

Title: Introduction to and Historical Overview of Quantum Logics

Date: Jul 17, 2005 02:00 PM

URL: <http://pirsa.org/05070090>

Abstract: Introductory lecture summary:

1. Finite dimensional hilbert spaces and (complemented) modular lattices; infinite-dimensional hilbert spaces and orthomodularity.
2. von Neumann's QL; von Neumann-Birkhoff (briefly!); reconstruction of QM from $P(H)$
3. Mackey's programme; some early axiomatics (e.g., Zierler); QLs as OMPs + order-determining sets of states
4. Piron's Theorem; some discussion of Piron's axioms
5. Keller's examples (maybe just a mention, though I'd like to indicate how they come up); Soler's theorem (just the statement)
6. Abstract OMLs; Greechie Diagrams and the Loop Lemma; brief mention of Harding's results on decompositions
7. Tensor products (the F-R example, showing OMLs not stable under tensor products); orthoalgebras.
8. Orthoalgebras from test spaces

II

1 Complements, distributivity
Boolean algebra

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2.

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2. Modular lattices $\leadsto$ projective geometry
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4. Piron's rep. Thm.

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Al

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Def: Let $(L, \leq)$ be
a lattice. A complement

for $a \in L$ is an element

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d.

||x||₀ :

) be
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* In $\mathcal{P}(X)$, each set $a \subseteq X$
has a unique complement - a^c

= 0

= 1

$\mathcal{P}(X)$:

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* In $L(X)$, an element and its complement

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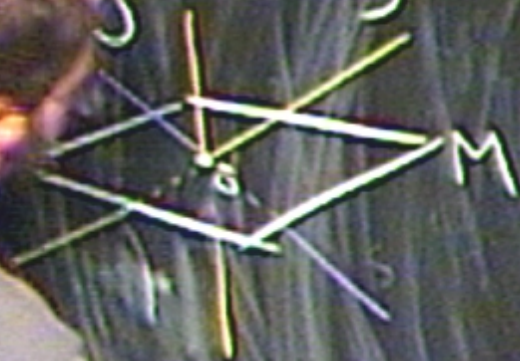
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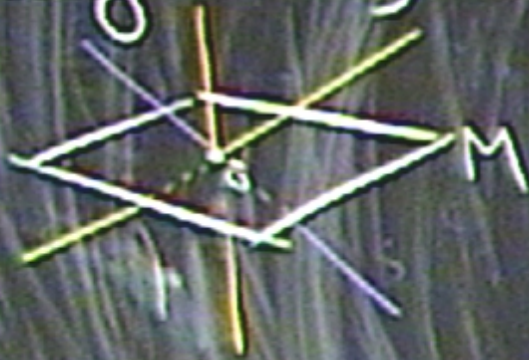
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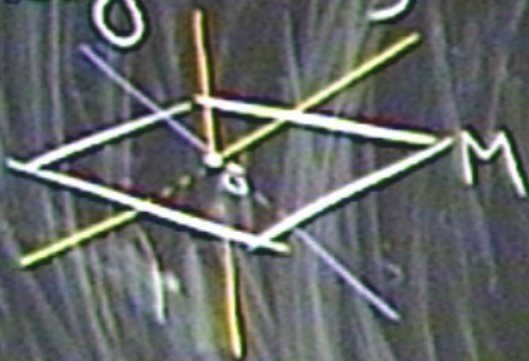
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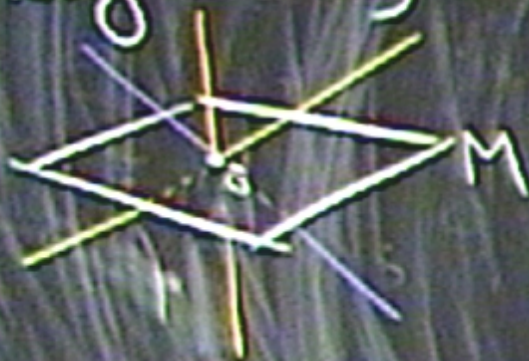
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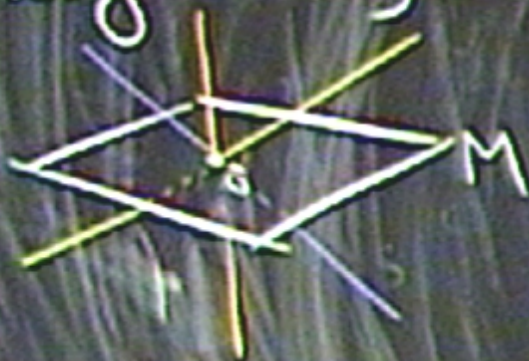


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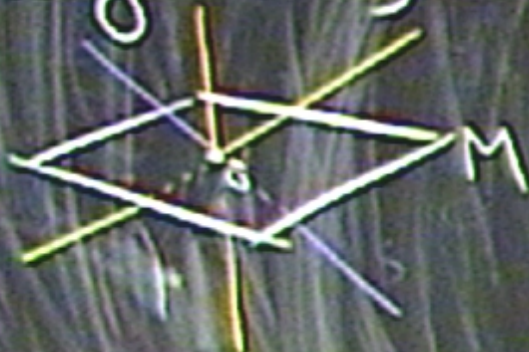


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Def: A lattice L is distributive

=

Def 2: A lattice L is distributive

$\Leftrightarrow \forall a, b, c$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

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Ex //

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$$1 \leq b \vee c \leq 1 \Rightarrow c \leq b.$$

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All lattices bounded.

Def: A distributive lattice in $\mathcal{L}$

ective
metry
modularity

$\sum x$

All lattices bounded.

Def: A distributive lattice in which every element has a comp. to a Boolean lattice, or Boolean.

ativity

projective
geometry

homomodularity

All lattices bounded.

Def: A distributive
lattice in which every
element has a
to a Boskian
or Boolean

$\mathbb{Z}_2$:

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All lattices bounded.

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Def: A distributive
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Thm (Birkhoff): Every BA is $\cong$ a sub-BA of $\mathcal{P}(X)$

Thm (Pumping Lemma): Every BA is not a sub-BA of $\mathcal{P}(\Sigma)$

Thm () Let $L = bc$

Thm (Birkhoff): Every BA is isomorphic a sub-BA of $\mathcal{P}(X)$

Thm () Let L be a complete, atomic

Thm (Rice's Thm): Every BA is $\cong$ a sub-BA of $\mathcal{P}(\Sigma)$

Thm () Let L be a complete, automatic
BA. Then $L \cong \mathcal{P}(\Sigma)$

(L is complete $\Leftrightarrow \forall A \in L$

Thm (Tarski's): Every BA is $\cong$ a sub-BA of $\mathcal{P}(X)$

Thm () Let L be a complete, atomic
BA Then $L \cong \mathcal{P}(X)$

(L is complete $\Leftrightarrow \forall A \subseteq L, \bigwedge A, \bigvee A \in L$)

Thm (Birkhoff): Every BA is $\cong$ a sub-BA of $\mathcal{P}(X)$

Thm () Let L be a complete, atomistic BA. Then $L \cong \mathcal{P}(X)$

(L is complete $\Leftrightarrow \forall A \subseteq L, \bigwedge A, \bigvee A \in L$)

(L is atomistic $\Leftrightarrow$ every $a \in L$ is a join of atoms)

— an atom being an element covering 0

Q. Is there a nice characterization
of lattices of the form

L

All

Def

lattice
element

to

or

Q. Is there a nice characterization
of lattices of the form
 $L(M)$

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Q. Is there a nice characterization
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A. Review of earlier Lemma,
NO the following

All

Def

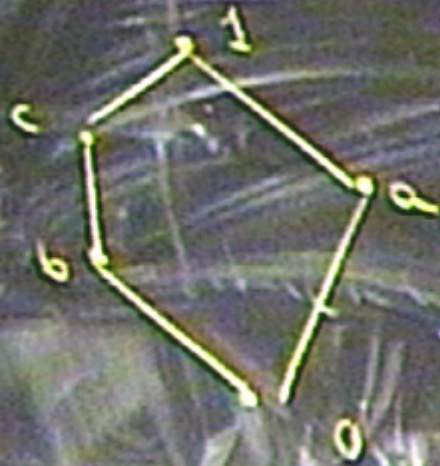
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N. In view of earlier Lemma,
None of the following are distributive.



All

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A: In view of earlier Lemma,
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we have a nice characterization
 of the form

lattice
Def: L is
modular $\Leftrightarrow$

$$c \leq a \Rightarrow$$

$$a \wedge (b \vee c) = (a \wedge b) \vee$$

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$$L(\mathcal{H})$$

characterization
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Ex: Let V be any vector

$$\text{Let } L_0(V) = \{ M \vdash M \}$$

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lattice

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$$\text{Let } L_0(V) = \{ M \mid M \leq V \}$$

A lattice w/.

$$M \wedge N = M \cap N$$

$$M \vee N = \text{span}$$

$$= M + N$$

Lemma: $L_0(V)$ is modular

That is: If $M, N, Q \in L_0(V)$

$Q \subseteq M$, then

$$M \cap (N + Q) =$$

any vector space:

$$L_0(V) = \{M \subseteq V\}$$

$$M \wedge N = M \cap N$$

$$M \vee N = \text{span}(M \cup N) \\ = \underline{M + N}$$

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$$L_0(V) = \{M \mid M \subseteq V\}$$

$$M \wedge N = M \cap N$$

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Let $x \in M \cap (N + Q)$:

$$x = n + q \quad (n \in N, q \in Q)$$

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any vector space

$$L_0(V) = \{M \subseteq V\}$$

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$$x \in M \cap (N + Q):$$

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$\Rightarrow$

Lemma: $L_0(V)$ is modular.

That is: If $M, N, Q \in L_0(V)$

$Q \subseteq M$, then

$$M \cap (N + Q) \subseteq (M \cap N) + Q$$

Let $x \in M \cap (N + Q)$:

$$x = n + q \in M \quad (n \in N, q \in Q)$$
$$\Rightarrow x - q = n \in M$$

$$\therefore x \in (M \cap N) + Q \quad \checkmark$$

any vector space:

$$L_0(V) = \{M \mid M \subseteq V\}$$

$$M \wedge N = M \cap N$$

$$M \vee N = \text{span}(M \cup N)$$
$$= \underline{M + N}$$

Thm (Veblen, Bder, ...)

Let L be a complete, atomistic
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Thm (Veblen, Bader, ...)

Let L be a complete, atomistic
modular lattice of finite height

Thm (Veblen, Bader, ...)

Let L be a complete, atomistic
modular lattice of finite height
satisfying Atomic Covering Law (AC)

is an atom

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modular lattice of finite height
satisfying Atomic Covering Law (AC).

Is x is an atom, $x \neq a$
Then $x \vee a$ covers a .

Lemma: $L_0(V)$ is modular.

That is: If $M, N, Q \in L_0(V)$

$Q \subseteq M$, then

$$M \cap (N + Q) \subseteq (M \cap N) + Q$$

Let $x \in M \cap (N + Q)$:

$$x = n + q \in M \quad (n \in N, q \in Q)$$

$$\Rightarrow x - q = n \in M$$

$$\text{so } x \in (M \cap N) + Q \quad \checkmark$$



vector

$M \cap N$
 $\text{par}(M \cup N)$
 $M + N$



$$xva = \text{span}(a \cup x) \\ = a + x$$

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$$M \wedge N = M \cap N$$

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$$x \vee a = \text{span}(a \cup x)$$

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On the complementations

Orthocomplementations

Def . An orthocomplementation on L is a map $L \rightarrow L$

$$* \quad a \leq b \Rightarrow b' \leq a'$$

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An ortholattice is a lattice w/ a specified orthocomplementation.

Orthocomplementations

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- * $a'' = a$

Ex On $L(\mathcal{K})$: $M \mapsto M^\perp$

An ortholattice is a lattice w/ a specified orthocomplementation

Def: An ortholattice $(L, \cdot, \leq)$ is orthomodular $\Leftrightarrow$

$$a \leq b \Rightarrow b = (b \wedge a') \vee a$$

Def: L is modular $\Leftrightarrow$

$$c \leq a \Rightarrow$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

Ex: Let $\vee$ be
Let $L_0(\vee)$

A lattice w/.

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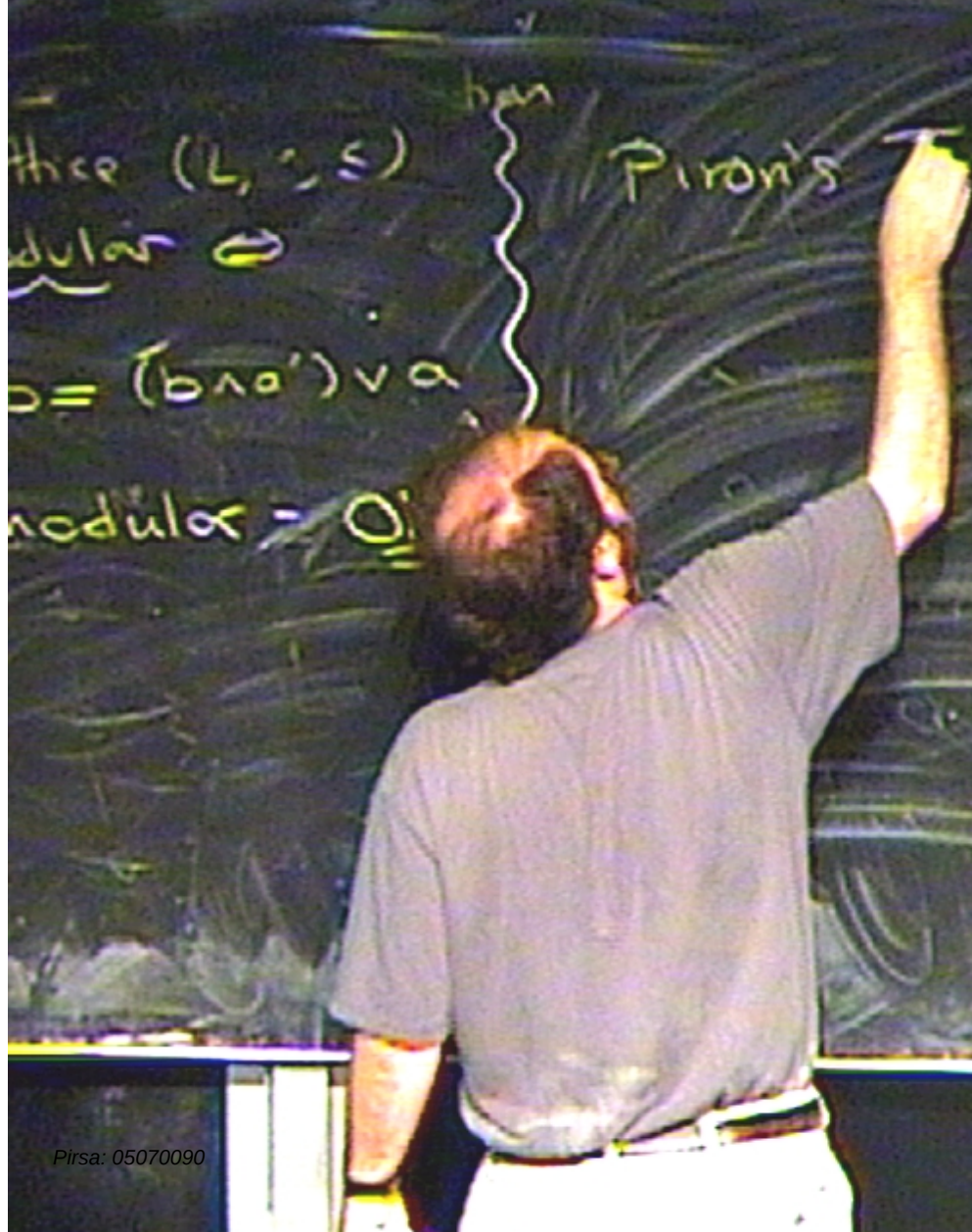
$\text{OL}(\mathcal{K}) \models \text{orthomodular} = \underline{\text{OML}}$

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Ex: Let $\mathcal{V}$ be
Let $\mathcal{L}_0(\mathcal{V})$
A lattice w/.



thrice (L, S) has

dular $\Leftrightarrow$

$o = (bno')va$

modular = 0

Piron's

$xva = \text{span}(a \cup x)$

$a \subset x$

any vector space

$M \subseteq M \subseteq V$

$M \cap N = M \cup N$

$M \vee N = \text{span}(M \cup N)$

$M + N$

$\text{thice } (L, \cdot, \leq)$
 $\text{dular} \Leftrightarrow$
 $\alpha = (b \wedge a') \vee a$
 $\text{modular} = \underline{\underline{OML}}$

Piron's Thm

Let L be a
 complete
 orthomodular
 modular

height



$$\begin{aligned}
 \chi \vee a &= \text{span}(a \cup \chi) \\
 &= a \vee \chi
 \end{aligned}$$

any vector space:
 $\mathcal{L}(V) = \{M \mid M \leq V\}$

$$M \wedge N = M \cap N$$

$$\begin{aligned}
 M \vee N &= \text{span}(M \cup N) \\
 &= \underline{\underline{M + N}}
 \end{aligned}$$

lattice $(L, \leq)$
 modular $\Leftrightarrow$
 $(b \wedge a') \vee a = (b \vee a') \wedge a$
 $\alpha = \underline{OML}$

Piron's Thm

Let L be a
 complete
 atomistic
 orthomodular
 AC
 of height ≥ 4



$$x \vee a = \text{span}(a \cup x) = a + x$$

any vector space
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then $(L, \cdot, \varepsilon)$ has
 modular $\Leftrightarrow$
 $\phi = (bna') \vee a$
 modular - OML

Piron's Thm

Let L be a
 complete
 atomistic
 orthomodular
 AC
 of height ≥ 4
 & irreducible

Then $\exists$

$\ast$ -ID - div. ring

Lemma: $L_0(V)$ is modular

That is: If $M, N, Q \in L_0(V)$

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$\star$ $\mathbb{D}$ - div. ring

$\mathcal{H}$ - a $\mathbb{D}$ -vector space $Q \in \mathcal{H}$, then

$\langle, \rangle$

such that

$$L \cong L(\mathcal{H})$$

$$N = \{M \leq \pi \mid M = M^{\perp\perp}\}$$

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Amemiya-Araki (67)

IS $\mathbb{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}$

Then $\mathcal{H}$ is complete.

Then $\exists$

$\ast\text{-}\mathbb{D}$ - div. ring

$\mathcal{M}$ - a $\mathbb{D}$ -vector space

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IS $\mathbb{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}$

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Solèr (95)

IS $\mathcal{H}$ has an
 ∞ orthonormal basis,

then $\mathbb{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}$

Then $\exists$

$\star$ - $\mathbb{D}$ - div. ring

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$\langle, \rangle$

such that

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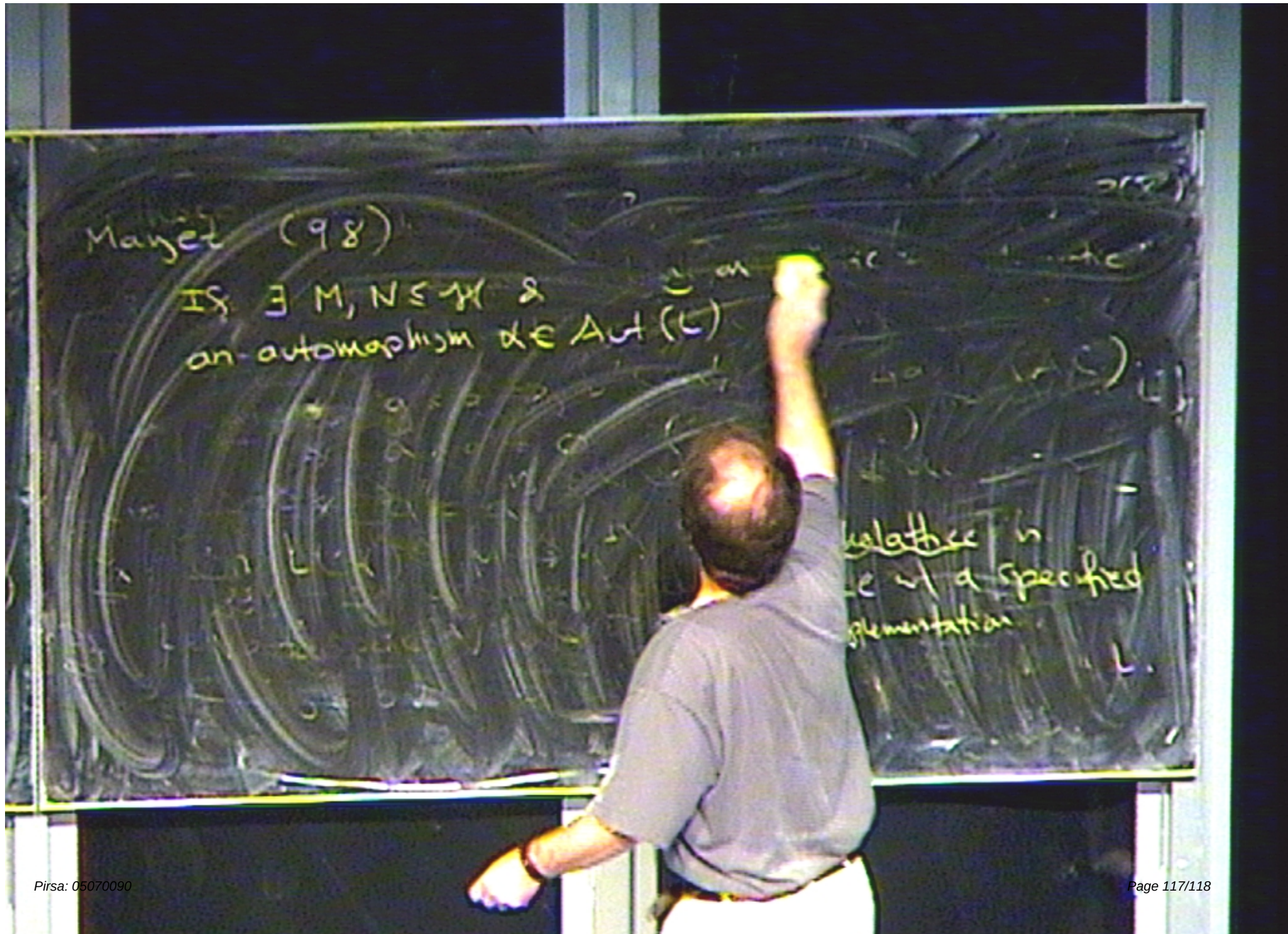
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Mayet (98)

IS $\exists M, N \in \mathcal{M}$ & $M \leq N$
an automorphism of $\text{Aut}(L)$

lattice in
of a specified
presentation

Mañé (98)

IS $\exists M, N \in \mathcal{H}$ &
an automorphism $\alpha \in \text{Aut}(L)$
such that

(i) $\dim N \geq 2$, $\alpha(Q) = Q \ \forall \ Q \leq N$

(ii) $\alpha(M) \neq M$

$\Rightarrow \mathbb{D} \cong \mathbb{R}, \mathbb{H}, \mathbb{C}$

& $\dim(\mathcal{H}) = \infty$.