).

4.) $a \perp \mathbb{1} \Rightarrow a = 0$.

(R1) (R2)

Examples on board

Def. of "Sub-effect algebra": F is a sub-EA of E iff:

$F \subseteq E$, [illegible]

(R3) $\mathbb{1} \in F$, $a \in F \Rightarrow a' \in F$, & $a, b \in F$ & $a \perp b \Rightarrow a \oplus b \in F$.

(subset containing $\mathbb{1}$, & closed under $'$ and $\oplus$).

①

Effect Algebras

Motivate by: POVM elements, fuzzy sets special case, general operational?

Definition: Set E, partial binary operation $\oplus$, distinguished element $\mathbb{1} \in E$ satisfying:

(Notation: $a \perp b := a \oplus b$ exists)

1.) Strong commutativity: $a \perp b \Rightarrow (b \perp a$ and $a \oplus b = b \oplus a)$.

2.) Strong associativity: $b \oplus c$ exists and $a \oplus (b \oplus c)$ exists $\Longrightarrow (a \oplus b)$ exists and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

3.) Orthosupplementation: $\forall a \in E, \exists! b : a \oplus b = \mathbb{1}$.

(Notation: $\exists! :=$ "there exists a unique"; call the unique b corresponding to a, 'a'' aka the "orthosupplement of a". $0 := \mathbb{1}'$).

4.) $a \perp \mathbb{1} \Rightarrow a = 0$.

R1 R2

Examples on board

Def. of "Sub-effect algebra": F is a sub-EA of E iff:

$F \subseteq E$, ~~[crossed out]~~

R3 $\mathbb{1} \in F$, $a \in F \Rightarrow a' \in F$, & $a, b \in F$ & $a \perp b \Rightarrow a \oplus b \in F$.

(subset containing $\mathbb{1}$, & closed under $'$ and $\oplus$).

(1)

Effect Algebras

Motivate by: POVM elements
fuzzy sets special case
general operational?

Definition: Set E, partial binary operation $\oplus$, distinguished element $\mathbb{1} \in E$ satisfying:

(Notation: $a \perp b := a \oplus b$ exists)

1.) Strong commutativity: $a \perp b \Rightarrow (b \perp a \text{ and } a \oplus b = b \oplus a)$.

2.) Strong associativity: $b \oplus c$ exists and $a \oplus (b \oplus c)$ exists
$\Longrightarrow (a \oplus b)$ exists and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

3.) Orthosupplementation: $\forall a \in E, \exists! b : a \oplus b = \mathbb{1}$.

(Notation: $\exists! :=$ "there exists a unique";
call the unique b corresponding to a, a'
aka the "orthosupplement of a".
$0 := \mathbb{1}'$).

4.) $a \perp \mathbb{1} \Rightarrow a = 0$.

(R1)
(R2)

Examples on board

Def. of "Sub-effect algebra": F is a sub-EA of E iff:
$F \subseteq E$,
(R3) $\mathbb{1} \in F$, $a \in F \Rightarrow a' \in F$, & $a, b \in F$ & $a \perp b \Rightarrow a \oplus b \in F$.
(subset containing $\mathbb{1}$, & closed under $'$ and $\oplus$).

①

Effect Algebras

Motivate by: POVM elements, fuzzy sets special case, general operational?

Definition: Set E, partial binary operation $\oplus$, distinguished element $\mathbb{1} \in E$ satisfying:

(Notation: $a \perp b := a \oplus b$ exists)

1.) Strong commutativity: $a \perp b \Rightarrow (b \perp a \text{ and } a \oplus b = b \oplus a)$.

2.) Strong associativity: $b \oplus c$ exists and $a \oplus (b \oplus c)$ exists $\Longrightarrow (a \oplus b)$ exists and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

3.) Orthosupplementation: $\forall a \in E,\ \exists! b : a \oplus b = \mathbb{1}$.

(Notation: $\exists! :=$ "there exists a unique"; call the unique b corresponding to a, 'a'' aka the "orthosupplement of a". $0 := \mathbb{1}'$).

4.) $a \perp \mathbb{1} \Rightarrow a = 0$.

R1 R2

Examples on board

Def. of "Sub-effect algebra": F is a sub-EA of E iff:

$F \subseteq E$,

R3 $\mathbb{1} \in F$, $a \in F \Rightarrow a' \in F$, & $a, b \in F$ & $a \perp b \Rightarrow a \oplus b \in F$.

(subset containing $\mathbb{1}$, & closed under $'$ and $\oplus$).

①

Effect Algebras

Motivate by: POVM elements, fuzzy sets special case, general operational?

Definition: Set E, partial binary operation $\oplus$, distinguished element $\mathbb{1} \in E$ satisfying:

(Notation: $a \perp b := a \oplus b$ exists)

1.) Strong commutativity: $a \perp b \Rightarrow (b \perp a$ and $a \oplus b = b \oplus a)$.

2.) Strong associativity: $b \oplus c$ exists and $a \oplus (b \oplus c)$ exists

$\Longrightarrow (a \oplus b)$ exists and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

3.) Orthosupplementation: $\forall a \in E,\ \exists! b : a \oplus b = \mathbb{1}$.

(Notation: $\exists! :=$ "there exists a unique"; call the unique b corresponding to a, 'a'' aka the "orthosupplement of a". $0 := \mathbb{1}'$).

4.) $a \perp \mathbb{1} \Rightarrow a = 0$.

Examples on board

R1 R2

Def. of "Sub-effect algebra": F is a sub-EA of E iff:

$F \subseteq E$, [illegible]

R3 $\mathbb{1} \in F$, $a \in F \Rightarrow a' \in F$, & $a, b \in F$ & $a \perp b \Rightarrow a \oplus b \in F$.

(subset containing $\mathbb{1}$, & closed under $'$ and $\oplus$).

(1)

Effect Algebras

Motivate by: POVM elements, fuzzy sets special case, general operational?

Definition: Set E, partial binary operation $\oplus$, distinguished element $\mathbb{1} \in E$ satisfying:

(Notation: $a \perp b := a \oplus b$ exists)

1.) Strong commutativity: $a \perp b \Rightarrow (b \perp a$ and $a \oplus b = b \oplus a)$.

2.) Strong associativity: $b \oplus c$ exists and $a \oplus (b \oplus c)$ exists $\Longrightarrow (a \oplus b)$ exists and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

3.) Orthosupplementation: $\forall a \in E,\ \exists! b : a \oplus b = \mathbb{1}$.

(Notation: $\exists! :=$ "there exists a unique"; call the unique b corresponding to a, a', aka the "orthosupplement of a". $0 := \mathbb{1}'$).

4.) $a \perp \mathbb{1} \Rightarrow a = 0$.

R1 R2

Examples on board

Def. of "Sub-effect algebra": F is a sub-EA of E iff:

$F \subseteq E$,

R3

$\mathbb{1} \in F$, $a \in F \Rightarrow a' \in F$, & $a, b \in F$ & $a \perp b \Rightarrow a \oplus b \in F$.

(subset containing $\mathbb{1}$, & closed under $'$ and $\oplus$).

①

Effect Algebras

Motivate by: POVM elements, fuzzy sets special case, general operational?

Definition: Set E, partial binary operation $\oplus$, distinguished element $\mathbb{1} \in E$ satisfying:

(Notation: $a \perp b := a \oplus b$ exists)

1.) Strong commutativity: $a \perp b \Rightarrow (b \perp a$ and $a \oplus b = b \oplus a)$.

2.) Strong associativity: $b \oplus c$ exists and $a \oplus (b \oplus c)$ exists $\Longrightarrow (a \oplus b)$ exists and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

3.) Orthosupplementation: $\forall a \in E,\ \exists! b : a \oplus b = \mathbb{1}$.

(Notation: $\exists! :=$ "there exists a unique"; call the unique b corresponding to a, a' aka the "orthosupplement of a". $0 := \mathbb{1}'$).

4.) $a \perp \mathbb{1} \Rightarrow a = 0$.

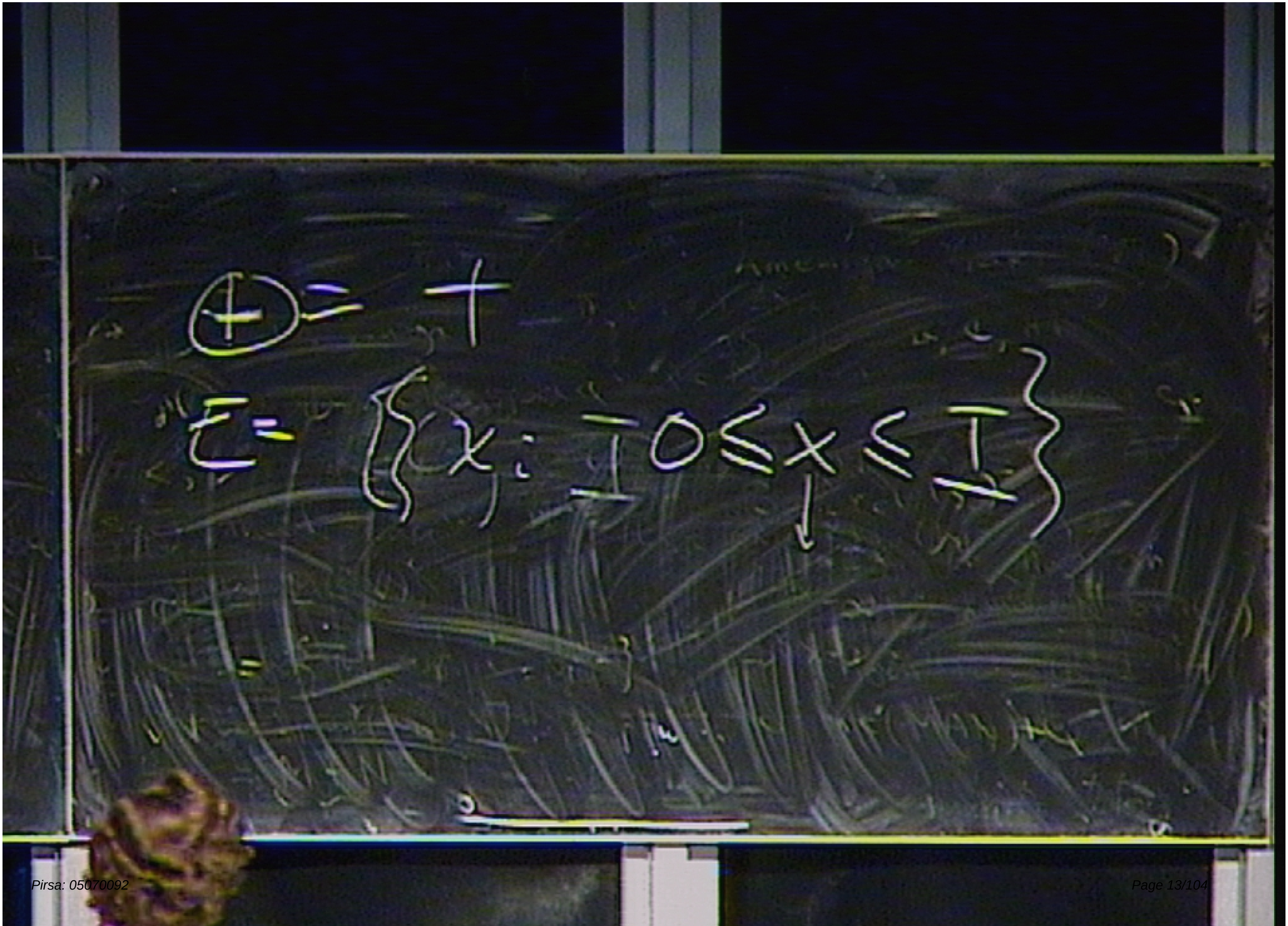
Examples on board

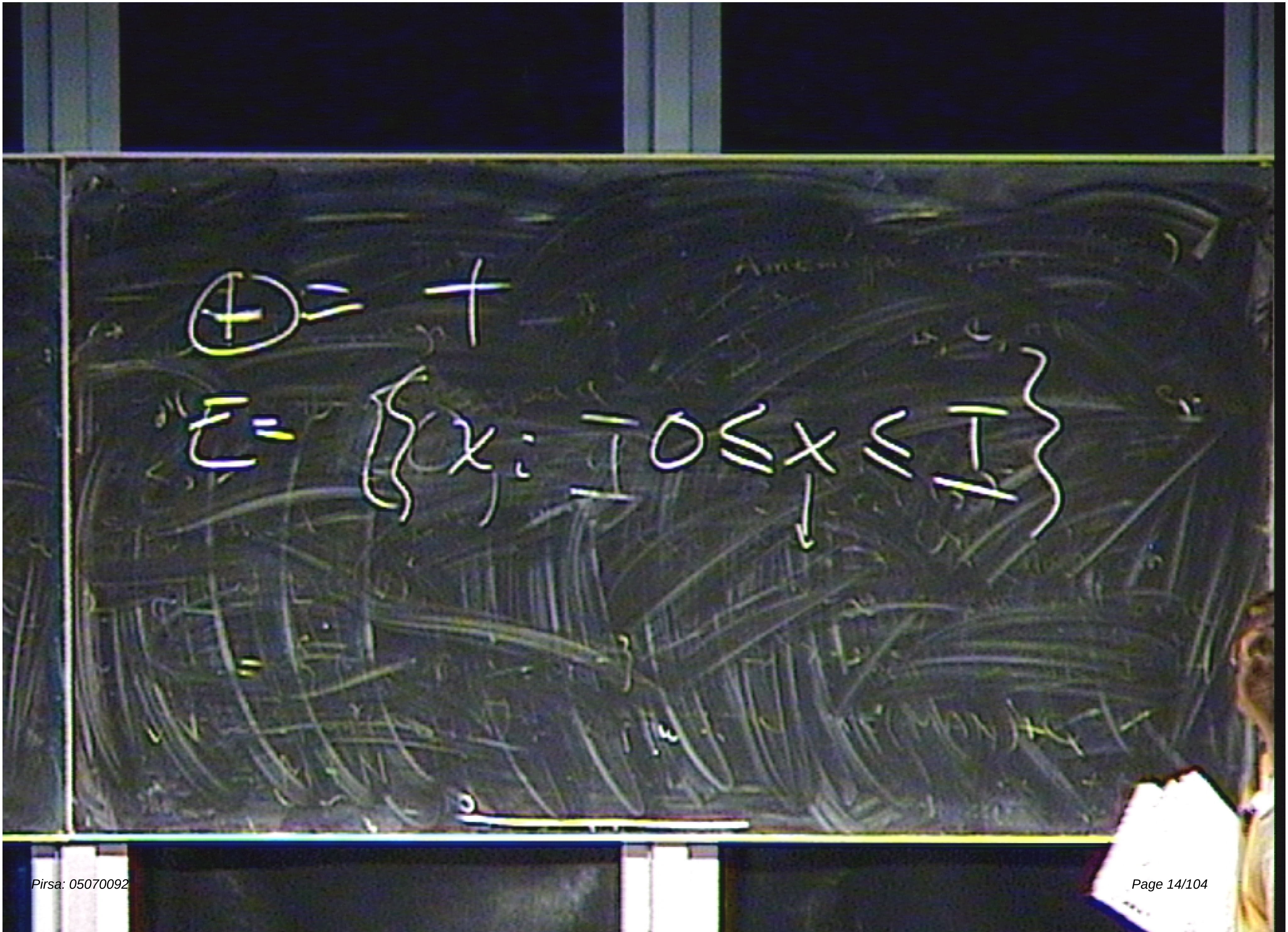
R1 R2 Def. of "Sub-effect algebra": F is a sub-EA of E iff:

$F \subseteq E$,

R3 $\mathbb{1} \in F$, $a \in F \Rightarrow a' \in F$, & $a, b \in F$ & $a \perp b \Rightarrow a \oplus b \in F$.

(subset containing $\mathbb{1}$, & closed under $'$ and $\oplus$).




⊕ = +
E = {x_i : 0 ≤ x ≤ I}

$$\oplus = +$$

$$E = \{ x_i : I \; 0 \leq x \leq I \}$$

(1)

Effect Algebras

Motivate by: POVM elements, fuzzy sets special case, general operational?

Definition: Set E, partial binary operation $\oplus$, distinguished element $\mathbb{1} \in E$ satisfying:

(Notation: $a \perp b := a \oplus b$ exists)

1.) Strong commutativity: $a \perp b \Rightarrow (b \perp a$ and $a \oplus b = b \oplus a)$.

2.) Strong associativity: $b \oplus c$ exists and $a \oplus (b \oplus c)$ exists $\Longrightarrow (a \oplus b)$ exists and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

3.) Orthosupplementation: $\forall a \in E,\ \exists! b : a \oplus b = \mathbb{1}$.

(Notation: $\exists! :=$ "there exists a unique"; call the unique b corresponding to a, a', aka the "orthosupplement of a". $0 := \mathbb{1}'$).

4.) $a \perp \mathbb{1} \Longrightarrow a = 0$.

R1 R2

Examples on board

Def. of "Sub-effect algebra": F is a sub-EA of E iff:

$F \subseteq E$,

R3 $\mathbb{1} \in F$, $a \in F \Rightarrow a' \in F$, & $a, b \in F$ & $a \perp b \Rightarrow a \oplus b \in F$.

(subset containing $\mathbb{1}$, & closed under $'$ and $\oplus$).

(1)

Effect Algebras

Motivate by: POVM elements, fuzzy sets special case, general operational?

Definition: Set E, partial binary operation $\oplus$, distinguished element $\mathbb{1} \in E$ satisfying:

(Notation: $a \perp b := a \oplus b$ exists)

1.) Strong commutativity: $a \perp b \Rightarrow (b \perp a$ and $a \oplus b = b \oplus a)$.

2.) Strong associativity: $b \oplus c$ exists and $a \oplus (b \oplus c)$ exists $\Longrightarrow (a \oplus b)$ exists and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

3.) Orthosupplementation: $\forall a \in E,\ \exists! b : a \oplus b = \mathbb{1}$.

(Notation: $\exists! :=$ "there exists a unique"; call the unique b corresponding to a, a', aka the "orthosupplement of a". $0 := \mathbb{1}'$).

4.) $a \perp \mathbb{1} \Rightarrow a = 0$.

Examples on board

R1

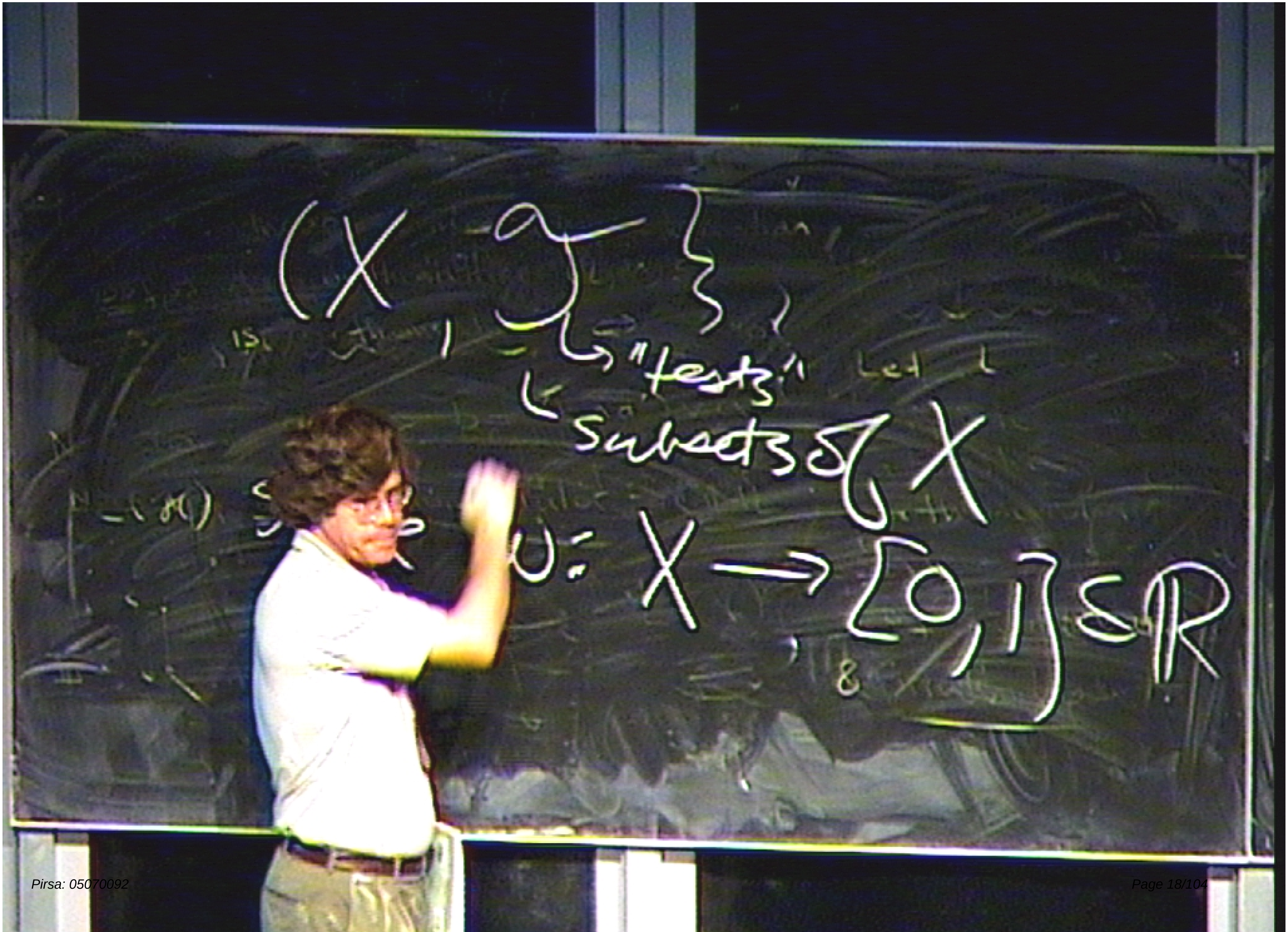
R2

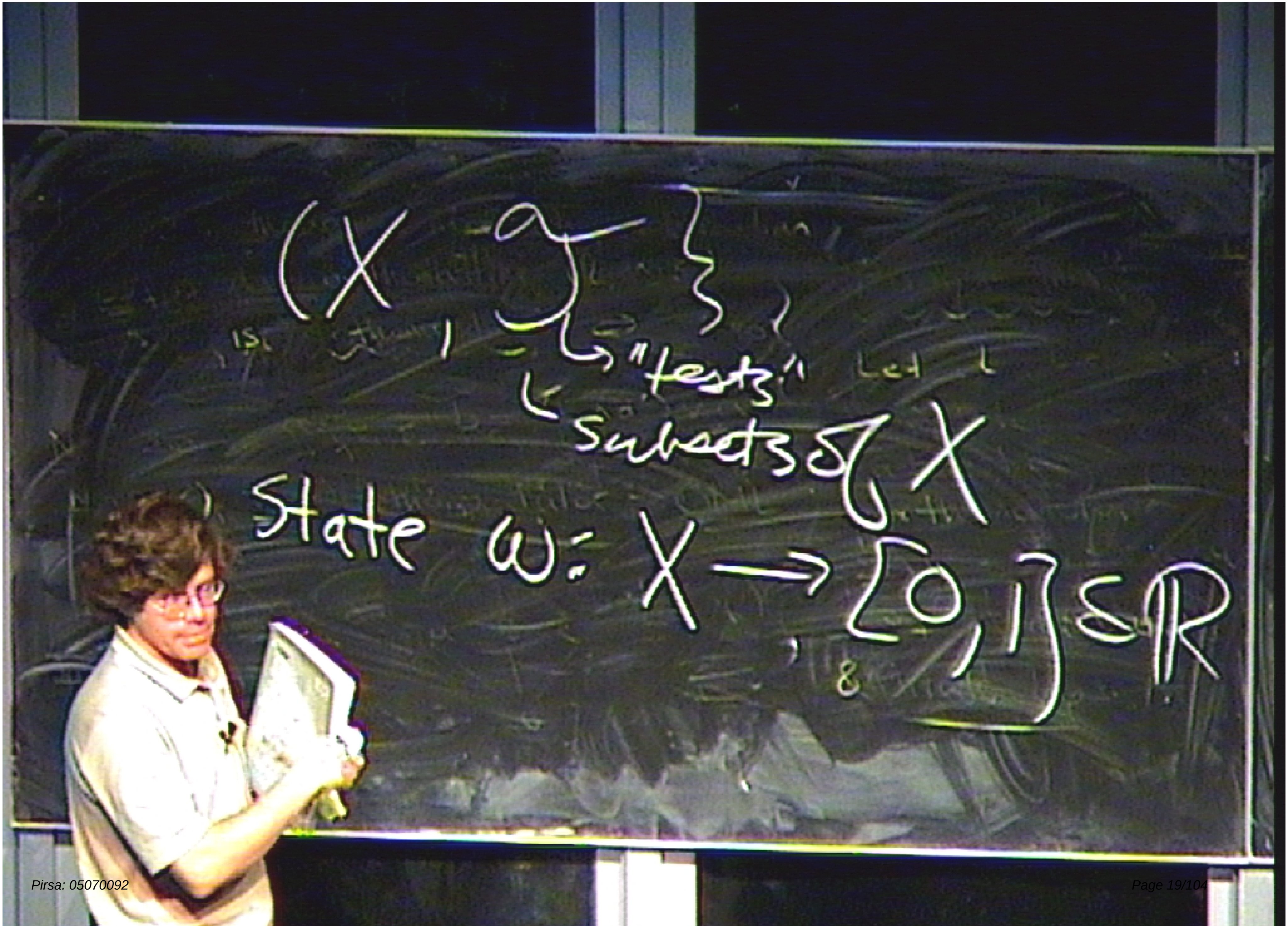
Def. of "Sub-effect algebra": F is a sub-EA of E iff:

$F \subseteq E$, [illegible]

R3 $\mathbb{1} \in F$, $a \in F \Rightarrow a' \in F$, & $a, b \in F$ & $a \perp b \Rightarrow a \oplus b \in F$.

(subset containing $\mathbb{1}$, & closed under $'$ and $\oplus$).


(X, a)
"tests"
Let
subsets of X
u: X → {0,1} ⊆ R


(X, 𝒜)
"tests"
subsets of X
State ω: X → [0,1] ⊆ ℝ

$(X, \mathcal{T})$

$\mathcal{T}$ → "tests"

↳ subsets of X

State $\omega : X \to [0,1] \subseteq \mathbb{R}$

$\forall T \in \mathcal{T} \quad \sum_{x \in T} \omega(x) = 1$

$(X, \mathcal{J})$

$\mathcal{J}$ → "tests"

↳ subsets of X

State $\omega : X \to [0,1] \subseteq \mathbb{R}$

$$\forall T \in \mathcal{J} \quad \sum_{x \in T} \omega(x) = 1$$

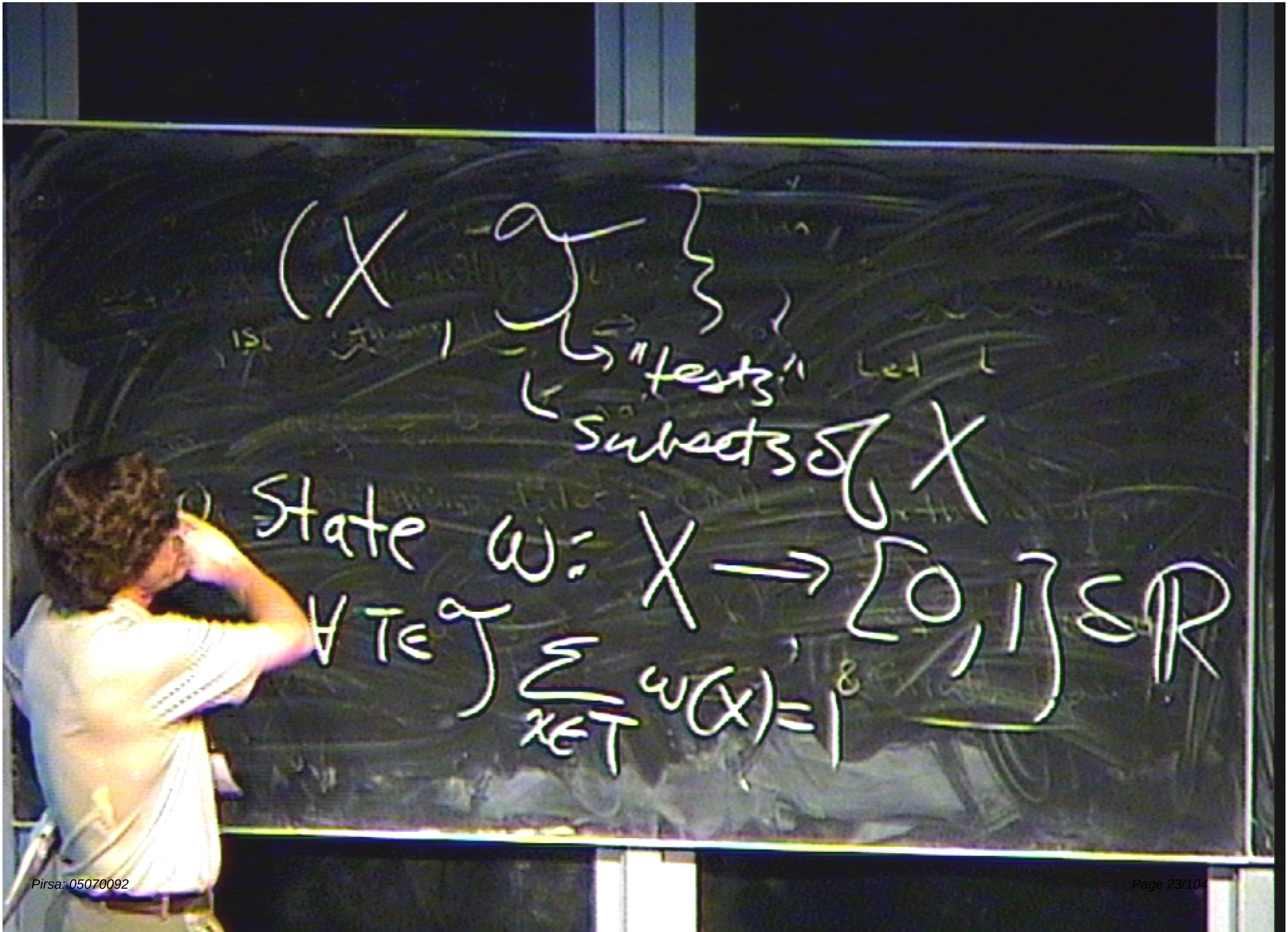
$(X, \mathcal{T})$

"tests"

subsets of X

State $\omega : X \to [0,1] \subseteq \mathbb{R}$

$\forall T \in \mathcal{T} \quad \sum_{x \in T} \omega(x) = 1$

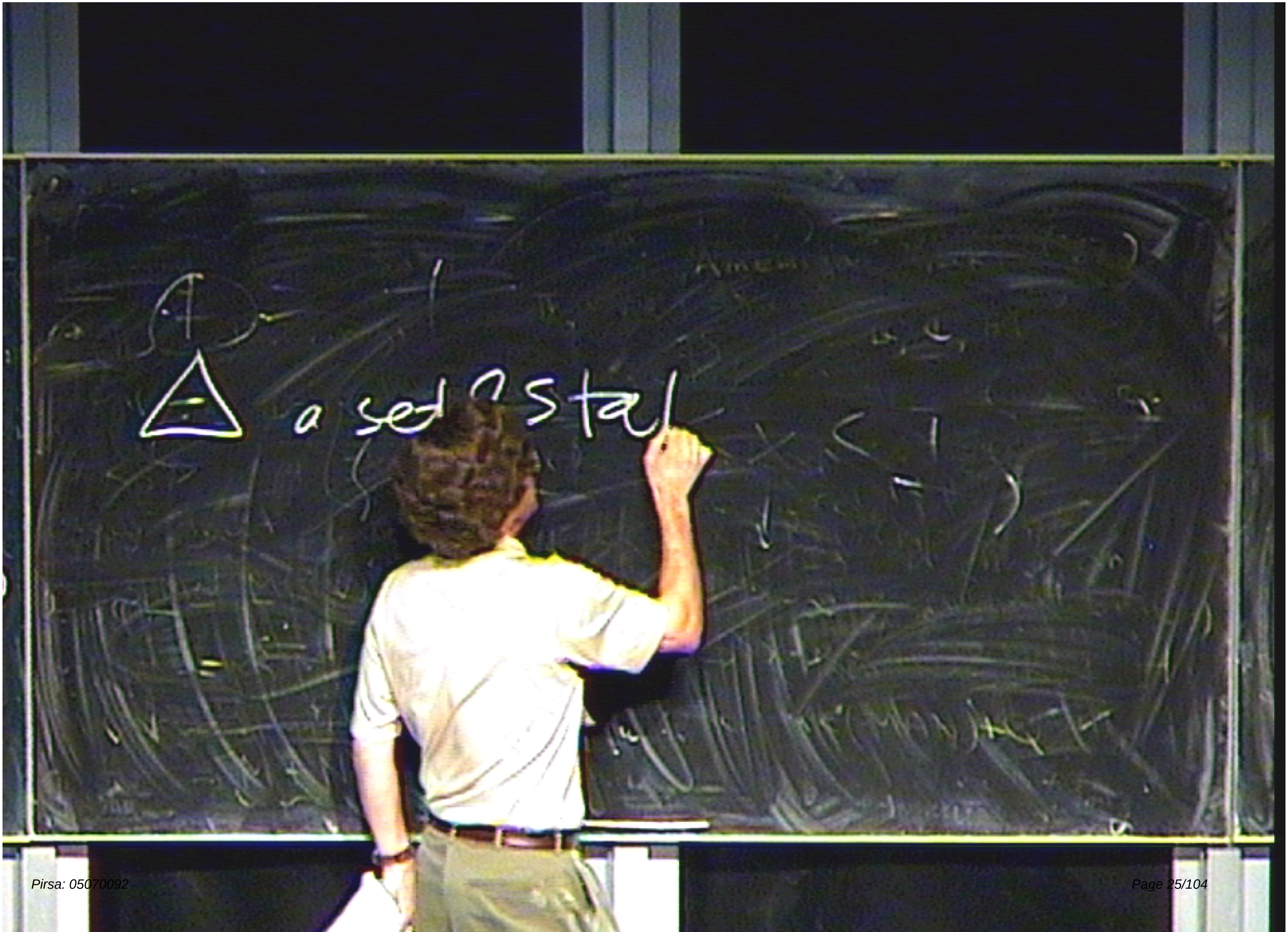

(X, 𝒯)
"tests"
subsets of X
State ω: X → [0,1] ⊆ R
∀T ∈ 𝒯 Σ x∈T ω(x) = 1

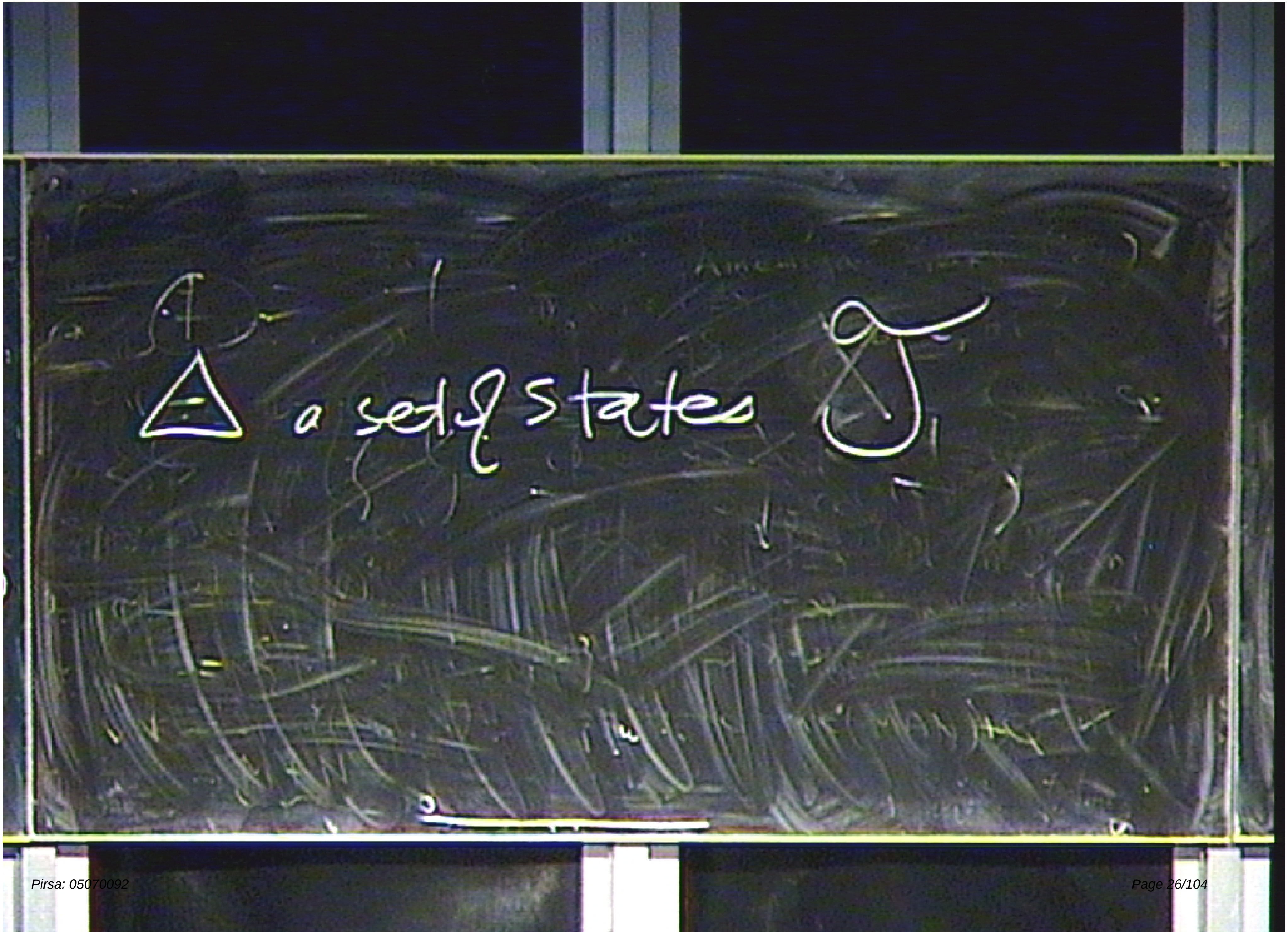
$(X, \mathcal{T})$

$\mathcal{T}$ → "tests" → subsets of X

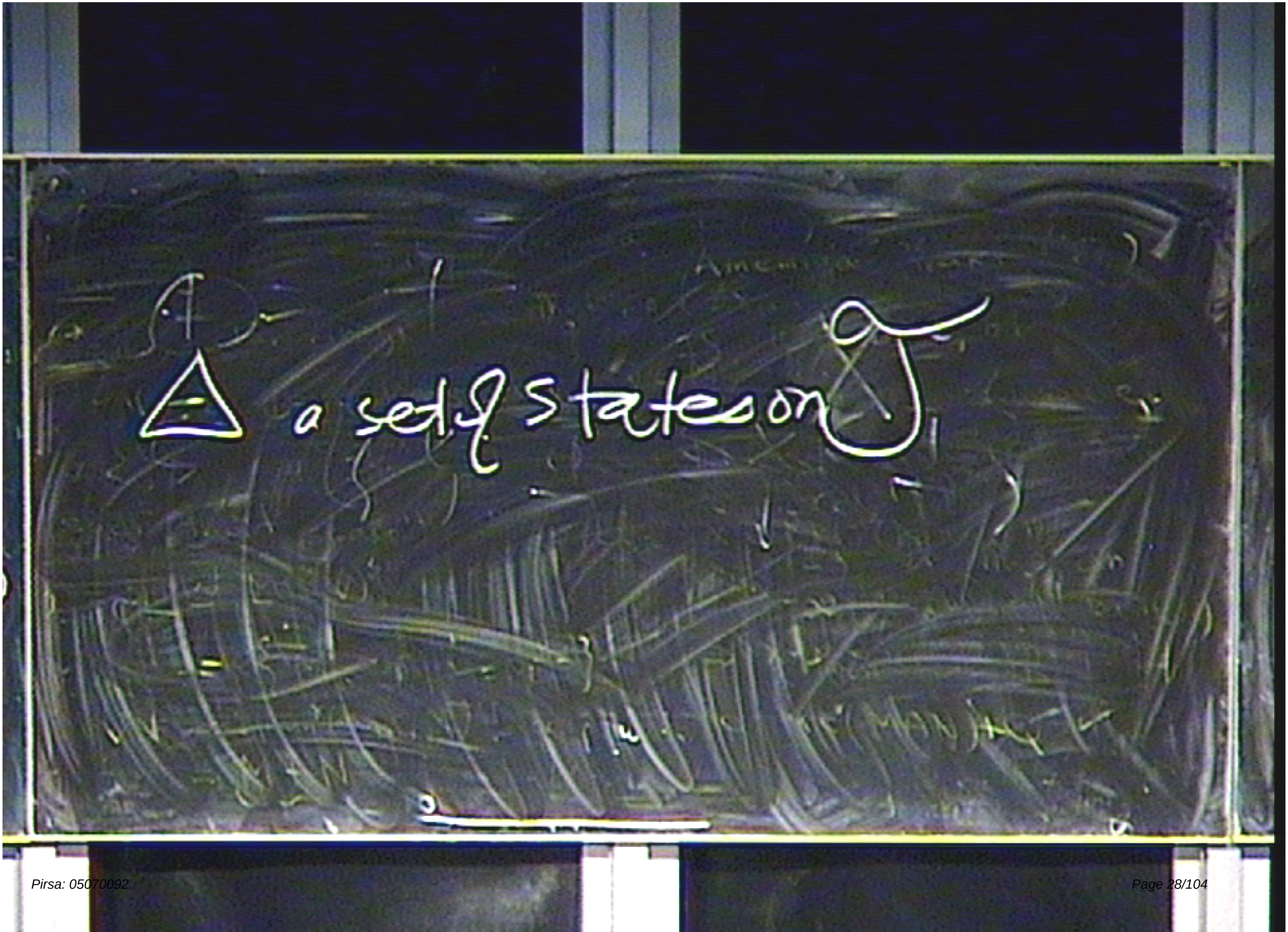
State $\omega : X \to [0,1] \subseteq \mathbb{R}$

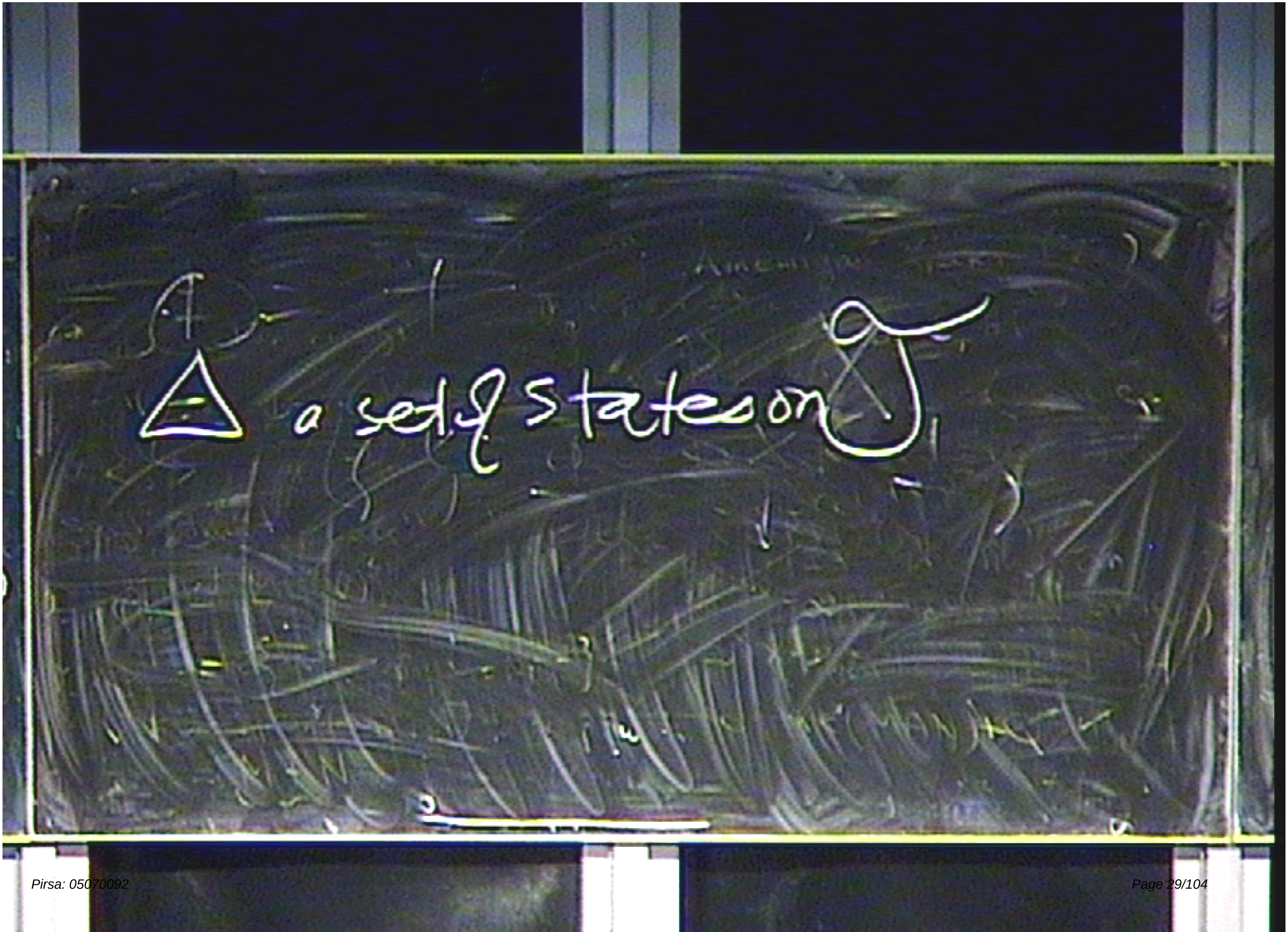
$\forall T \in \mathcal{T} \quad \sum_{x \in T} \omega(x) = 1$

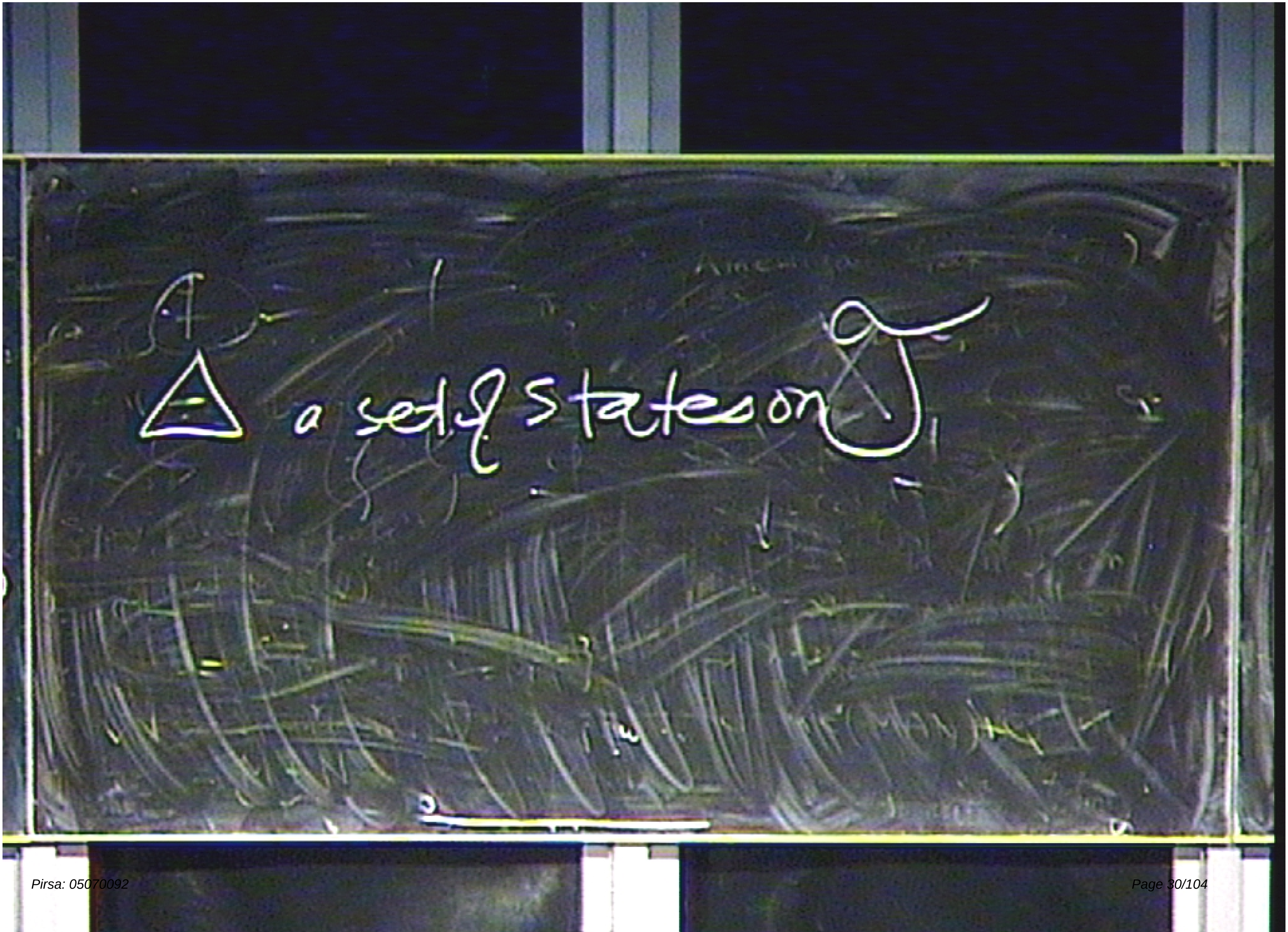


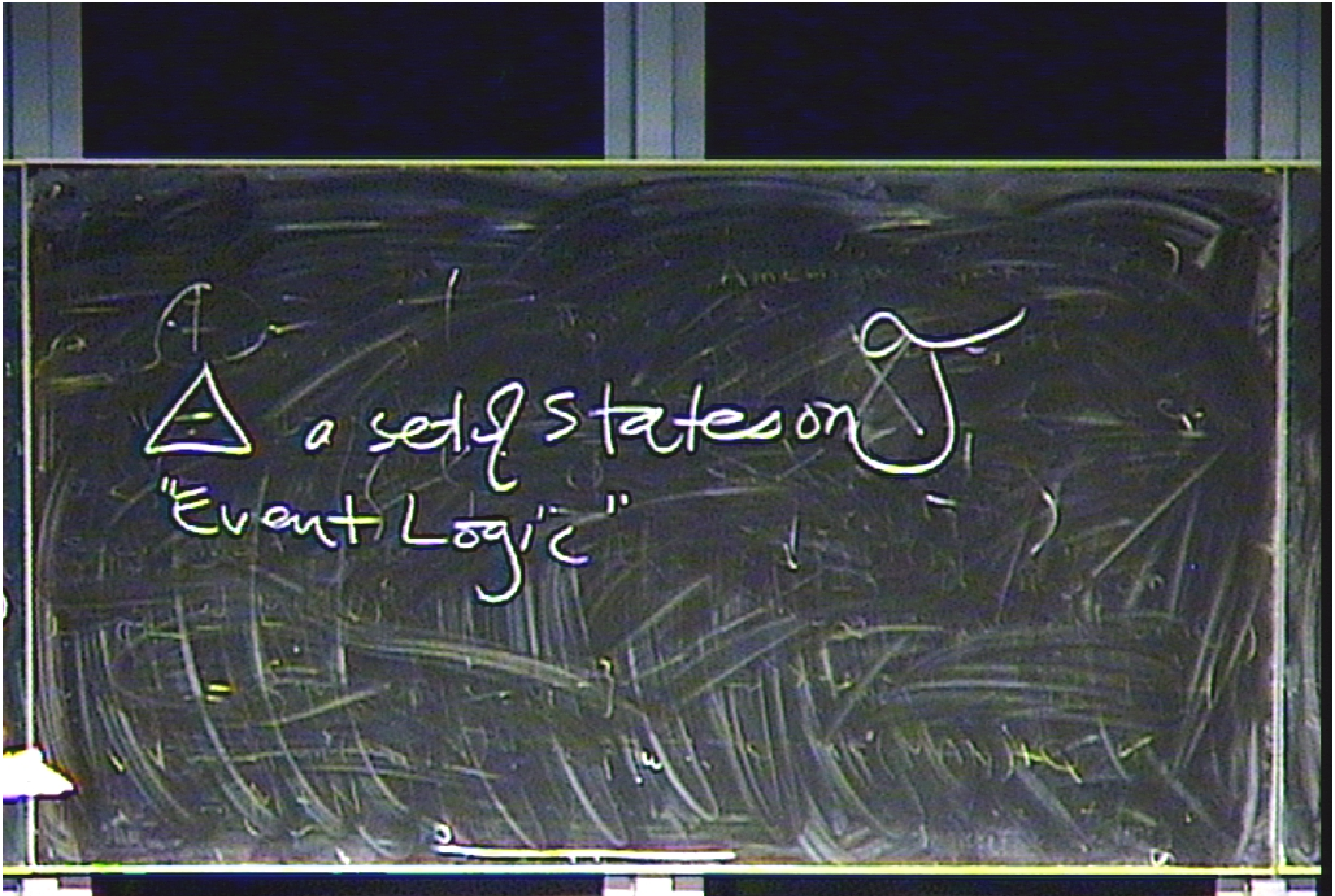

a set of states

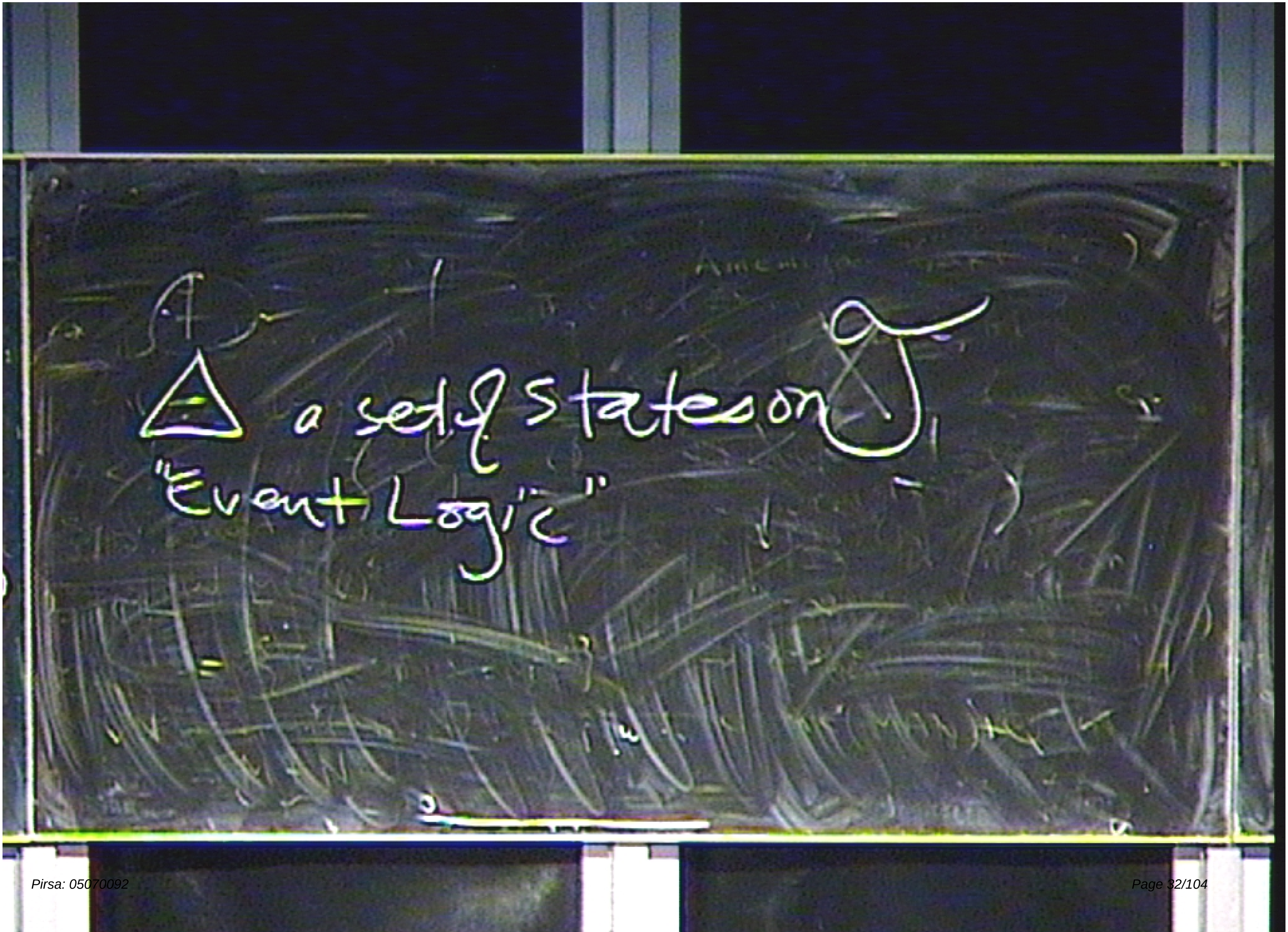
Δ a set of states on $\mathcal{J}$


Δ a set of states on


△ a set of states on 𝒥


△ a set of states on


Δ a set of states on 𝒥
"Event Logic"

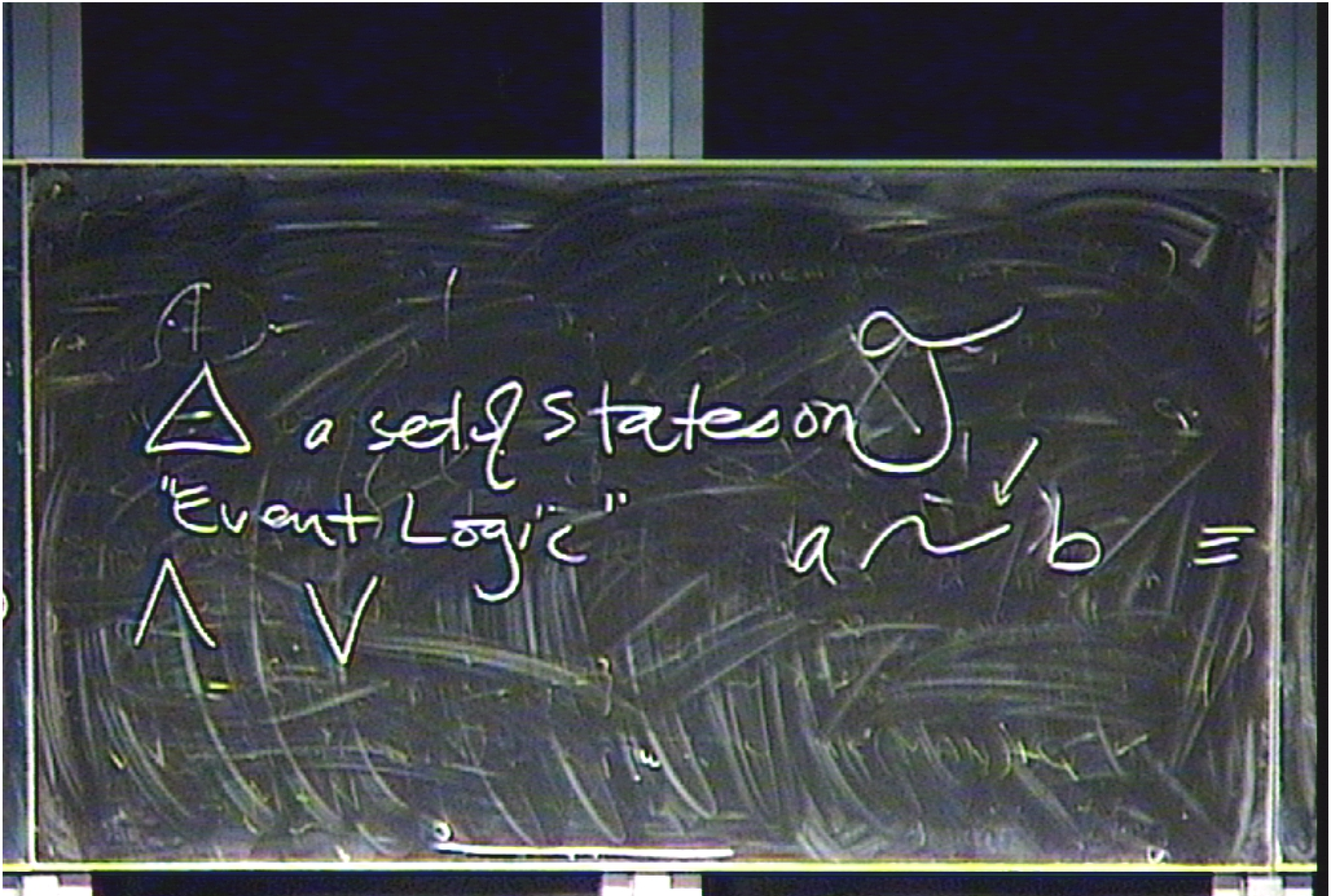

a set of states on
"Event Logic"

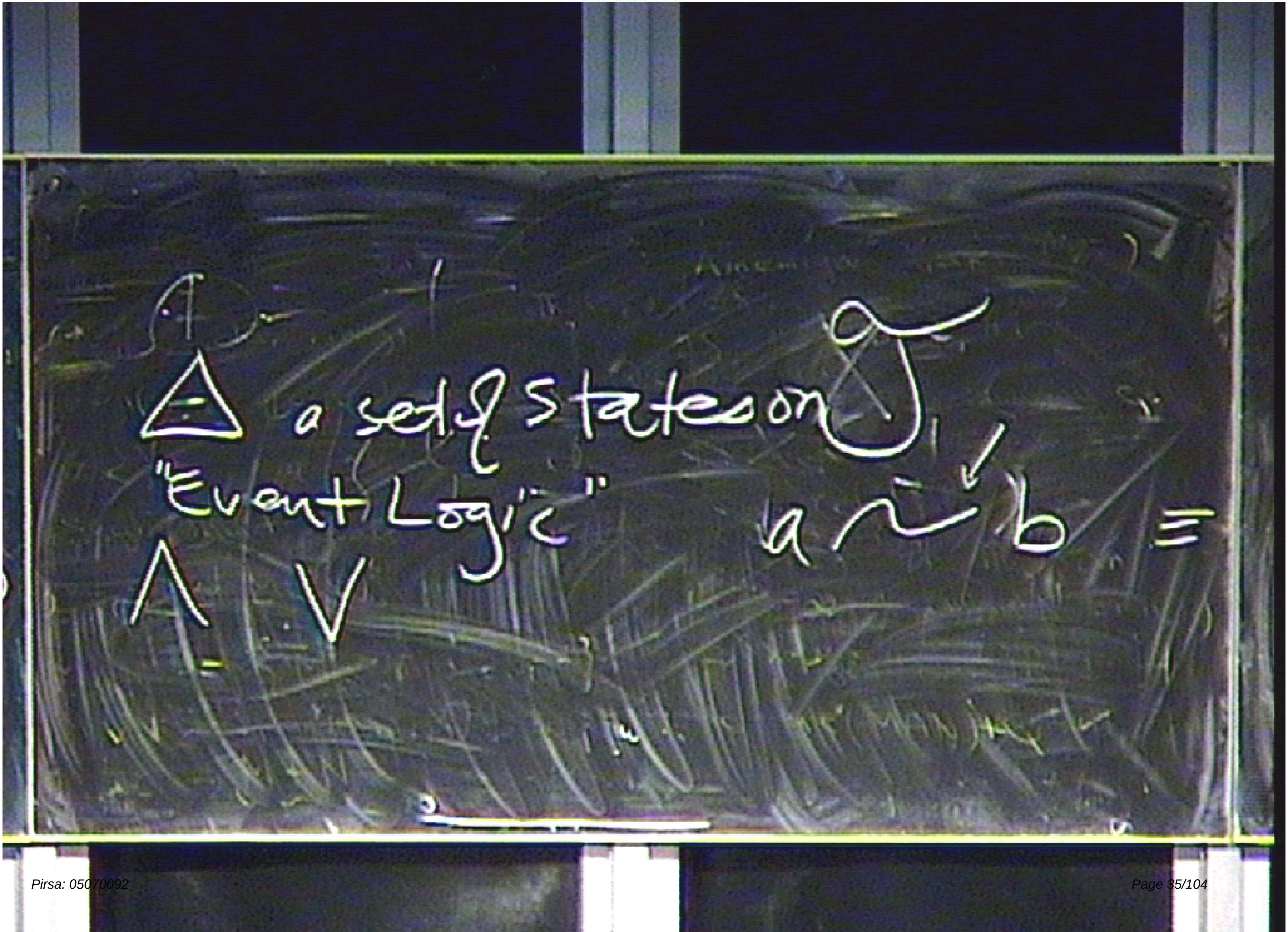
Δ a set of states on $\mathcal{A}$

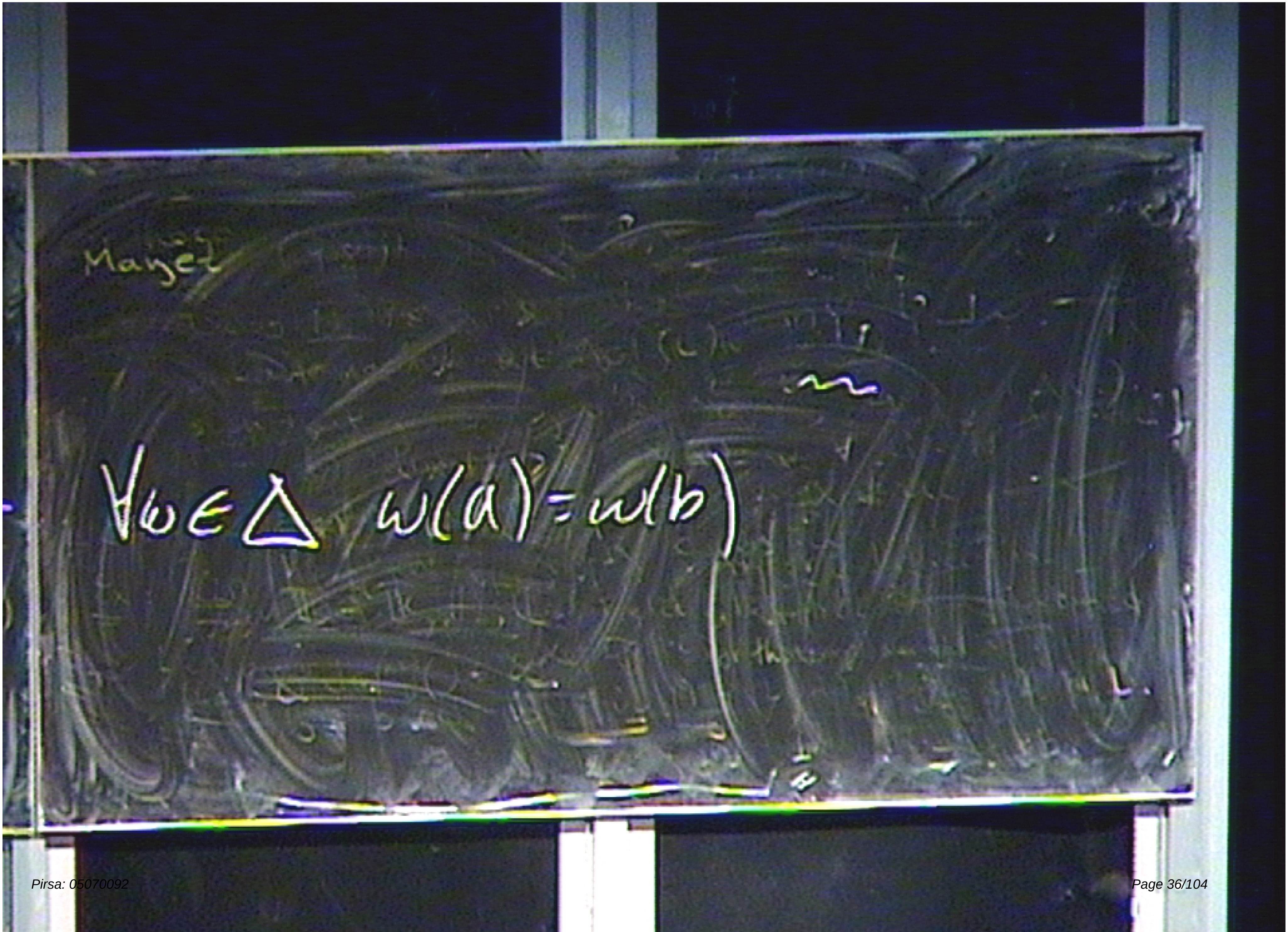
"Event Logic"

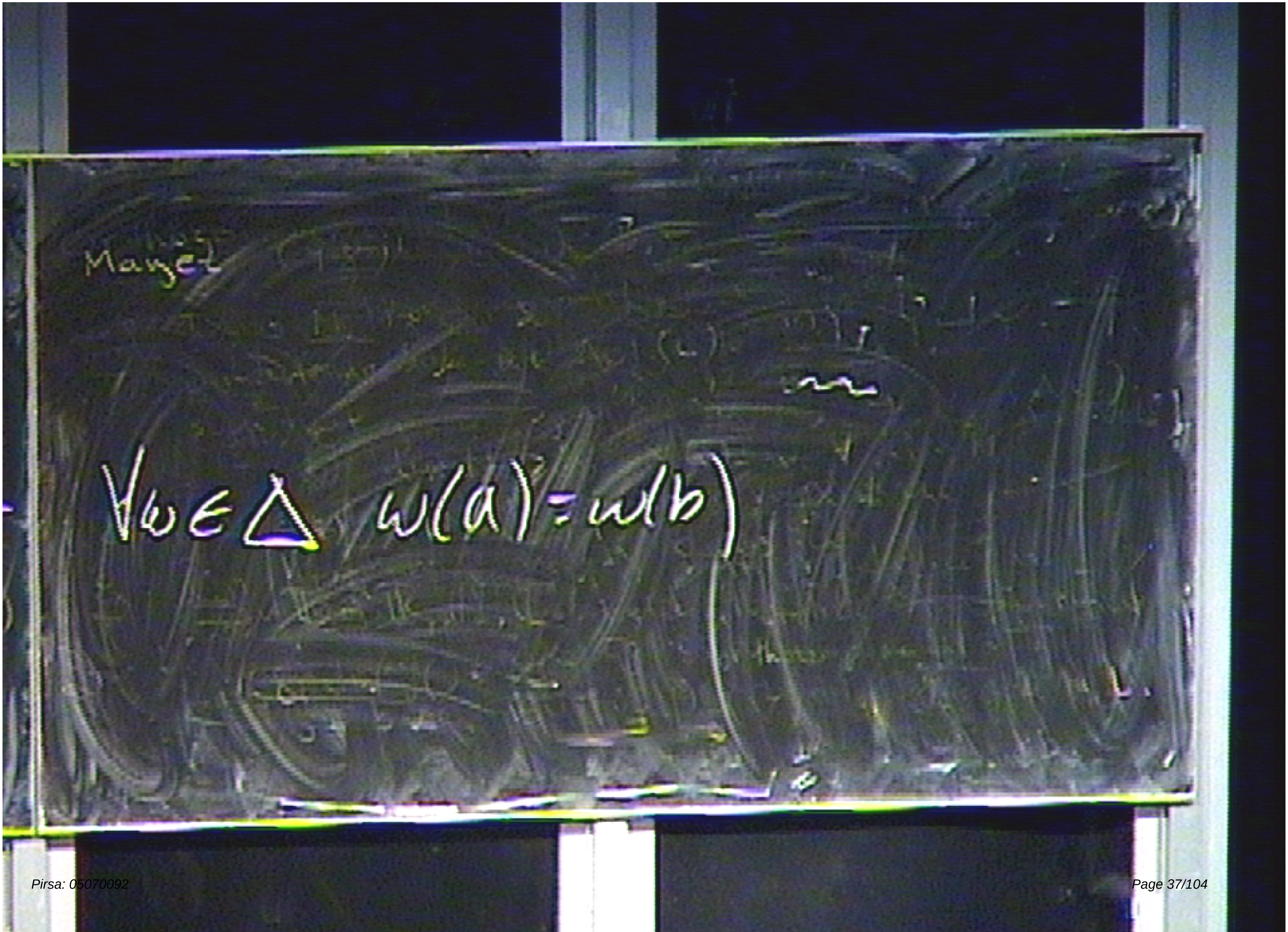
$\wedge \quad \vee$

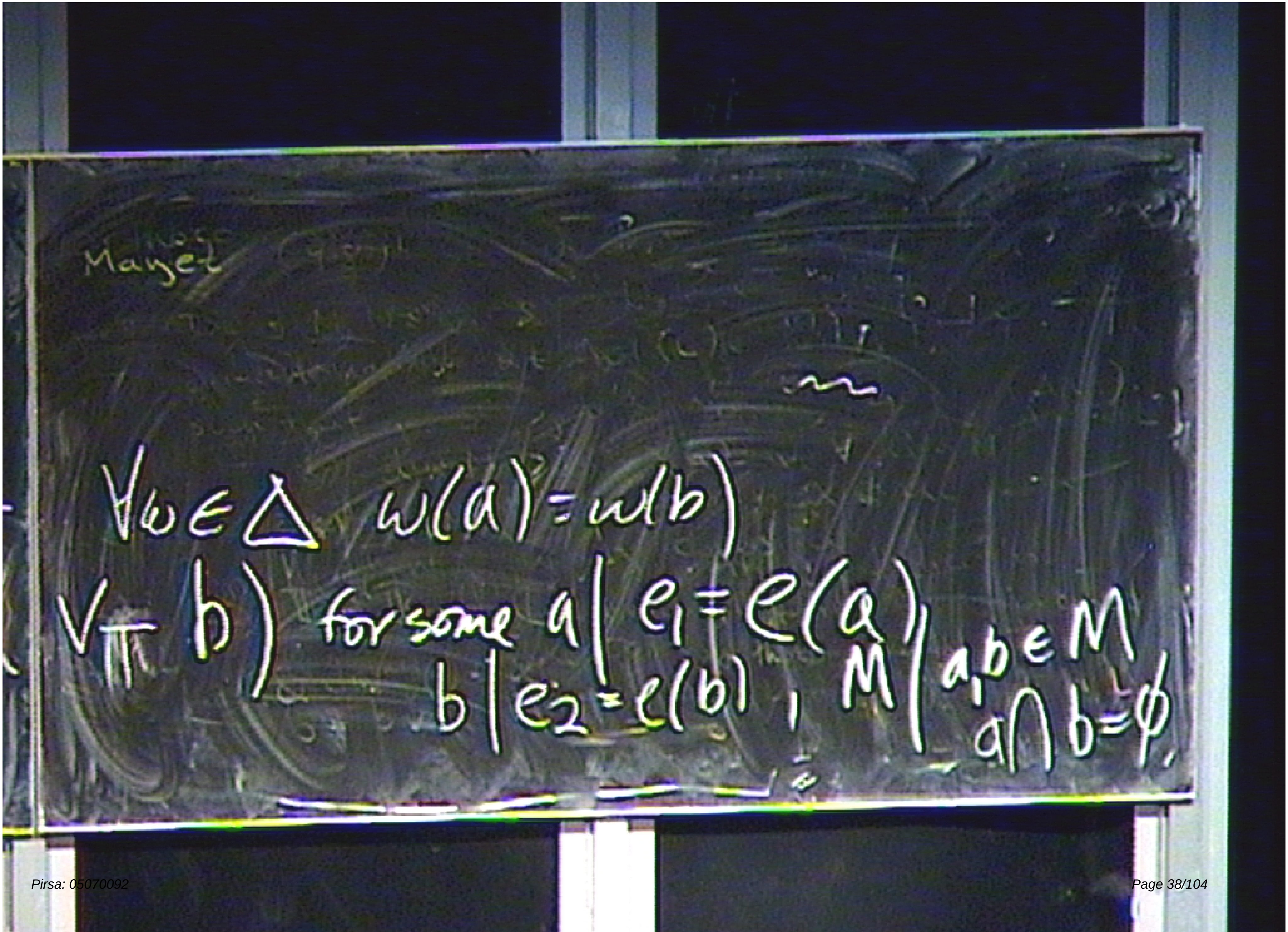
$a \sim b \equiv$

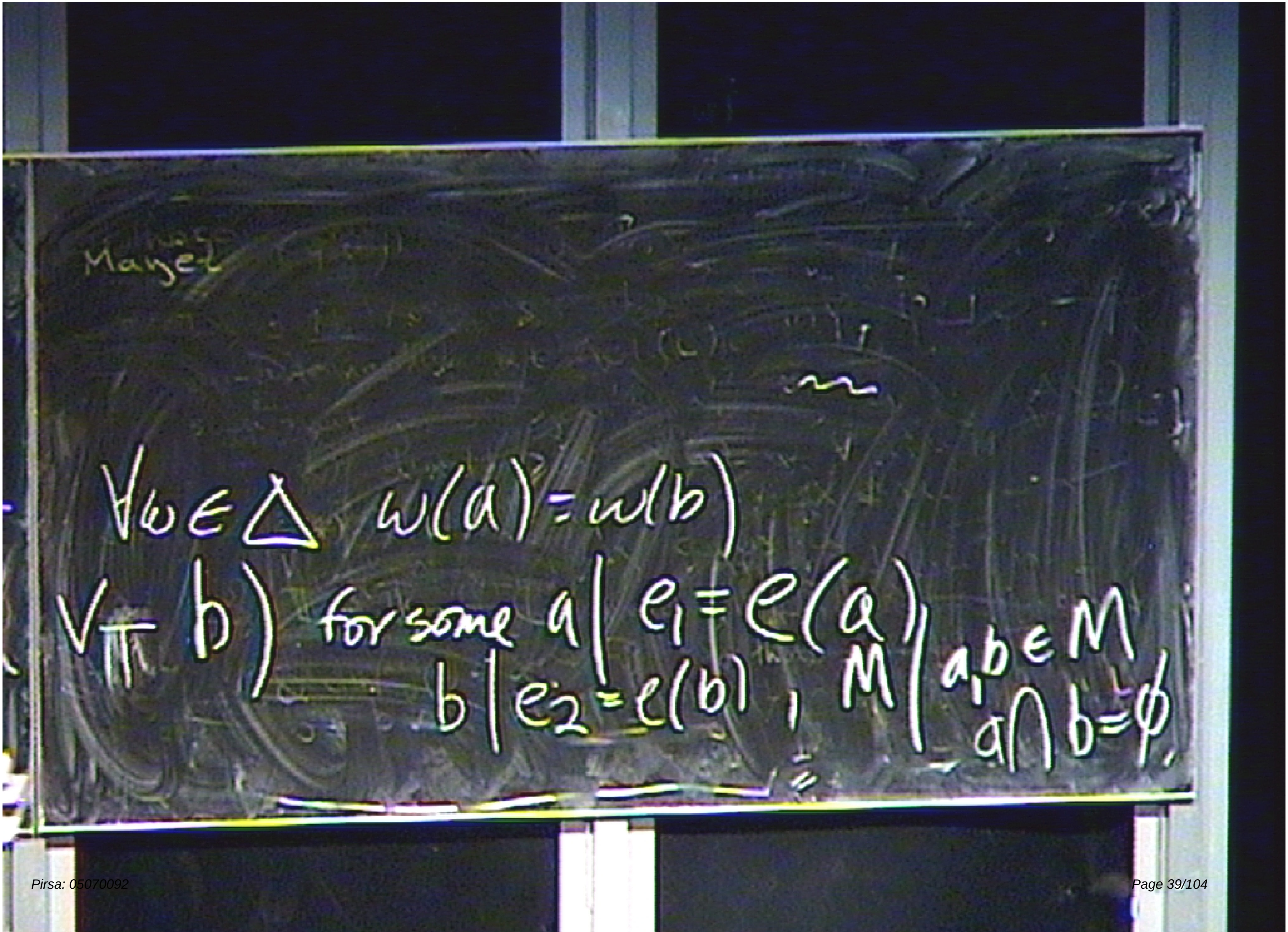

a set of states on
"Event Logic"
∧ ∨
a b ≡

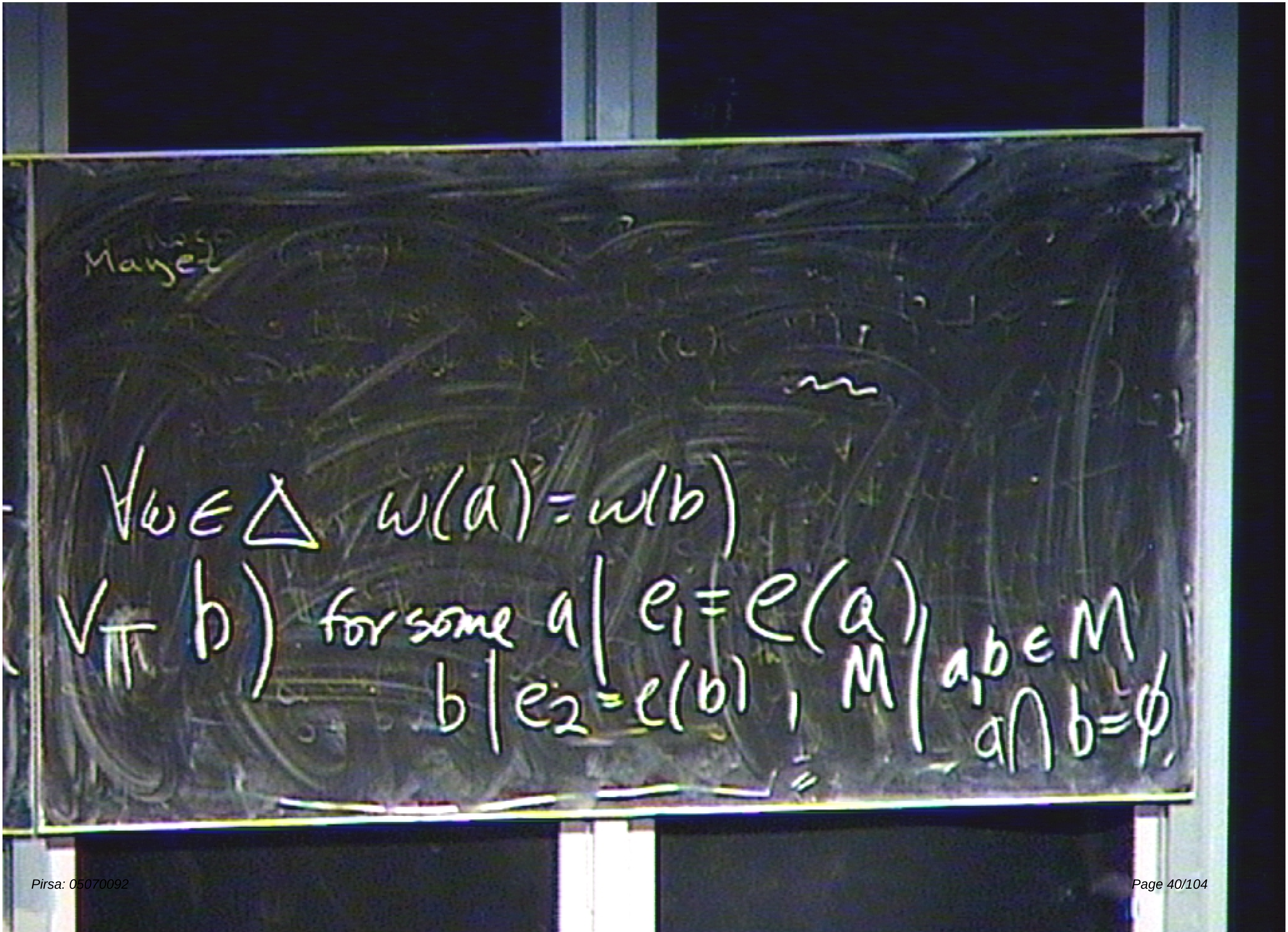

Δ a set of states on 𝒮
"Event Logic"
∧ ∨
a ~′ b ≡


$\forall \omega \in \Delta \quad \omega(a) = \omega(b)$


∀w ∈ Δ   w(a) = w(b)


∀ω∈Δ ω(a)=ω(b)
V_π(b) for some a| e1=e(a), M| a,b∈M
b| e2=e(b), a∩b=φ.


∀ω∈△ ω(a)=ω(b)
V_π(b) for some a | e1 = e(a) | a,b ∈ M
b | e2 = e(b), M | a∩b = φ


∀ω∈Δ ω(a)=ω(b)
V_π(b) for some a | e_1 = e(a), a,b ∈ M
b | e_2 = e(b), M | a∩b=φ.

$$\forall \omega \in \Delta \quad \omega(a) = \omega(b)$$

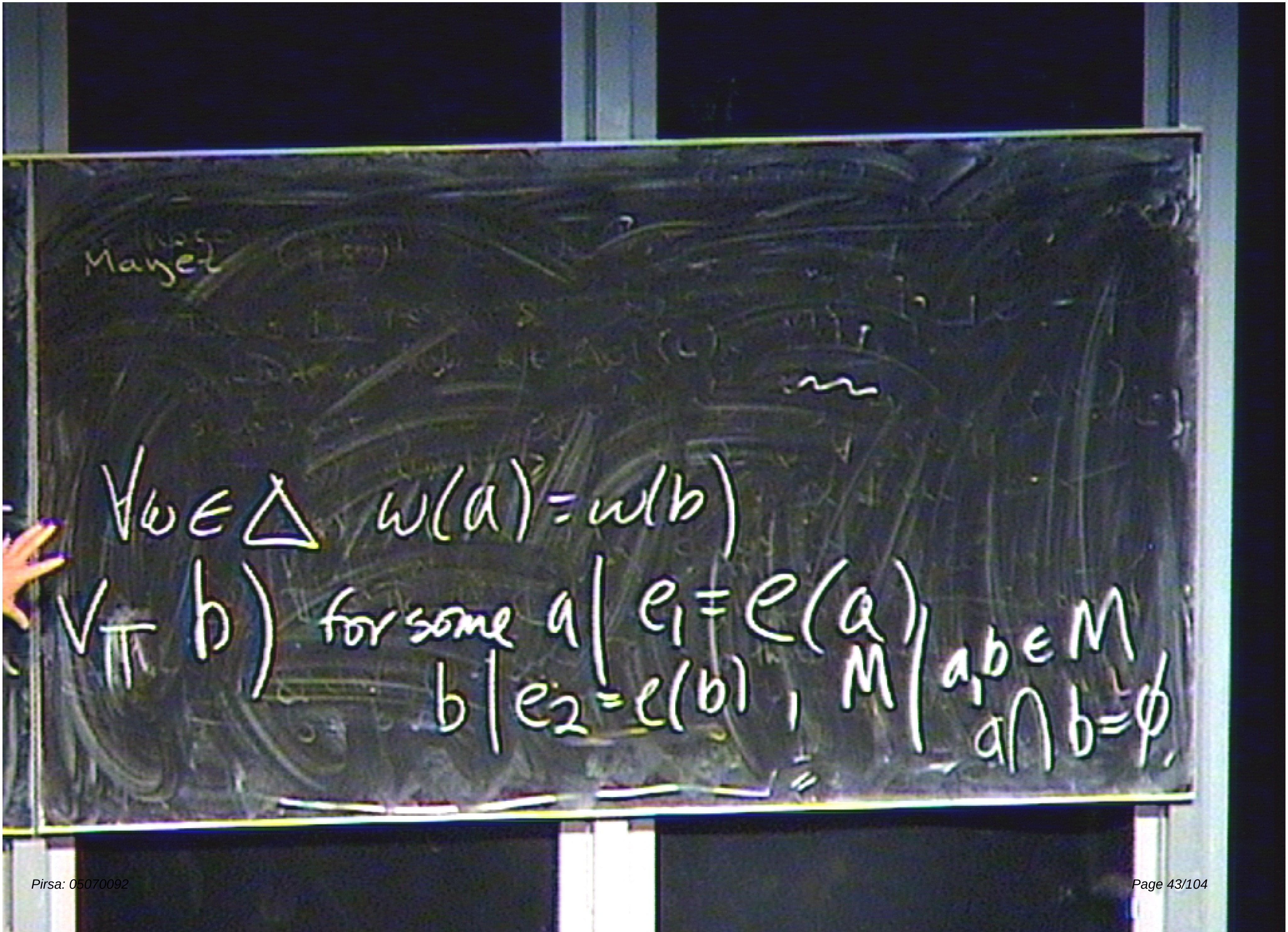
$$V_{\pi}(b) \text{ for some } a \mid e_1 = e(a), \quad M \mid a, b \in M$$

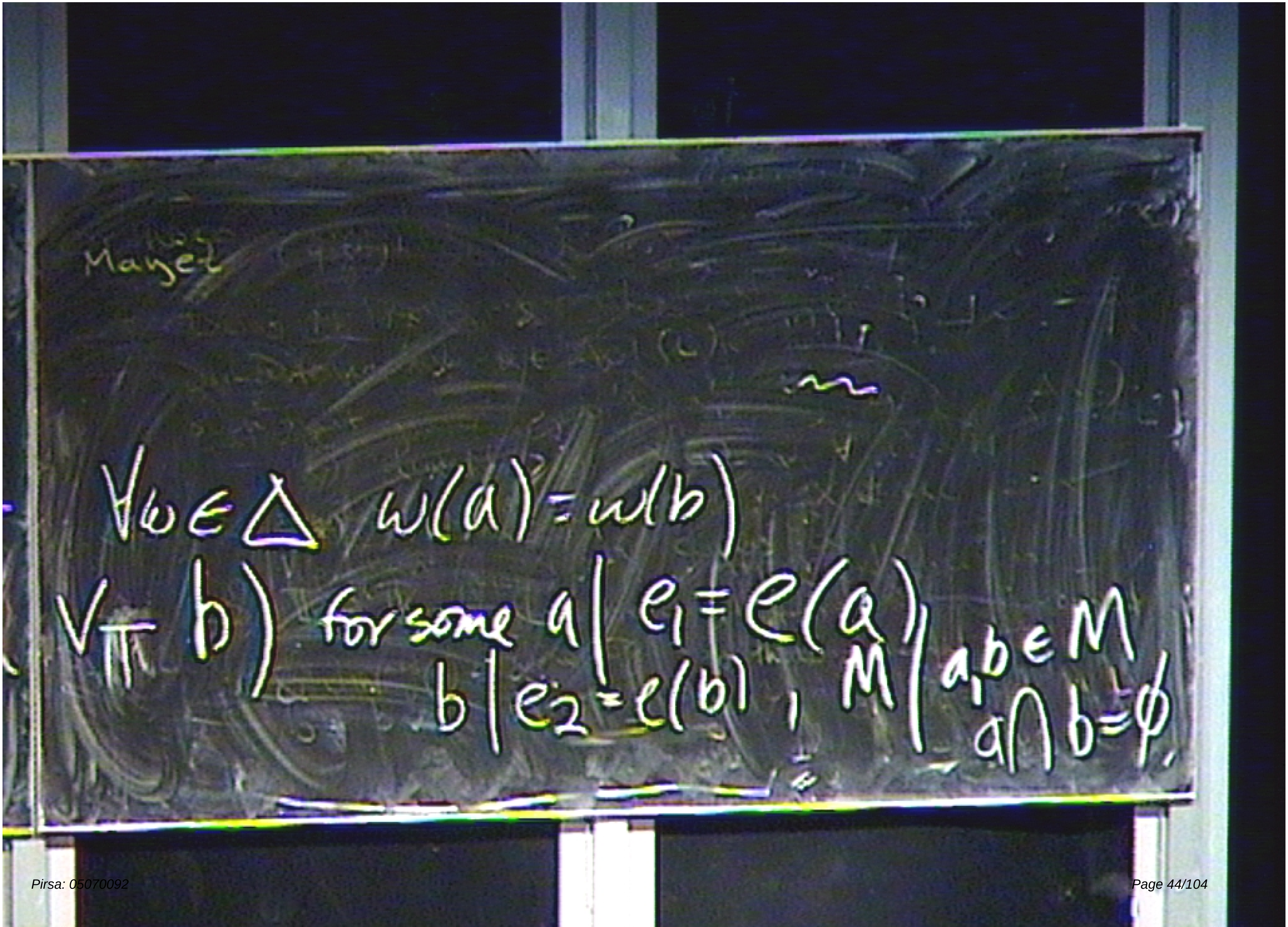
$$b \mid e_2 = e(b), \quad a \cap b = \phi.$$

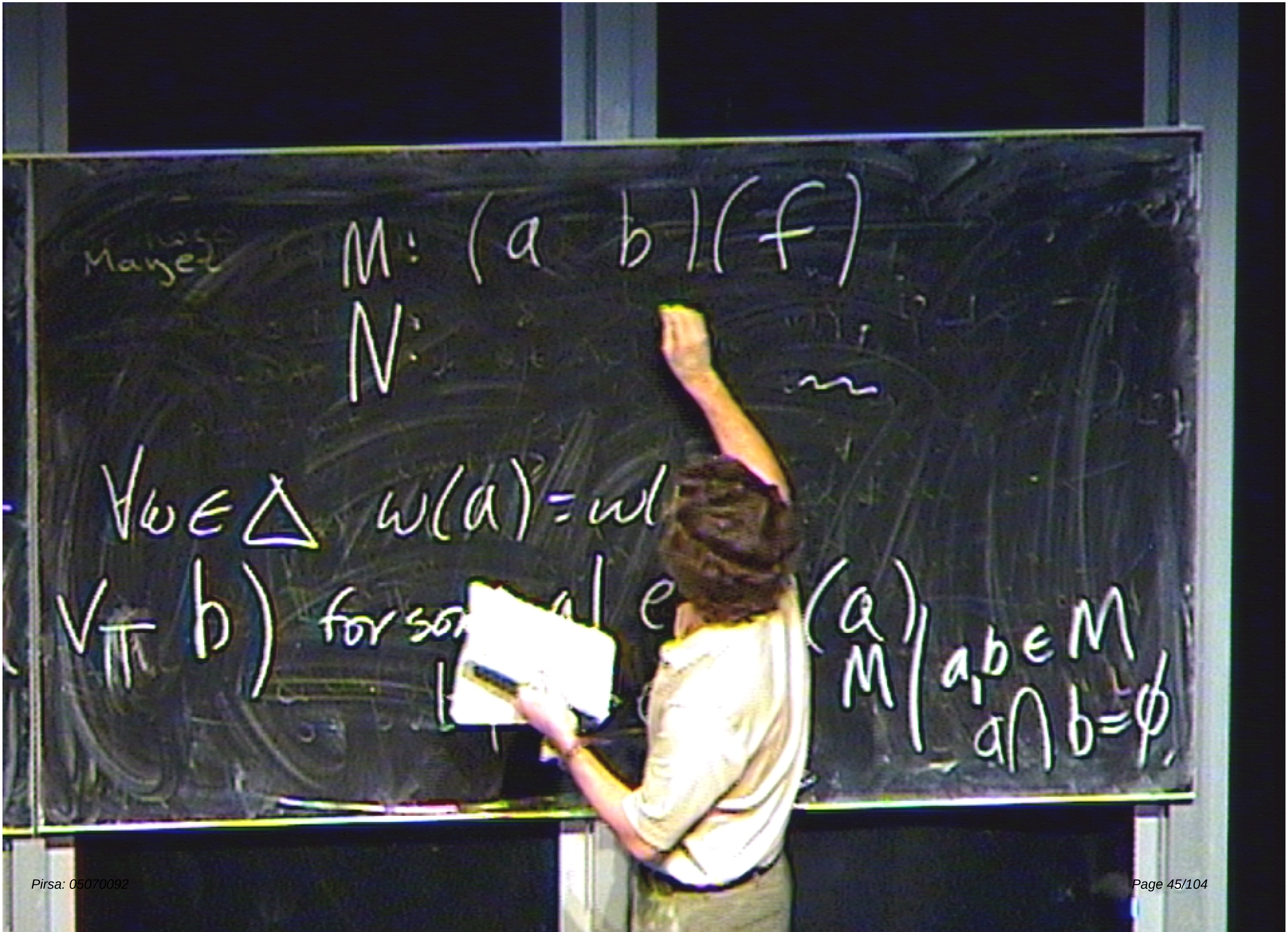
$$\forall \omega \in \Delta \quad \omega(a) = \omega(b)$$

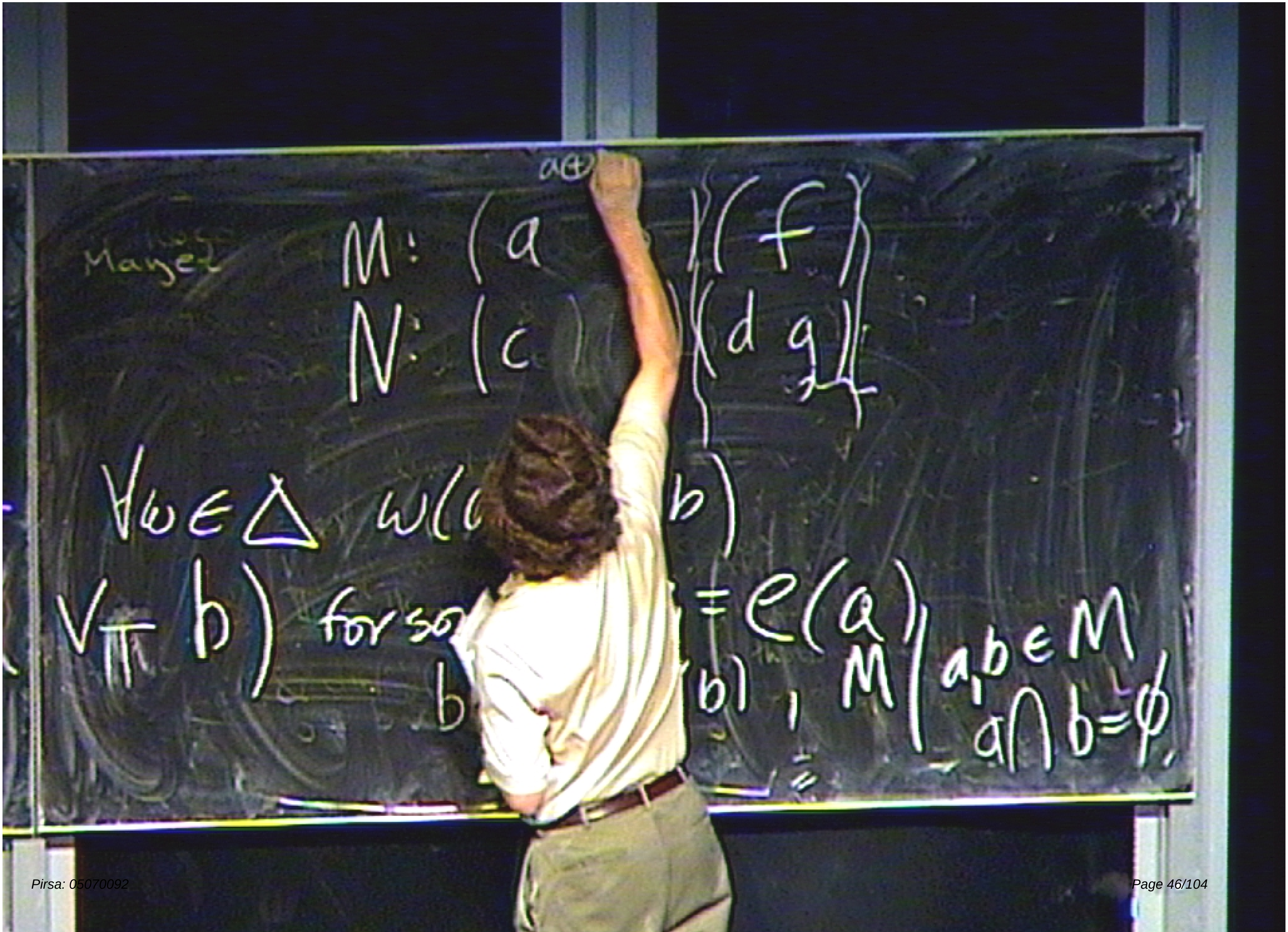
$$V_{\pi}(b) \quad \text{for some } a \mid e_1 = e(a), \quad M \mid a, b \in M$$

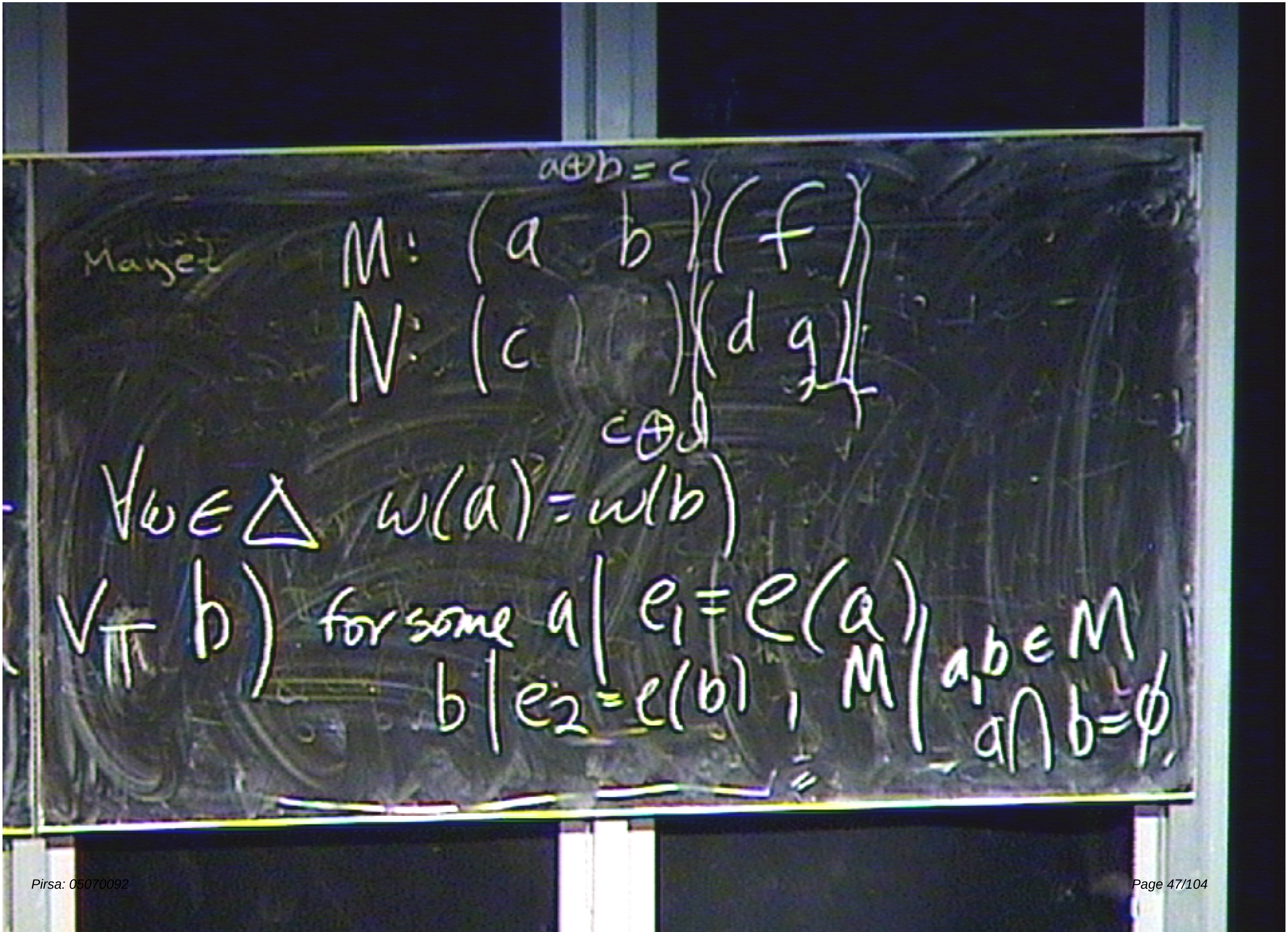
$$b \mid e_2 = e(b), \qquad a \cap b = \phi.$$

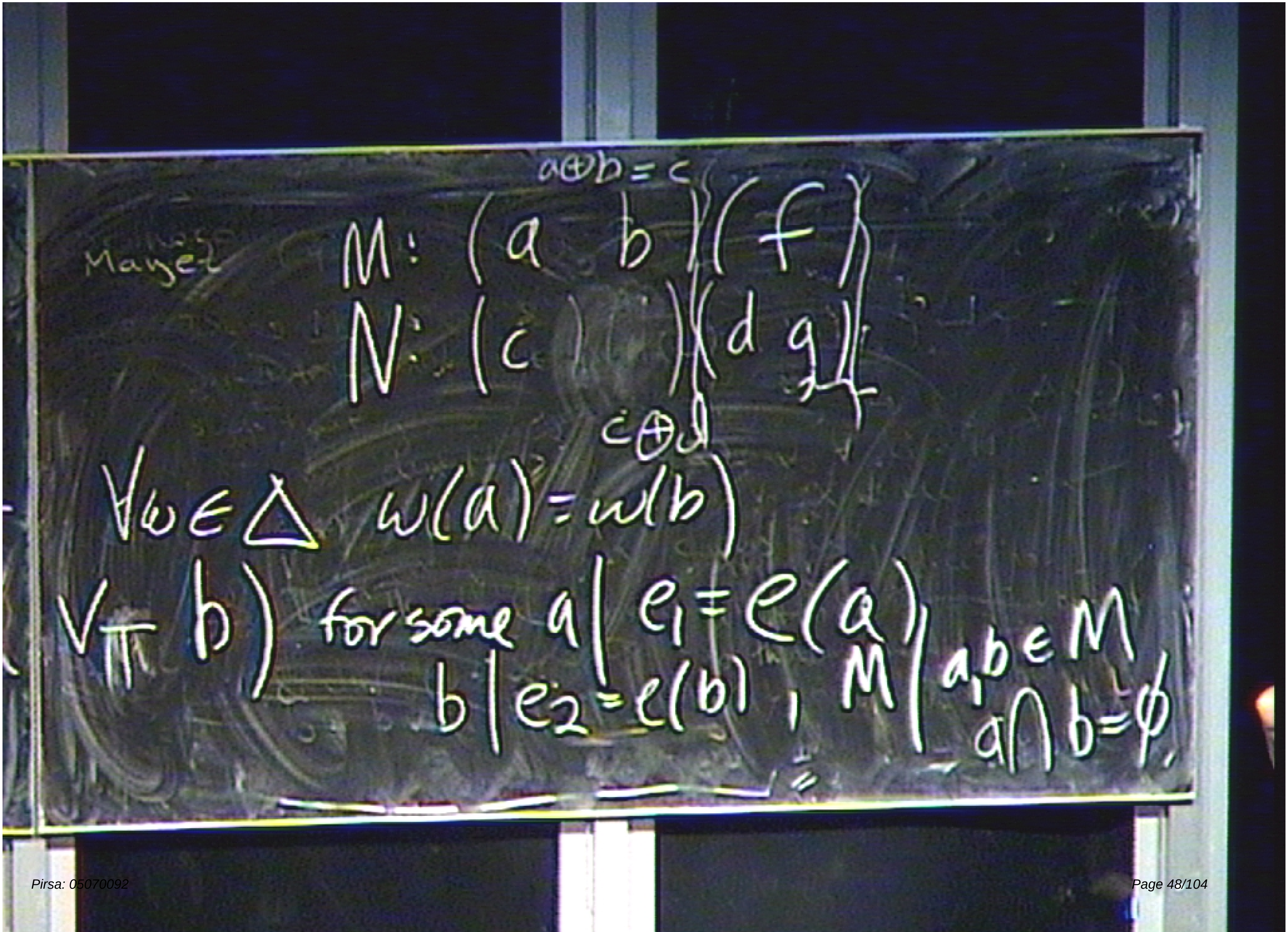

Mazet
∀ω∈Δ ω(a)=ω(b)
V_π(b) for some a| e1=e(a), M| a,b∈M
b| e2=e(b)
a∩b=φ

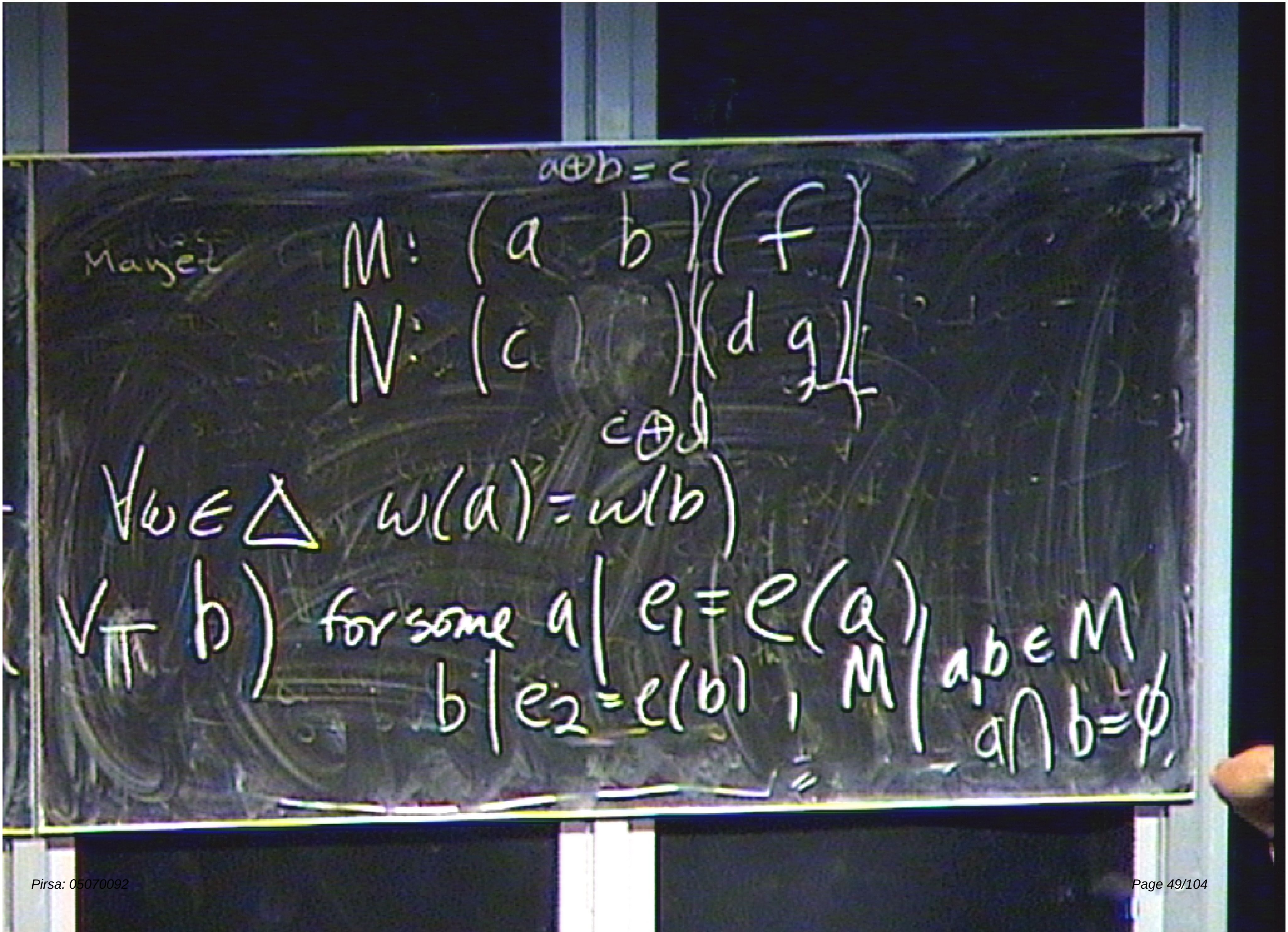

∀ω∈Δ ω(a)=ω(b)
V_π(b) for some a | e1=e(a) | a,b∈M
b | e2=e(b) , M | a∩b=φ.


M: (a b)(f)
N:
for so
a,b ∈ M
a∩b=φ

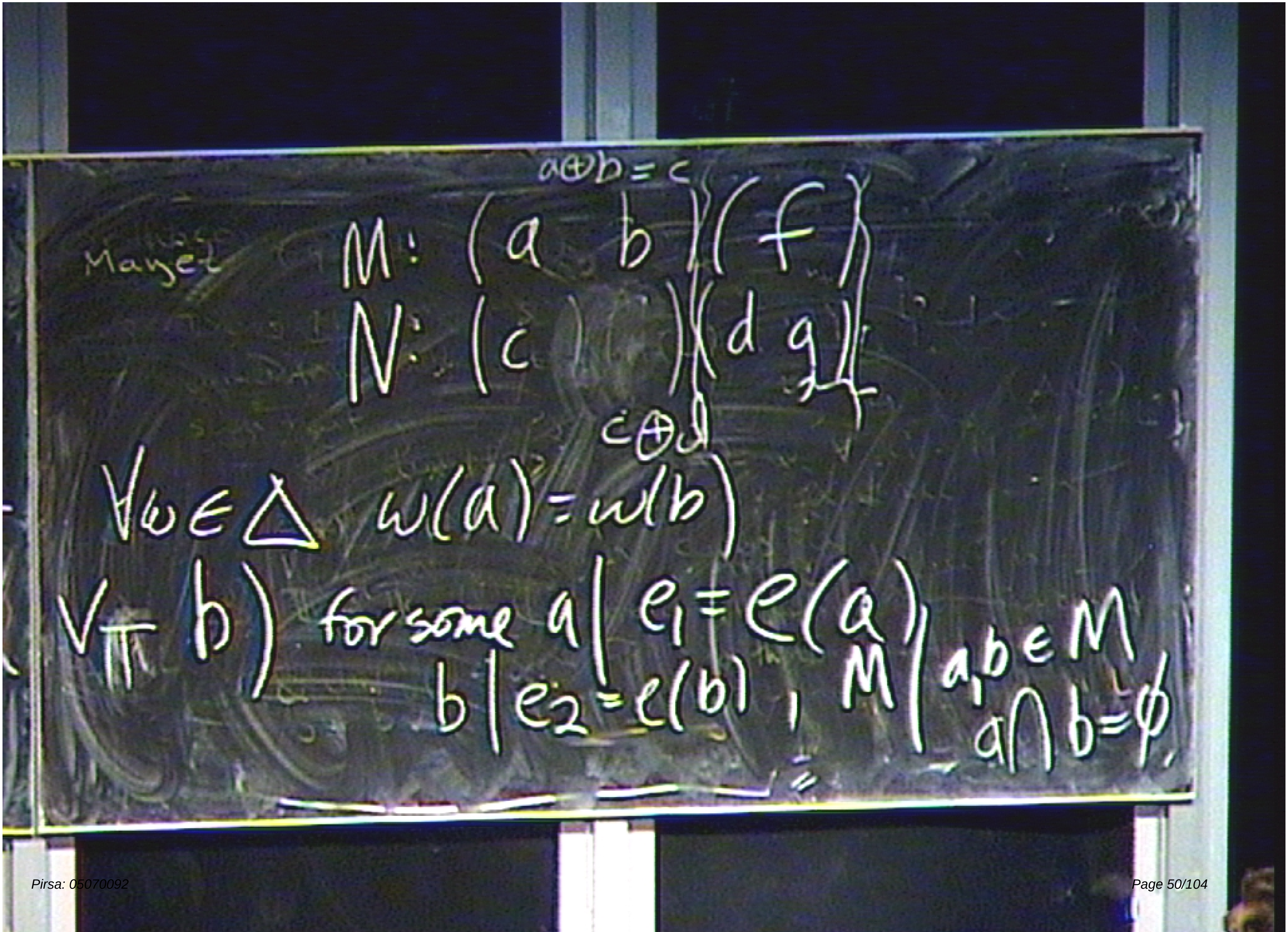

M: (a
N: (c
(d g
∀ω∈△ w(
for so
= e(a)
a, b ∈ M
a ∩ b = φ

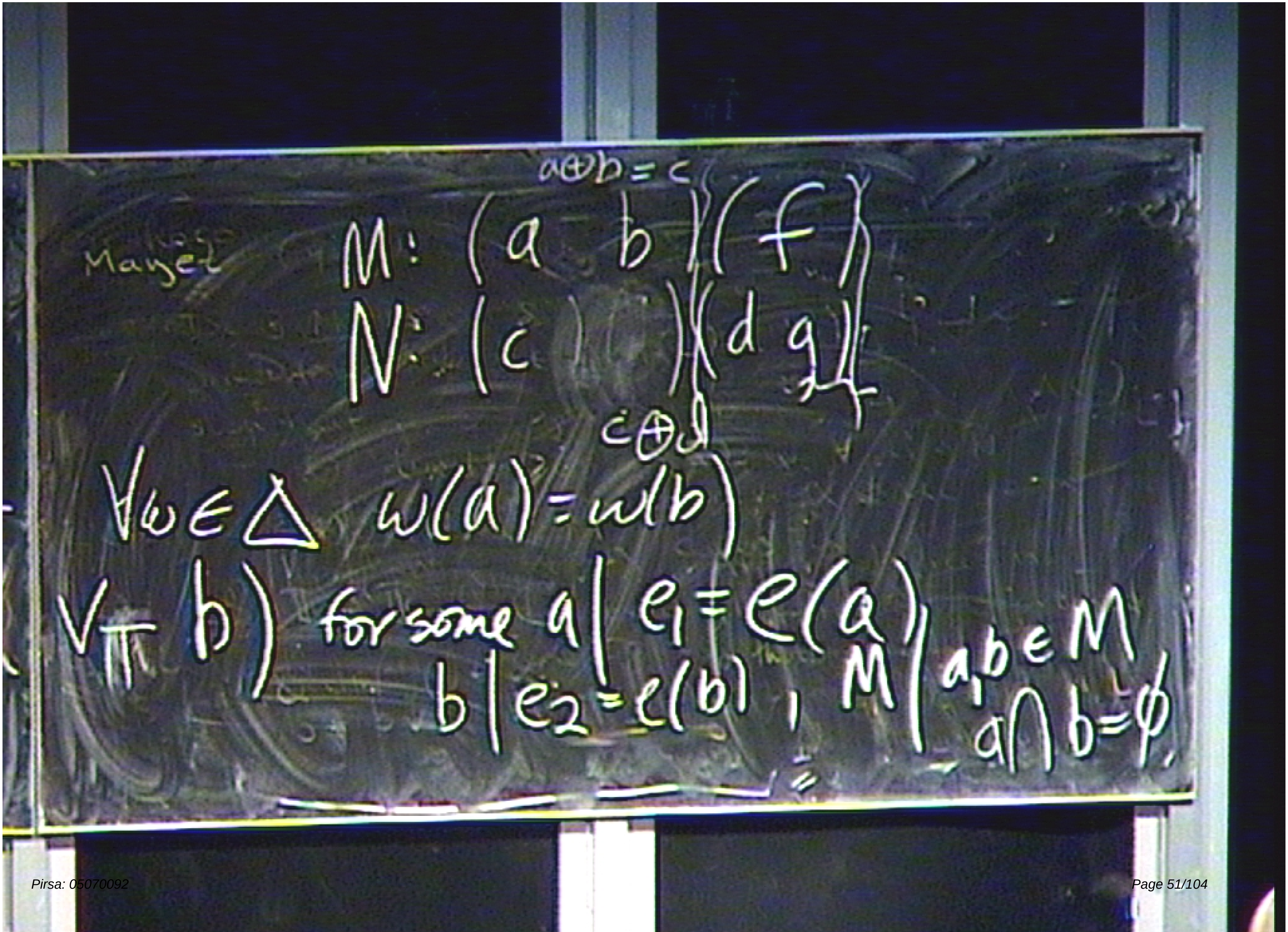

a⊕b = c
M: (a b)(f)
N: (c)(d g)
c⊕d
∀ω∈Δ ω(a) = ω(b)
Vπ b) for some a| e1 = e(a), M| a,b ∈ M
b| e2 = e(b), a∩b = φ

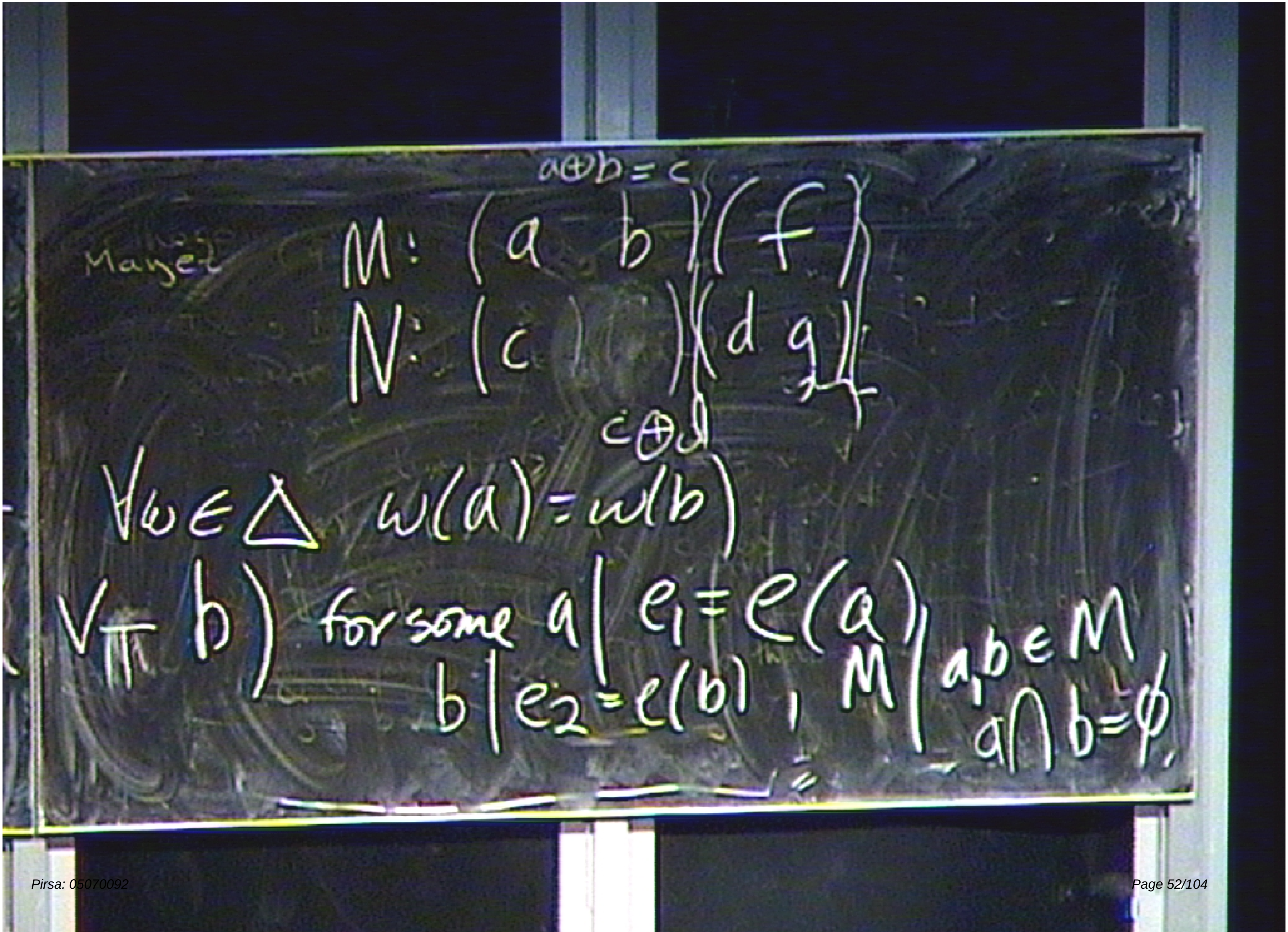

a⊕b = c
Mayer
M: (a b)(f)
N: (c)(d g)
c⊕d
∀ω∈Δ ω(a)=ω(b)
for some a| e1=e(a), M| a,b∈M
b| e2=e(b)
a∩b=φ

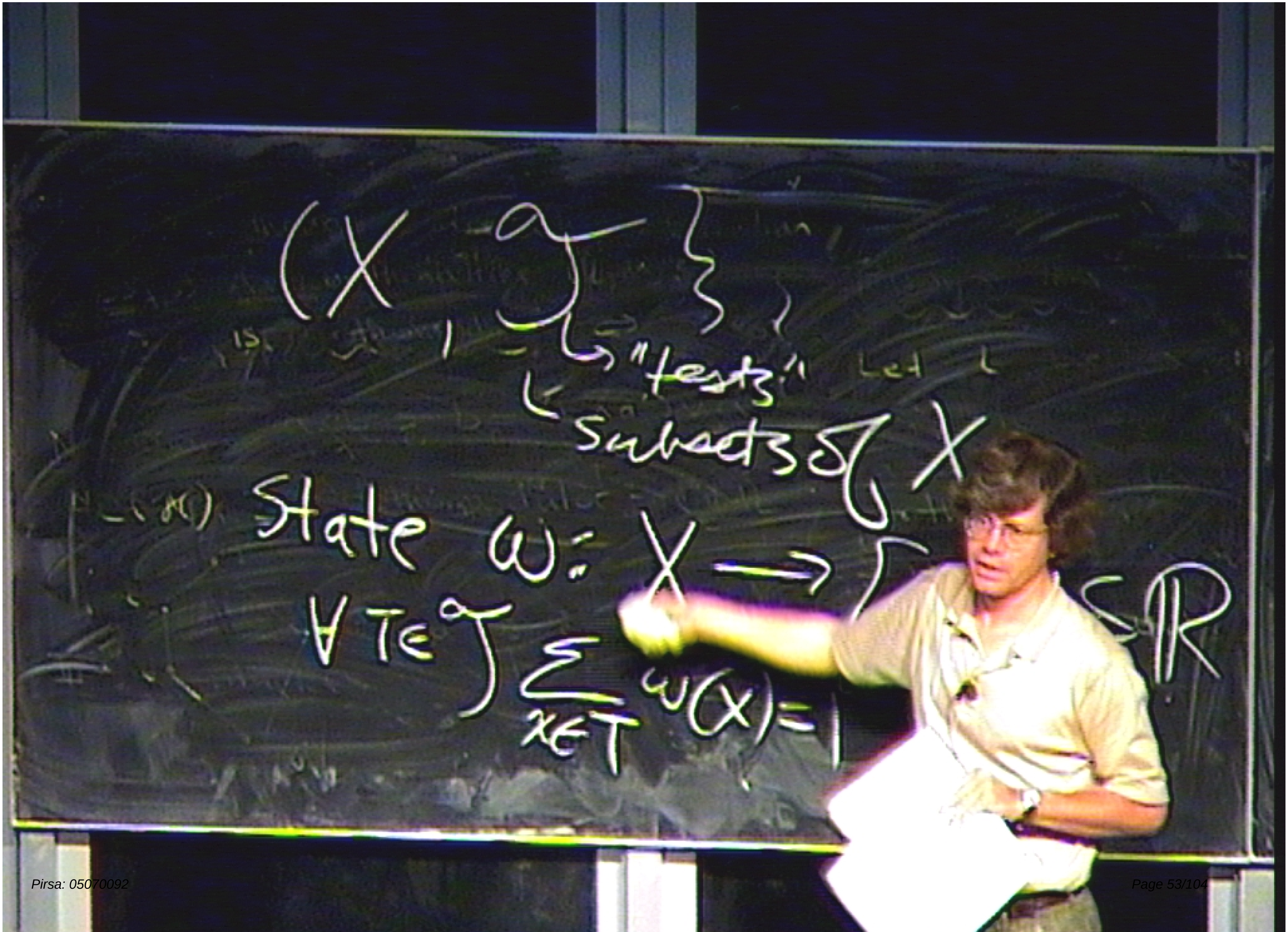


a⊕b=c
M: (a b)(f)
N: (c)(d g)
c⊕d
∀ω∈Δ ω(a)=ω(b)
V_π b) for some a | e_1 = e(a)
b | e_2 = e(b), M | a,b ∈ M
a∩b=φ


a⊕b = c
M: (a b)(f)
N: (c)(d g)
c⊕d
∀ω∈Δ ω(a) = ω(b)
V_π(b) for some a| e1 = e(a), M| a,b ∈ M
b| e2 = e(b)
a∩b = φ


a⊕b = c
M: (a b)(f)
N: (c)(d g)
c⊕d
∀ω∈Δ ω(a) = ω(b)
V_π(b) for some a | e_1 = e(a), a,b ∈ M
b | e_2 = e(b), M | a∩b = φ


a⊕b = c
M: (a b)(f)
N: (c)(d g)
c⊕d
∀ω∈Δ ω(a) = ω(b)
for some a | e1 = e(a), a, b ∈ M
b | e2 = e(b), M | a∩b = φ


(X, 𝒯)
"tests"
subsets of X
State ω: X →
⊆ ℝ
∀ T ∈ 𝒯 Σ_{x∈T} ω(x) = 1

$(X, \mathcal{T})$

$\mathcal{T}$ → "tests" ⊆ subsets of X

State $\omega : X \to [0,1] \subseteq \mathbb{R}$

$\forall T \in \mathcal{T} \quad \sum_{x \in T} \omega(x) = 1$

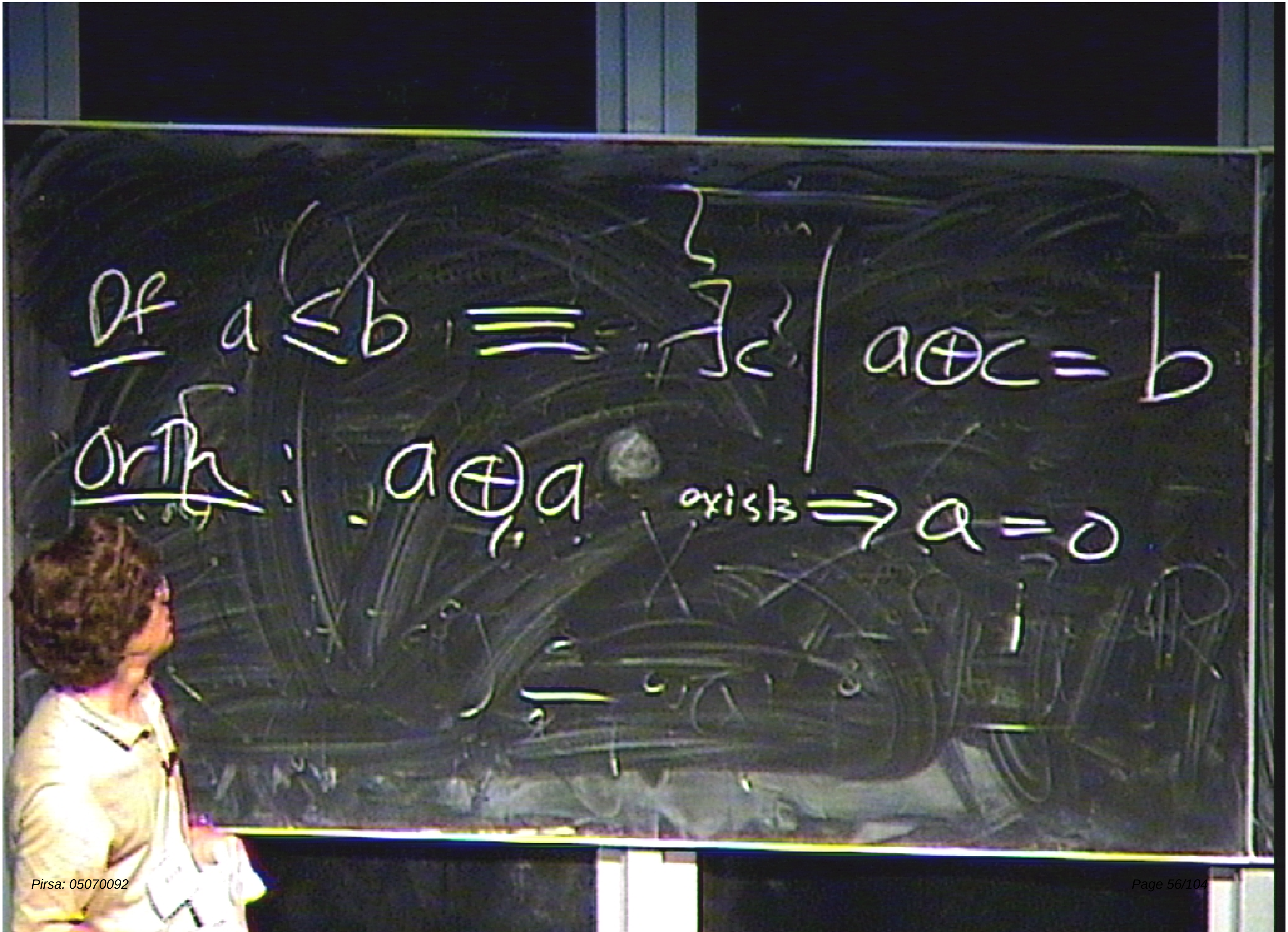
$(X, \mathcal{T})$

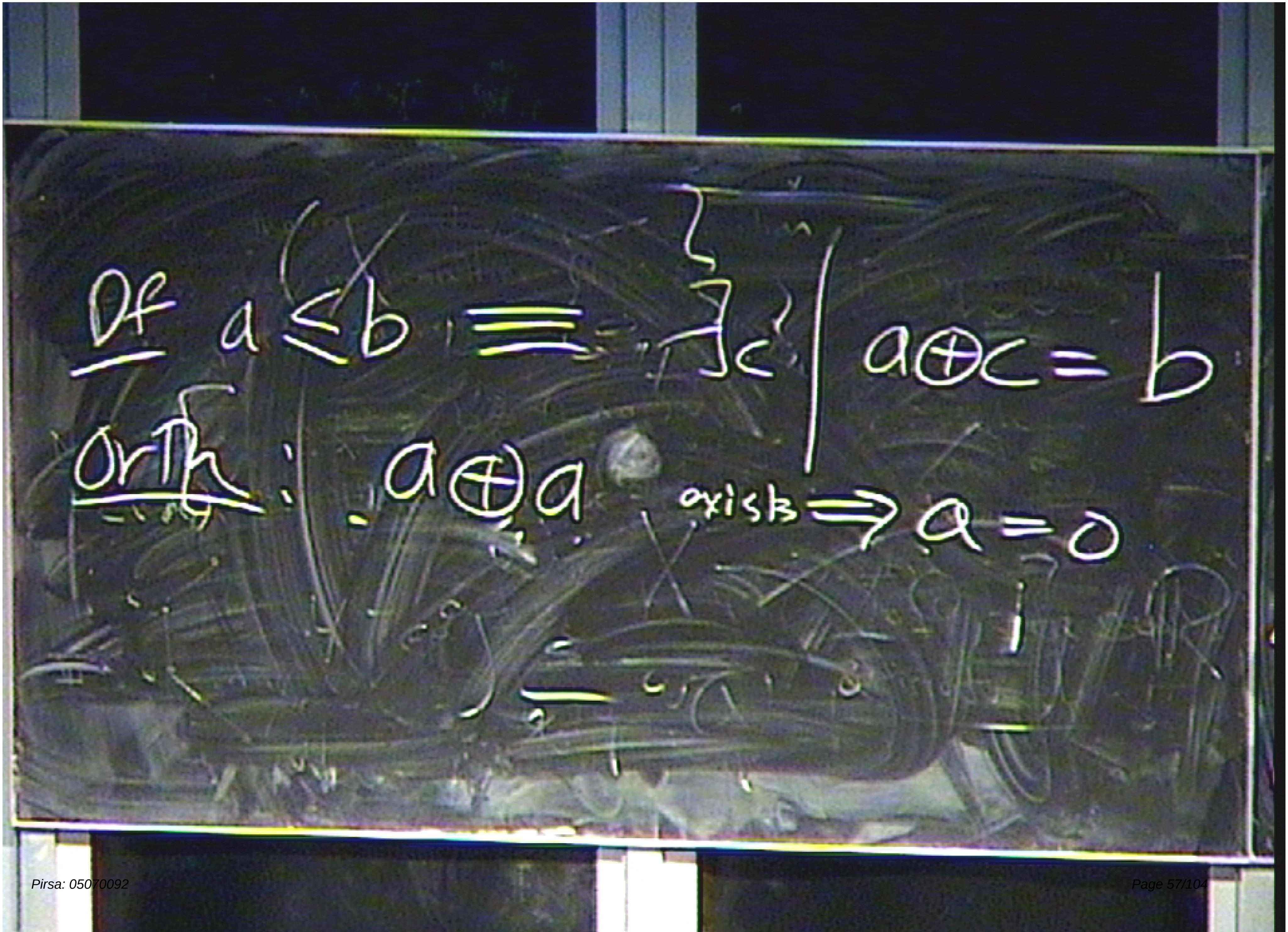
$\mathcal{T}$ → "tests"

⊆ subsets of X

State $\omega : X \to [0,1] \subseteq \mathbb{R}$

$\forall T \in \mathcal{T} \quad \sum_{x \in T} \omega(x) = 1$


Df a ≤ b ≡ ∃c | a⊕c = b
OrTh: a⊕a axioms ⇒ a = 0


Df a ≤ b ≡ ∃c | a⊕c = b
OrTh: a⊕a exists ⇒ a = 0

Df $a \leq b \equiv \exists c \mid a \oplus c = b$

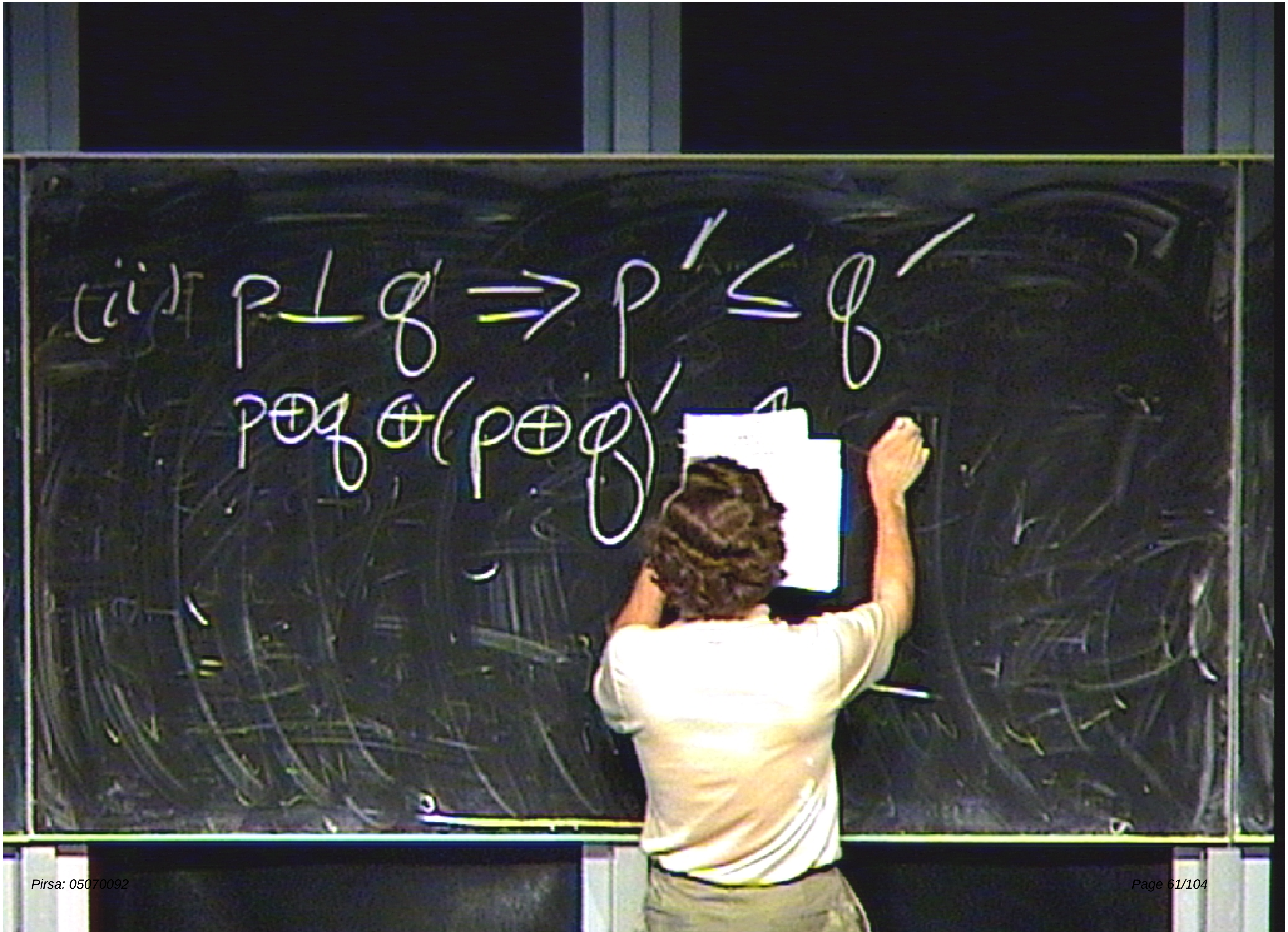
OrTh: $a \oplus a$ exists $\Rightarrow a = 0$

Df $a \leq b \equiv \exists c \mid a \oplus c = b$

OrTh: $a \oplus a$ exists $\Rightarrow a = 0$

(ii) If $p \perp q \Rightarrow p' \leq q'$

$p \oplus q \oplus (p \oplus q)'$


(ii) p ⊥ q ⇒ p' ≤ q'
p ⊕ q ⊕ (p ⊕ q)'

(ii) $p \perp q \Rightarrow p' \leq q'$

$p \oplus q \oplus (p \oplus q)' = 1 = q + q'$

(ii) $p \perp q \Rightarrow p' \leq q'$

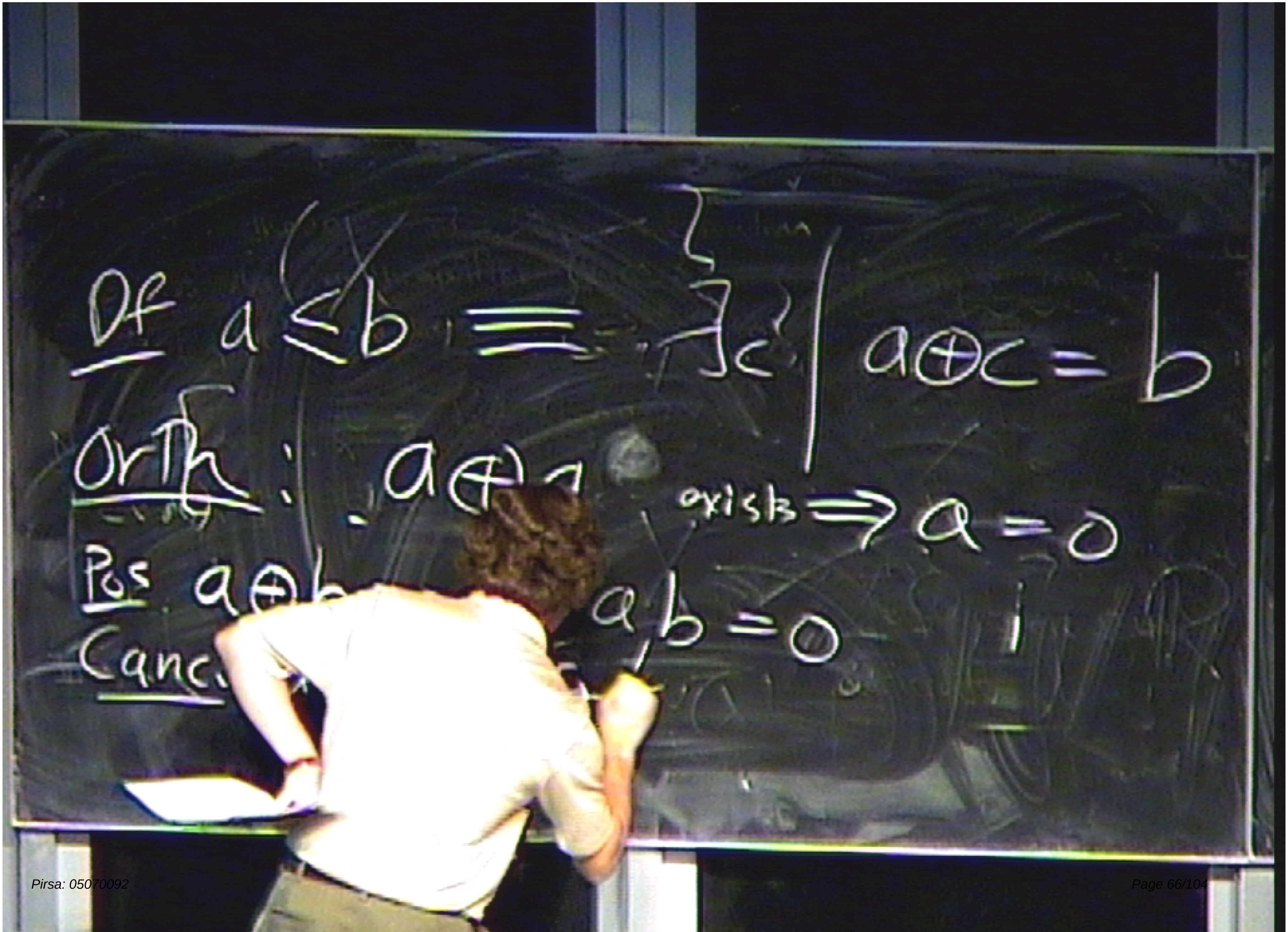
$p \oplus q \oplus (p \oplus q)' = 1 = q + q'$

(ii) $p \perp q \Rightarrow p' \le q'$

$p \oplus q \oplus (p \oplus q)' = 1 = q + q'$

(ii) $p \perp q \Rightarrow p' \leq q'$

$$p \oplus q \oplus (p \oplus q)' = 1 = q + q'$$

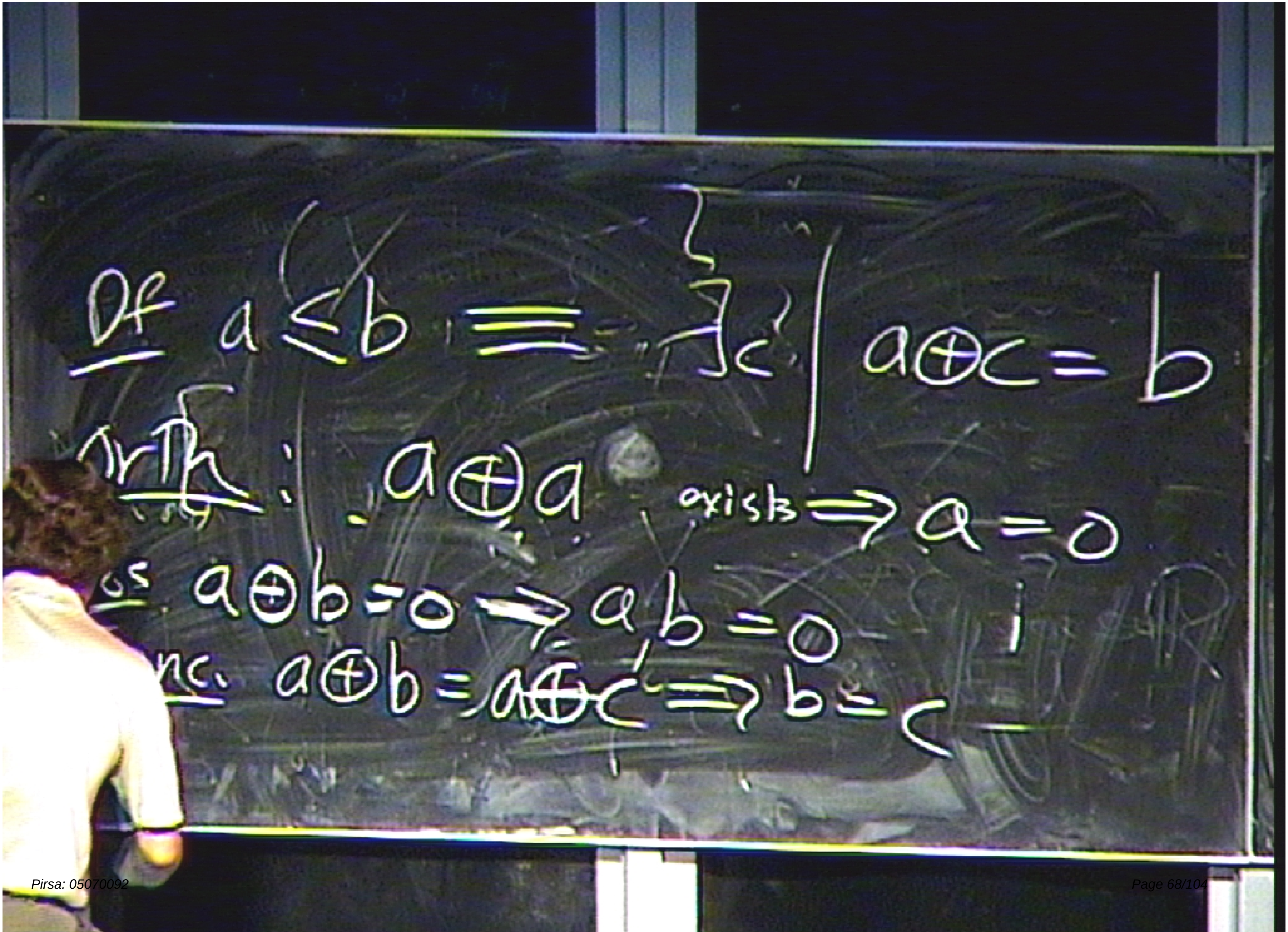

Df a ≤ b ≡ ∃c | a⊕c = b
OrTh : a⊕a
exists ⇒ a = 0
Pos a⊕b
a b = 0
Canc

Df $a \le b \equiv \exists c' \mid a \oplus c = b$

OrTh: $a \oplus a$ axists $\Rightarrow a = 0$

Pos $a \oplus b = 0 \Rightarrow a\ b = 0$

Canc. $a \oplus b = a \oplus c \Rightarrow b = c$


Df a ≤ b ≡ ∃c | a⊕c = b
Th: a⊕a exists ⇒ a = 0
a⊖b = 0 ⇒ a b = 0
a⊕b = a⊕c ⇒ b = c

Df $a \leq b \equiv \exists c \mid a \oplus c = b$

Orth: $a \oplus a$ exists $\Rightarrow a = 0$

Pos $a \oplus b = 0 \Rightarrow a, b = 0$

Canc. $a \oplus b = a \oplus c \Rightarrow b = c$

Unital: has a unit $u \mid \exists\, x \oplus 1 \Rightarrow x = 0.$

Df $a \leq b \equiv \exists c \mid a \oplus c = b$

Orth: $a \oplus a$ exists $\Rightarrow a = 0$

Pos $a \oplus b = 0 \Rightarrow a, b = 0$

Canc. $a \oplus b = a \oplus c \Rightarrow b = c$

Unital: has a unit $u \mid \exists\, x \oplus 1 \Rightarrow x = 0.$

Df $a \le b \equiv \exists c \mid a \oplus c = b$

Orth: $a \oplus a$ exists $\Rightarrow a = 0$

Pos $a \oplus b = 0 \Rightarrow a, b = 0$

Canc. $a \oplus b = a \oplus c \Rightarrow b = c$

Unital: has a unit $u \mid \exists\, x \oplus 1 \Rightarrow x = 0.$

(ii) $p \perp q \Rightarrow p \leq q'$

$$p \oplus q \oplus (p \oplus q)' = 1 = q + q'$$

$$p \oplus (p \oplus q)' = q'$$

(ii) $p \perp q \Rightarrow p \leq q'$

$p \oplus q \oplus (p \oplus q)' = 1 = q + q'$

$p \oplus (p \oplus q)' = q'$

(ii) $p \perp q \Rightarrow p \leq q'$

$p \oplus q \oplus (p \oplus q)' = 1 = q + q'$

$p \oplus (p \oplus q)' = q'$

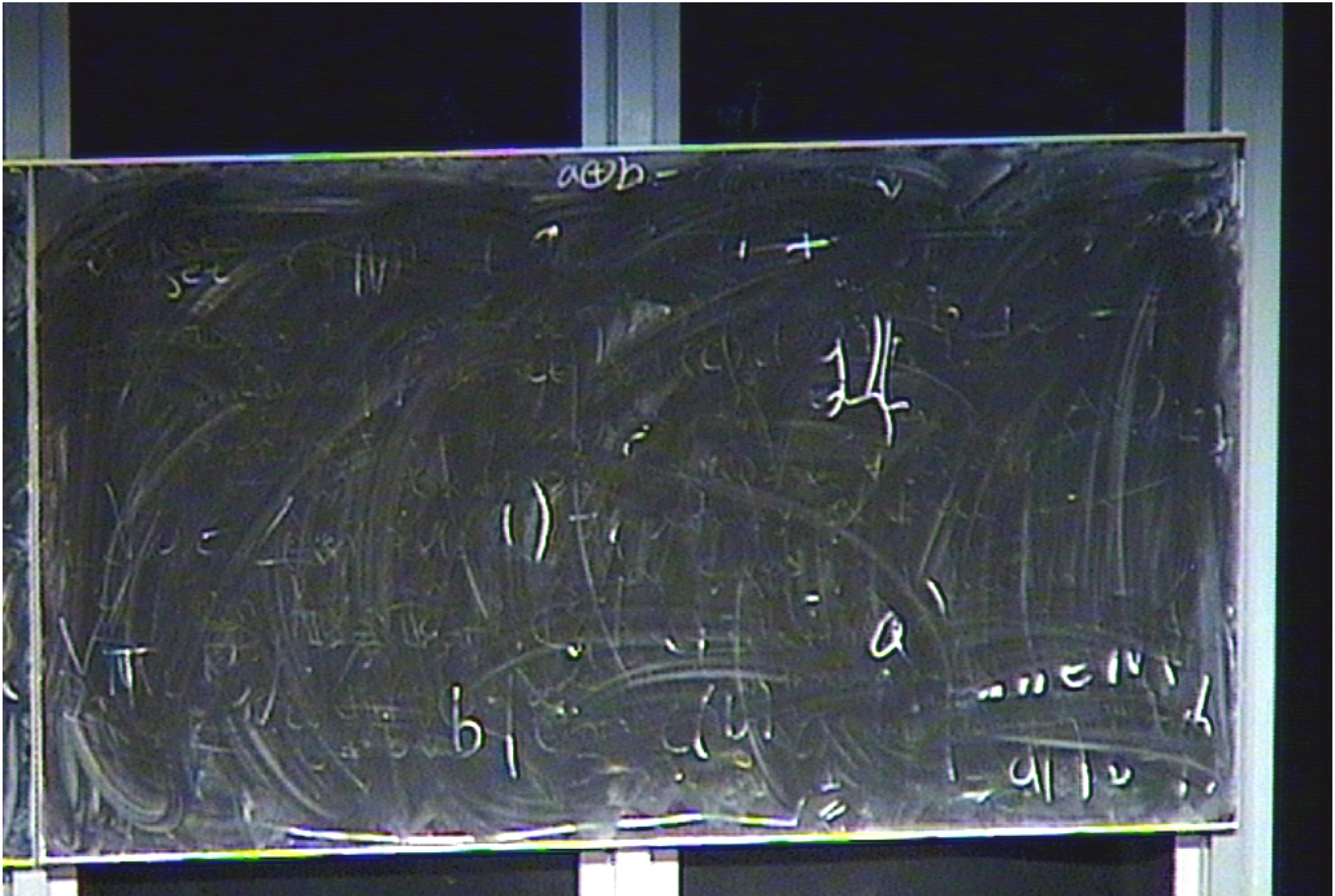
(iii) $p \leq q' \Longleftrightarrow q' < p'$

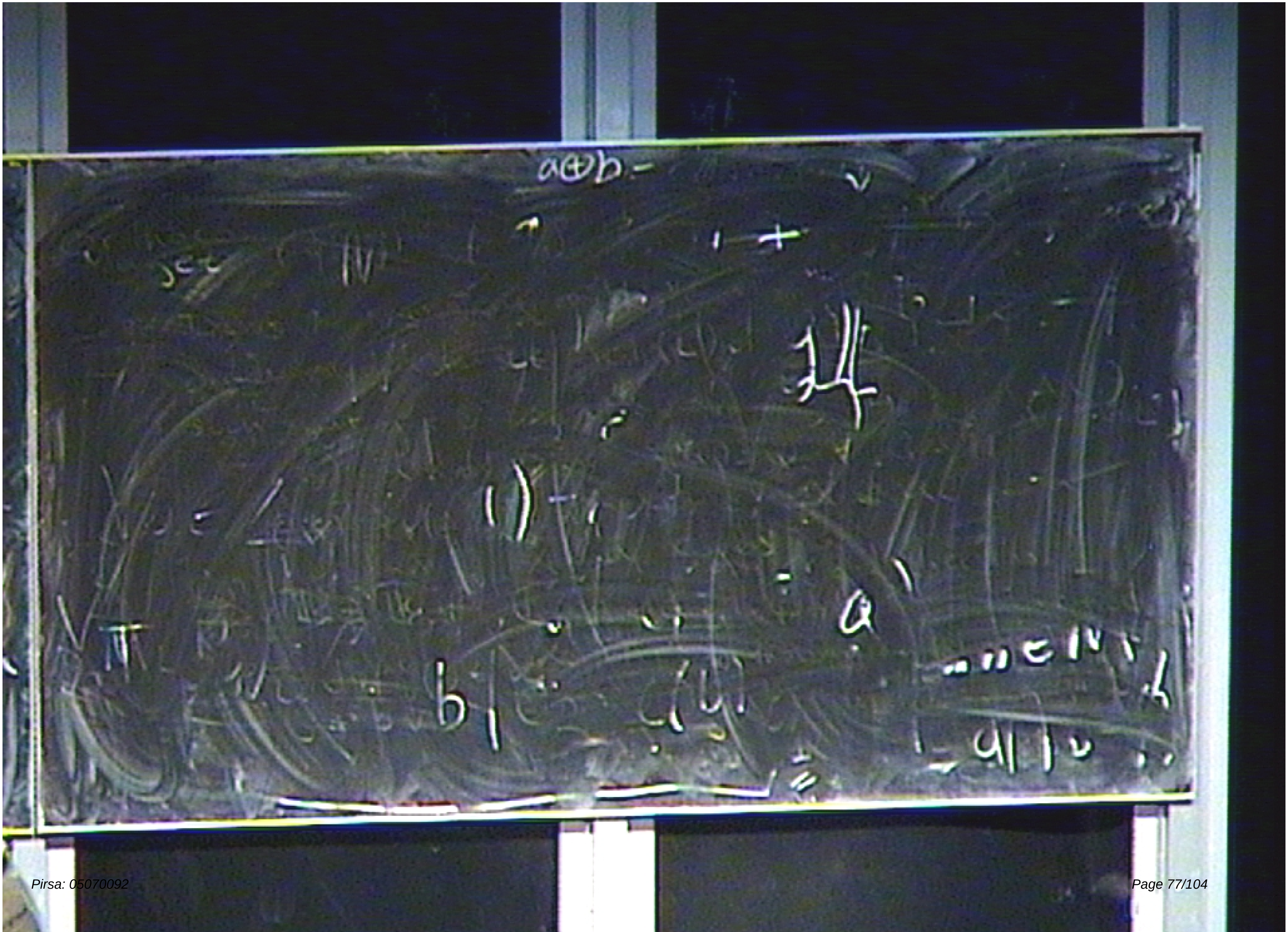
(ii) $p \perp q \Rightarrow p \leq q'$

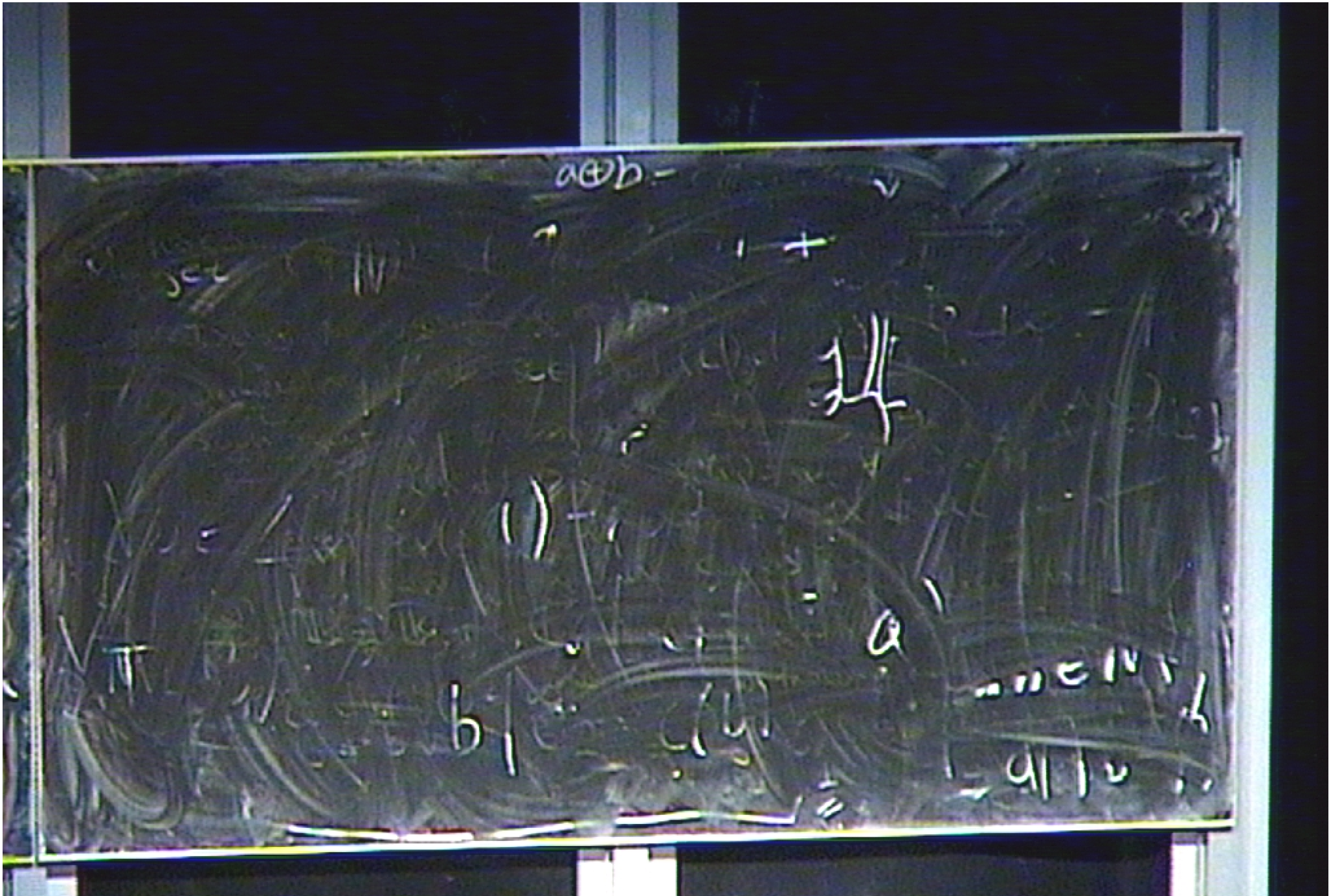
$$p \oplus q \oplus (p \oplus q)' = 1 = q + q'$$

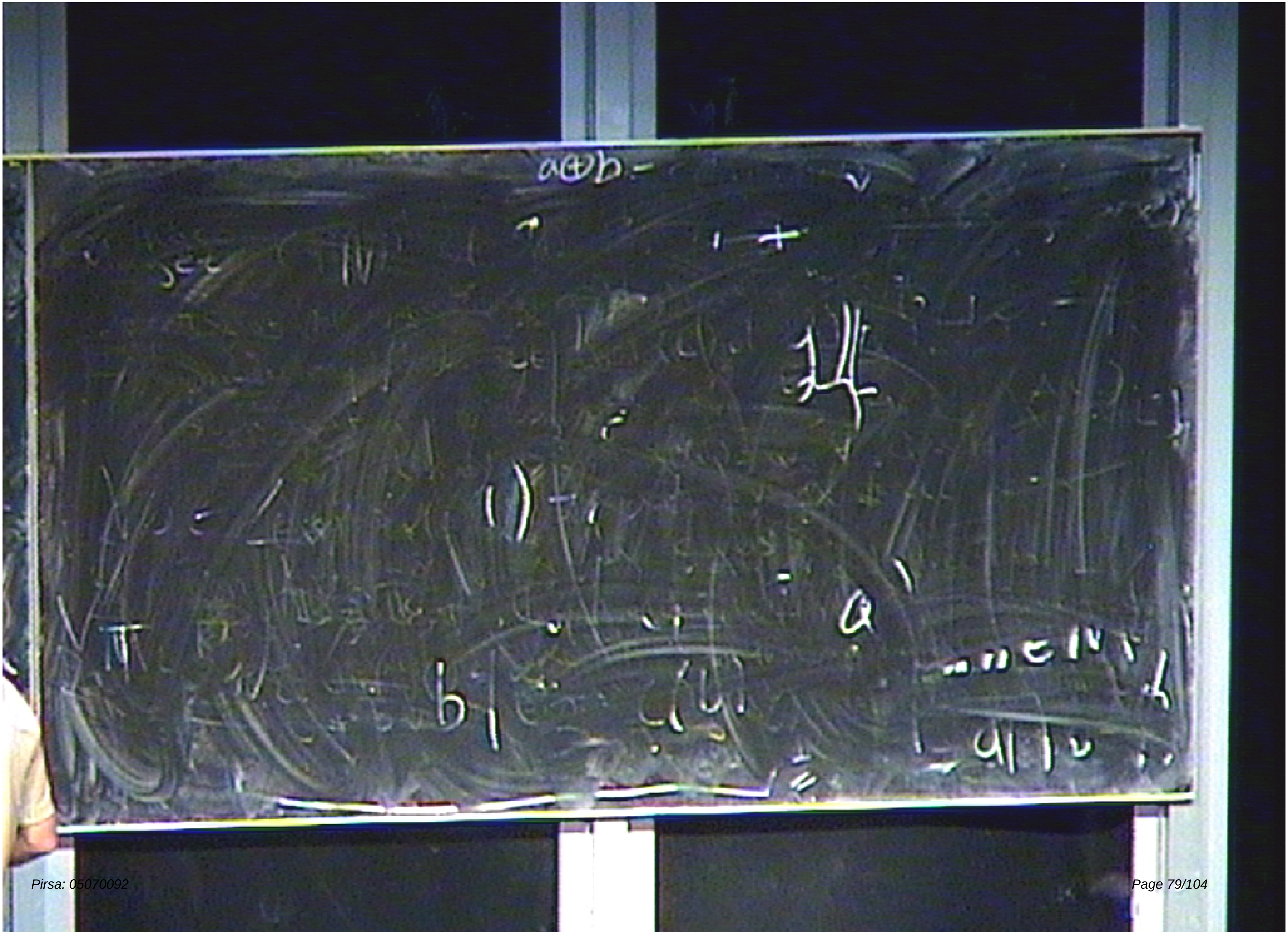
$$p \oplus (p \oplus q)' = q'$$

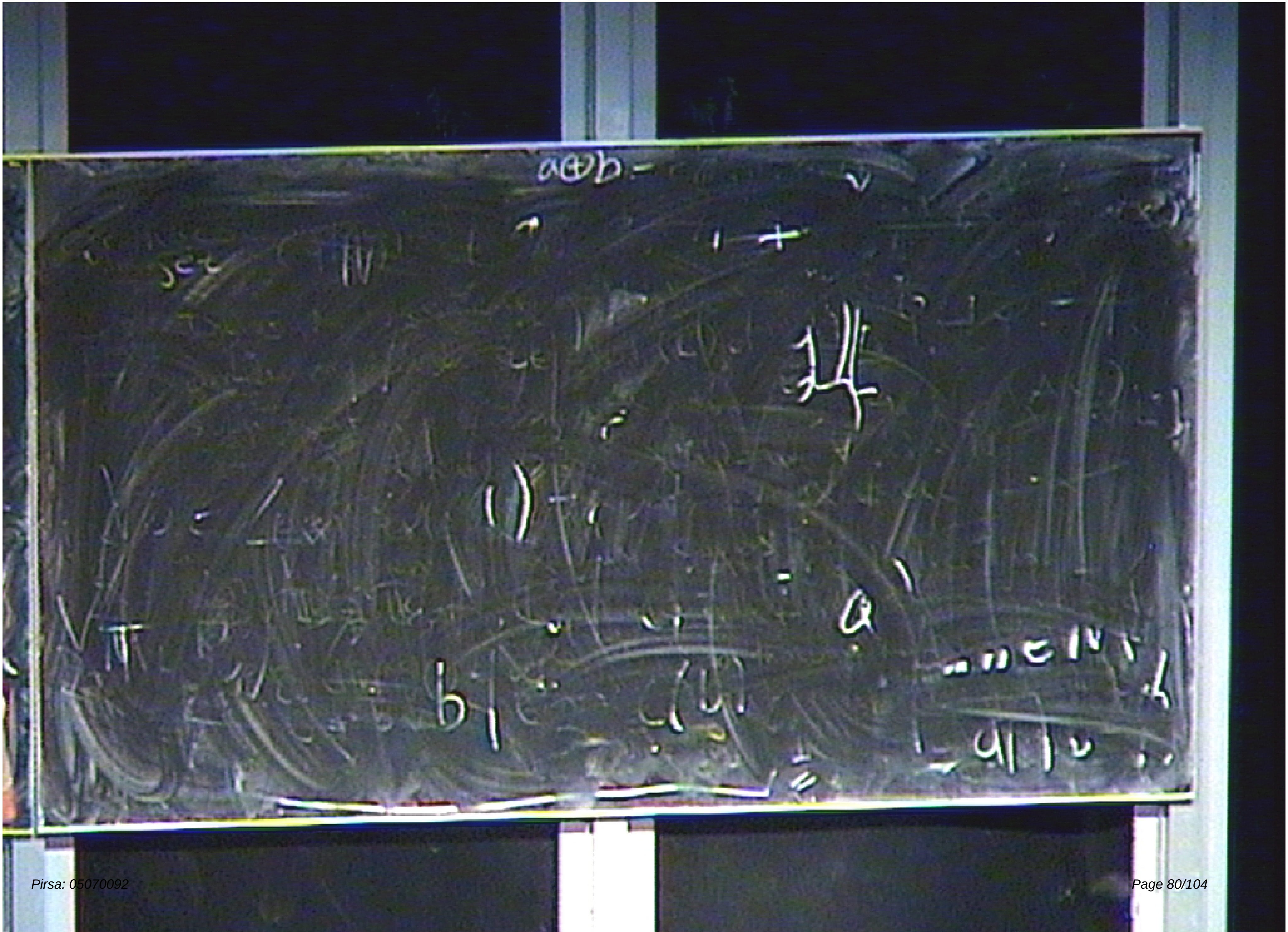
(iii) $p \leq q' \Longleftrightarrow q' < p'$

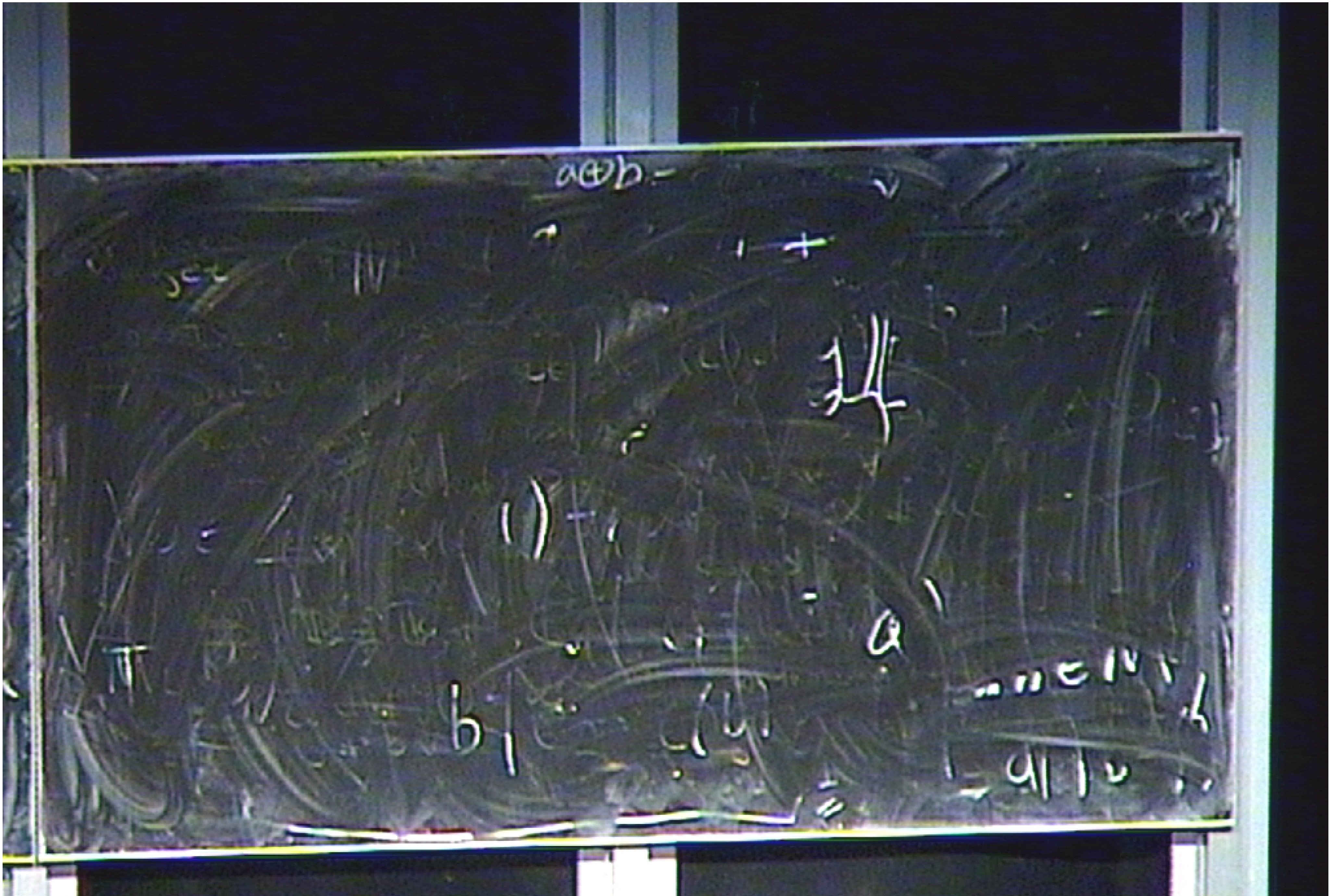


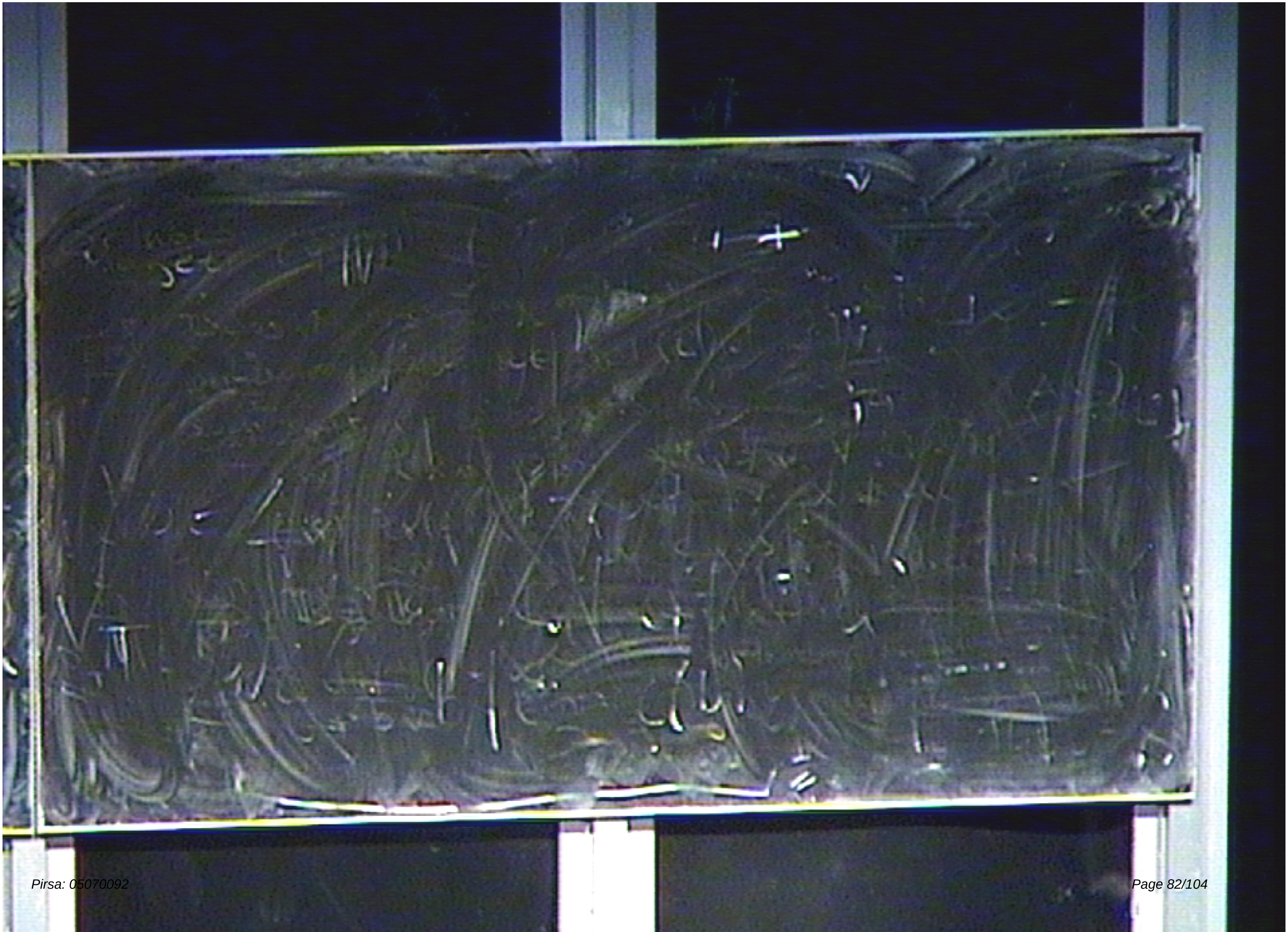


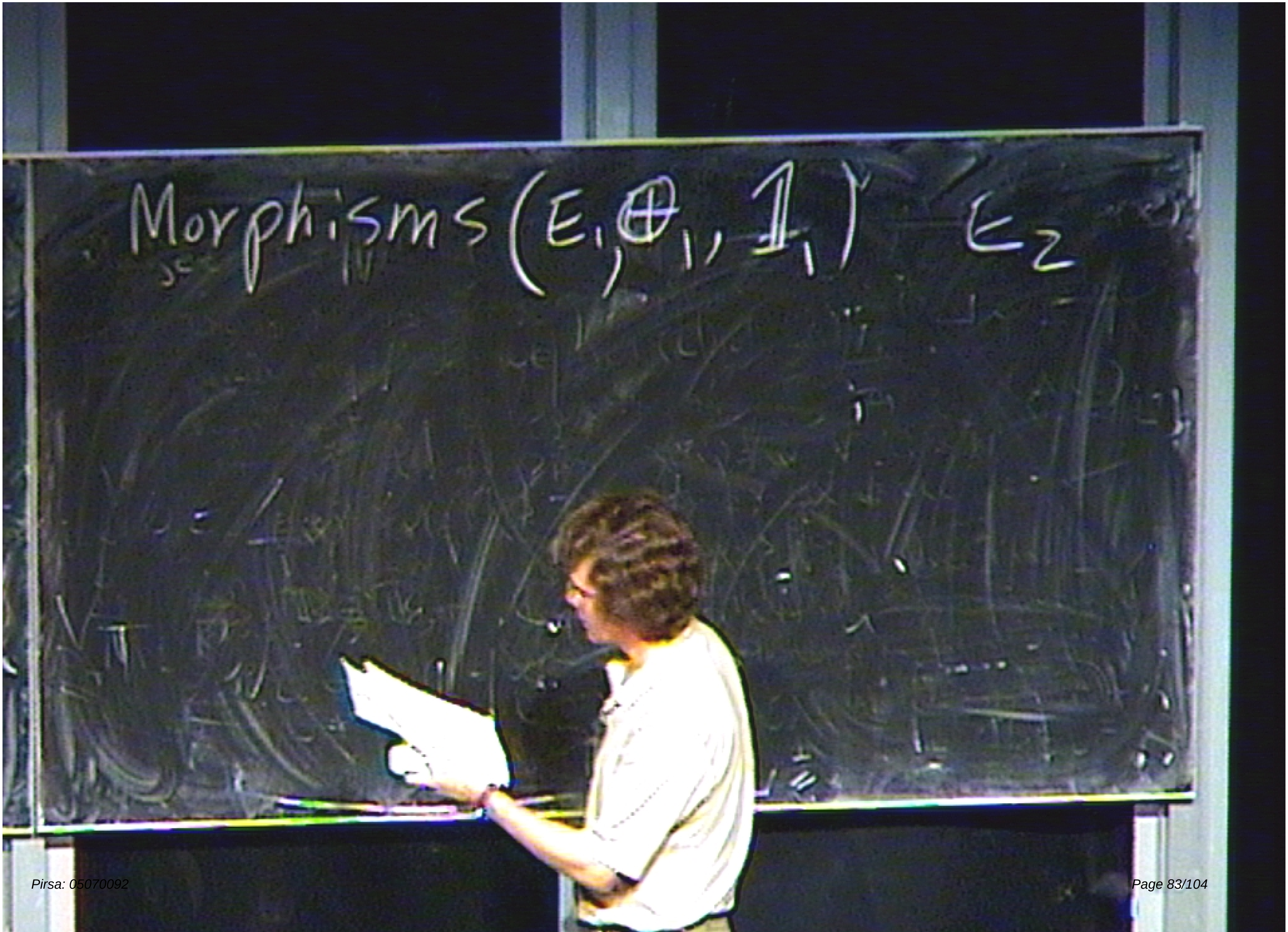











Morphisms (E1, ⊕1, 𝟙1)
E2

Morphisms $(E_1, \oplus_1, \mathbb{1}_1)$ E_2

$f : E_1 \to E_2$ is a morphism

Morphisms: Let $(E_1, \oplus_1, \mathbb{1}_1)$ & $(E_2, \oplus_2, \mathbb{1}_2)$ be EA's.

$f: E_1 \to E_2$ is a morphism :=

$f(\mathbb{1}_1) = \mathbb{1}_2$; $f(p) \oplus_2 f(q) = p \oplus_1 q$ when $p \oplus_1 q$ exists.

sometimes required only for "faithful" morphism.

monomorphism (sometimes also called "faithful"):

satisfies that $p \oplus_1 q$ exists whenever $f(p) \oplus_2 f(q)$ does.

iso := mono, onto

auto := iso from E to itself.

States: Morphisms to $[0,1]$ ← viewed as an EA, $x \oplus y = x + y$ except undefined if $x+y>1$.

i.e.: $\omega(\mathbb{1}) = 1$, $\omega(0) = 0$, $\omega(x \oplus y) = \omega(x) + \omega(y)$.

(c) Morphisms: Let $(E_1, \oplus_1, \mathbb{1}_1)$ & $(E_2, \oplus_2, \mathbb{1}_2)$ be EA's.

$f: E_1 \to E_2$ is a morphism :=

$\underbrace{f(\mathbb{1}_1) = \mathbb{1}_2}$; $f(p) \oplus_2 f(q) = p \oplus_1 q$ when $p \oplus_1 q$ ~~is de~~ exists.

sometimes required only for "faithful" morphism.

monomorphism (sometimes also called "faithful"):

satisfies that $p \oplus_1 q$ exists whenever $f(p) \oplus_2 f(q)$ does.

iso := mono, onto

auto := iso from E to itself.

States: Morphisms to $[0,1]$ ← viewed as an EA,

w/ $x \oplus y = x + y$ except undefined if $x+y > 1$.

i.e.: $\omega(\mathbb{1}) = 1$, $\omega(0) = 0$, $\omega(x \oplus y) = \omega(x) + \omega(y)$.

Morphisms: Let $(E_1, \oplus_1, \mathbb{1}_1)$ & $(E_2, \oplus_2, \mathbb{1}_2)$ be $\mathcal{E}a$'s.

$f: E_1 \to E_2$ is a morphism :=

$f(\mathbb{1}_1) = \mathbb{1}_2$; $f(p) \oplus_2 f(q) = f(p \oplus_1 q)$ when $p \oplus_1 q$ exists.

sometimes required only for "faithful" morphism.

monomorphism (sometimes also called "faithful"):
satisfies that $p \oplus_1 q$ exists whenever $f(p) \oplus_2 f(q)$ does.

iso := mono, onto

auto := iso from E to itself.

States: Morphisms to $[0,1]$ ← viewed as an $\mathcal{E}a$,
w/ $x \oplus y = x + y$ except undefined if $x+y > 1$.

i.e.: $\omega(\mathbb{1}) = 1$, $\omega(0) = 0$, $\omega(x \oplus y) = \omega(x) + \omega(y)$.

Sets of states on EA's

$\Omega(E)$:= the set of all states on E.

Defs. (Properties of state-sets)

SOD ⇒ OD ⇒ SEP ⇒ POS

- Δ Separating for E :=

  $\forall x, y \in E,\ x \neq y \Rightarrow \exists \omega \in \Delta\ \ \omega(x) \neq \omega(y)$.

- Δ order-determining :=

  $\left(\forall \mu \in \Delta,\ \mu(p) \leq \mu(q)\right) \Rightarrow p \leq q$.

- Δ positive :=

  $x \neq 0 \Rightarrow \exists \omega \in \Delta\ \ \omega(p) > 0$.

- Δ strongly order-determining :=

  $a \leq b \iff \forall \omega \in \Delta,\ \ \omega(a) = 1 \Rightarrow \omega(b) = 1$

  (Properly stronger than O.D.)

Sets of states on EA's

$\Omega(E) :=$ the set of all states on E.

Defs. (Properties of state-sets)   SOD $\Rightarrow$ OD $\Rightarrow$ SEP $\Rightarrow$ POS

- Δ Separating for E :=

  $$\forall x, y \in E,\ x \neq y \Rightarrow \exists \omega \in \Delta\ \ \omega(x) \neq \omega(y).$$

- Δ order-determining :=

  $$\left(\forall \mu \in \Delta,\ \mu(p) \leq \mu(q)\right) \Rightarrow p \leq q.$$

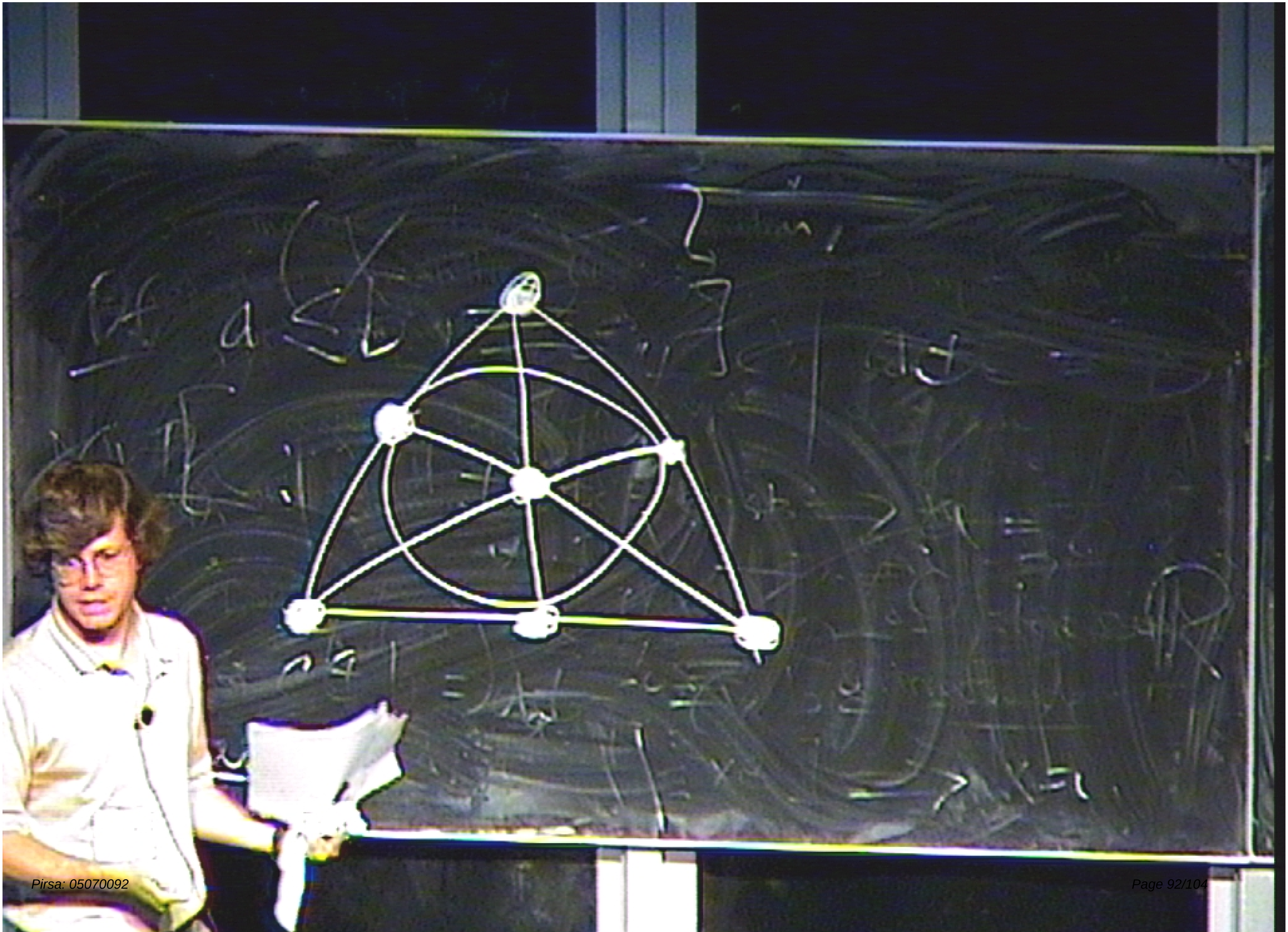
- Δ positive :=

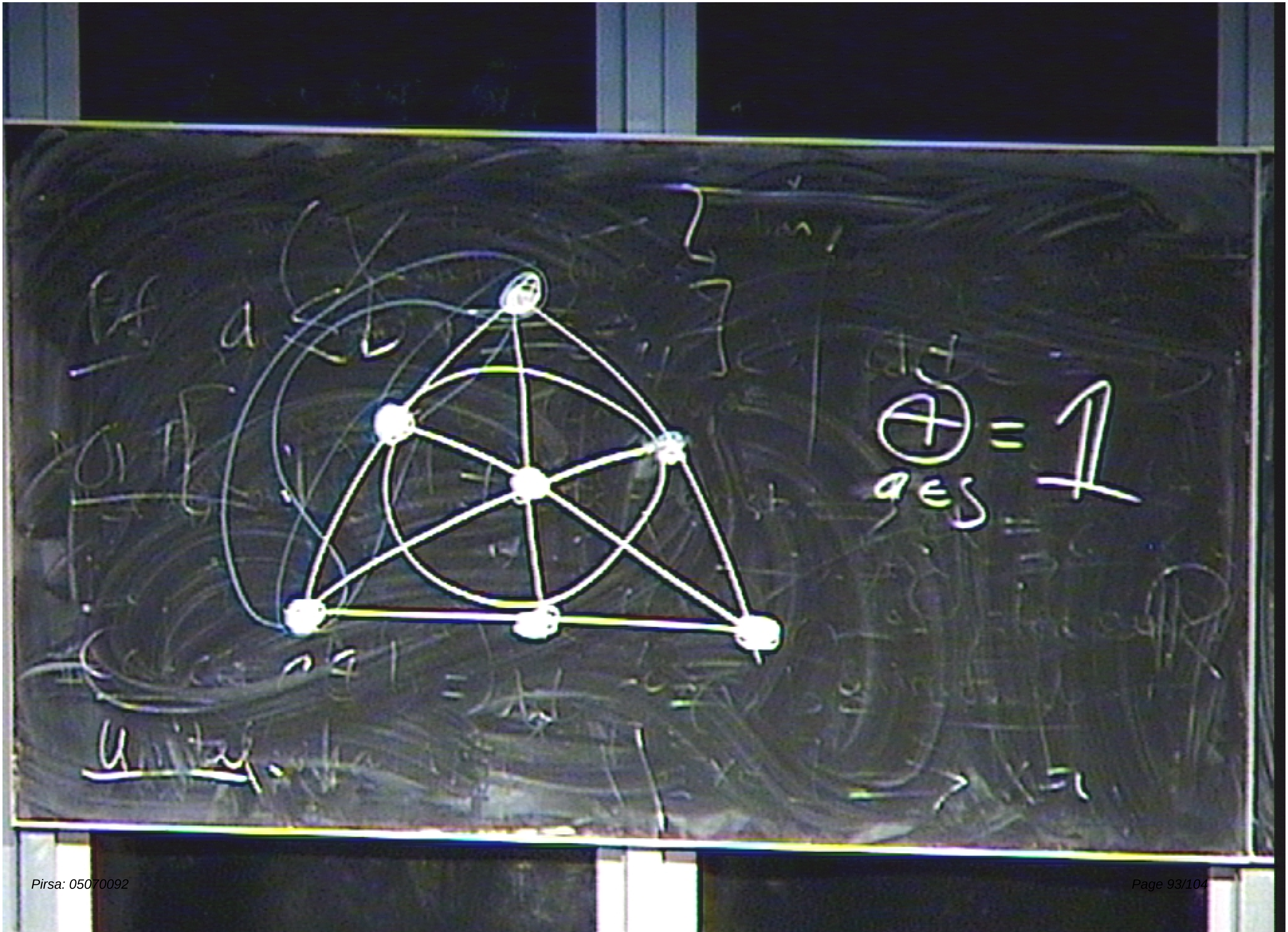
  $$x \neq 0 \Rightarrow \exists \omega \in \Delta\ \ \omega(p) > 0.$$

- Δ strongly order-determining :=

  $$a \leq b \Longleftrightarrow \forall \omega \in \Delta,\ \ \omega(a) = 1 \Rightarrow \omega(b) = 1$$

  (Properly stronger than O.D.)





Sharpness

Definitions

p is principal $:= (q, r \leq p, \exists q \oplus r) \Rightarrow q \oplus r \leq p$.

p is characteristic $:=$ $(q \leq p, p') \Longrightarrow q = 0$.
(i.e. $a \wedge a' = 0$).

Proposition: principal $\Longrightarrow$ characteristic.

PROOF: Suppose p non-characteristic, i.e $\exists q /$
$q \leq p, p'$ but $q \neq 0$.

$p \leq q'$, $p' \leq q'$ by (iii) "contraposition"
$p \oplus p'$ exists (it's $\mathbb{1}$). "Principal" would require
$\mathbb{1} \equiv p \oplus p' \leq q'$, but that implies $q = 0$.

Fact: In Hilbert-space POVM effect algebra,
principal $\equiv$ char. $\equiv$ projector.

Sharpness

Definitions

p is principal $:= (q, r \leq p,\ \exists q \oplus r) \Rightarrow q \oplus r \leq p$.

p is characteristic $:= (q \leq p, p') \Rightarrow q = 0$.

(i.e. $a \wedge a' = 0$).

Proposition: principal $\Rightarrow$ characteristic.

PROOF: Suppose p non-characteristic, i.e $\exists q /$

$q \leq p, p'$ but $q \neq 0$.

$p \leq q'$, $p' \leq q'$ by (iii) "contraposition"

$p \oplus p'$ exists (it's $\mathbb{1}$). "Principal" would require

$\mathbb{1} \equiv p \oplus p' \leq q'$, but that implies $q = 0$.

Fact: In Hilbert-space POVM effect algebra,

principal $\equiv$ char. $\equiv$ projector.

Sharpness

Definitions

p is principal $:= \left(q, r \leq p,\ \exists\, q \oplus r\right) \Rightarrow q \oplus r \leq p.$

p is characteristic $:=$ [redacted], $(q \leq p, p') \Longrightarrow q = 0.$

(i.e. $a \wedge a' = 0$).

Proposition: principal $\Longrightarrow$ characteristic.

PROOF: Suppose p non-characteristic, i.e $\exists\, q /$
$q \leq p, p'$ but $q \neq 0$.

$p \leq q'$, $p' \leq q'$ by (iii) "contraposition"
$p \oplus p'$ exists (it's $\mathbb{1}$). "Principal" would require
$\mathbb{1} \equiv p \oplus p' \leq q'$, but that implies $q = 0$.

Fact: In Hilbert-space POVM effect algebra,
principal $\equiv$ char. $\equiv$ projector.

Sharpness

Definitions

p is principal $:= \left(q, r \leq p, \exists q \oplus r\right) \Rightarrow q \oplus r \leq p$.

p is characteristic $:=$ [illegible], $(q \leq p, p') \Longrightarrow q = 0$.

(i.e. $a \wedge a' = 0$).

<u>Proposition</u>: principal $\Longrightarrow$ characteristic.

<u>PROOF</u>: Suppose p non-characteristic, i.e $\exists q /$

$q \leq p, p'$ but $q \neq 0$.

$p \leq q'$, $p' \leq q'$ by (iii) "contraposition"

$p \oplus p'$ exists (it's $\mathbb{1}$). "Principal" would require

$\mathbb{1} \equiv p \oplus p' \leq q'$, but that implies $q = 0$.

<u>Fact</u>: In Hilbert-space POVM effect algebra,

principal $\equiv$ char. $\equiv$ projector.

Sharpness

Definitions

p is principal $:= \left(q, r \leq p,\ \exists q \oplus r\right) \Rightarrow q \oplus r \leq p$.

p is characteristic $:=$ [redacted], $(q \leq p, p') \Rightarrow q = 0$.
(i.e. $a \wedge a' = 0$).

Proposition: principal $\Rightarrow$ characteristic.

PROOF: Suppose p non-characteristic, i.e $\exists q /$
$q \leq p, p'$ but $q \neq 0$.

$p \leq q'$, $p' \leq q'$ by (iii) "contraposition"
$p \oplus p'$ exists (it's $\mathbb{1}$). "Principal" would require
$\mathbb{1} \equiv p \oplus p' \leq q'$, but that implies $q = 0$.

Fact: In Hilbert-space POVM effect algebra,
principal $\equiv$ char. $\equiv$ projector.

Sharpness

Definitions

p is principal $:= \left(q, r \leq p,\ \exists\, q \oplus r\right) \Rightarrow q \oplus r \leq p.$

p is characteristic $:= \left(q \leq p, p'\right) \Longrightarrow q = 0.$
(i.e. $a \wedge a' = 0$).

Proposition: principal $\Longrightarrow$ characteristic.

PROOF: Suppose p non-characteristic, i.e $\exists\, q /$
$q \leq p, p'$ but $q \neq 0$.

$p \leq q'$, $p' \leq q'$ by (iii) "contraposition"
$p \oplus p'$ exists (it's $\mathbb{1}$). "Principal" would require
$\mathbb{1} \equiv p \oplus p' \leq q'$, but that implies $q = 0$.

Fact: In Hilbert-space POVM effect algebra,
(R5) principal $\equiv$ char. $\equiv$ projector.

Sharpness

Definitions

p is principal $:= \left(q, r \leq p, \ \exists q \oplus r \right) \Rightarrow q \oplus r \leq p.$

p is characteristic $:= \ , \left(q \leq p, p' \right) \Longrightarrow q = 0.$

(i.e. $a \wedge a' = 0$).

Proposition: principal $\Longrightarrow$ characteristic.

PROOF: Suppose p non-characteristic, i.e $\exists q /$

$q \leq p, p'$ but $q \neq 0$.

$p \leq q'$, $p' \leq q'$ by (iii) "contraposition"

$p \oplus p'$ exists (it's $\mathbb{1}$). "Principal" would require

$\mathbb{1} \equiv p \oplus p' \leq q'$, but that implies $q = 0$.

Fact: In Hilbert-space POVM effect algebra,

(15) principal $\equiv$ char. $\equiv$ projector.

Sharpness

Definitions

p is principal $:= \left(q, r \leq p, \exists q \oplus r\right) \Rightarrow q \oplus r \leq p$.

p is characteristic $:=$ [redacted], $(q \leq p, p') \Longrightarrow q = 0$.

(i.e. $a \wedge a' = 0$).

Proposition: principal $\Longrightarrow$ characteristic.

PROOF: Suppose p non-characteristic, i.e $\exists q /$ $q \leq p, p'$ but $q \neq 0$.

$p \leq q'$, $p' \leq q'$ by (iii) "contraposition"

$p \oplus p'$ exists (it's $\mathbb{1}$). "Principal" would require $\mathbb{1} \equiv p \oplus p' \leq q'$, but that implies $q = 0$.

Fact: In Hilbert-space POVM effect algebra, principal $\equiv$ char. $\equiv$ projector.

Sharpness

Definitions

p is principal $:= \left(q, r \leq p,\ \exists\, q \oplus r\right) \Rightarrow q \oplus r \leq p.$

p is characteristic $:= \left(q \leq p, p'\right) \Rightarrow q = 0.$
(i.e. $a \wedge a' = 0$).

Proposition: principal $\Rightarrow$ characteristic.

PROOF: Suppose p non-characteristic, i.e $\exists\, q /$
$q \leq p, p'$ but $q \neq 0$.

$p \leq q'$, $p' \leq q'$ by (iii) "contraposition"
$p \oplus p'$ exists (it's $\mathbb{1}$). "Principal" would require
$\mathbb{1} \equiv p \oplus p' \leq q'$, but that implies $q = 0$.

Fact: In Hilbert-space POVM effect algebra,
principal $\equiv$ char. $\equiv$ projector.