Title: Ashoke Sen (Harish Chandra Research Institute)

Date: Jul 07, 2005 04:30 AM

URL: http://pirsa.org/05070080

Abstract:

Black Holes, attractors, elementary strings etc.

Plan:

Lecture 1: Develop some general techniques for studying entropy of extremal black holes in gravity coupled to matter with higher derivative terms.

Lecture 2. Apply this method to the study of the correspondence between elementary strings and black holes in string theory.

Reference for lecture 1:

A.S. , hep-th/0506177 (also hep-th/0505122)

Related (complementary) work:

Kraus and Larsen, hep-th/0506176

Black Holes, attractors, elementary strings etc.

Plan:

Lecture 1: Develop some general techniques for studying entropy of extremal black holes in gravity coupled to matter with higher derivative terms.

Lecture 2. Apply this method to the study of the correspondence between elementary strings and black holes in string theory.

Reference for lecture 1:

A.S. , hep-th/0506177 (also hep-th/0505122)

Related (complementary) work:

Kraus and Larsen, hep-th/0506176

Motivation:

String theory contains gravity + matter fields.

Low energy dynamics → described by an effective action with two derivatives terms.

e.g. $\int d^D x \sqrt{-\det g} R$, $\int d^D x \sqrt{-\det g} \partial^\mu \phi \partial_\mu \phi$, $\int d^D x \sqrt{-\det g} F_{\mu\nu} F^{\mu\nu}$ etc.

The full effective action also contains terms with higher derivatives.

e.g. $\int d^D x \sqrt{-det g}\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$

→ studying black holes in string theory, → studying black holes in higher derivative theories of gravity + matter fields.

Typically black holes Hawking radiate → are unstable.

For a special class of black holes the Hawking temperature =0.

→ extremal black holes.

We shall focus on these black holes.

This leads us to the analysis of extremal black holes in higher derivative gravity in D dimensions

→ describes string theory with $(10-D)$ dimensions compactified

We shall restrict our analysis to spherically symmetric black holes.

In our analysis we shall not assume the theory to be supersymmetric.

– keeps the analysis as general as possible

– in general it is difficult to supersymmetrize higher derivative terms

As a result we cannot use BPS condition to define extremal black holes.

How do we recognize extremal black holes in a higher derivative theory?

Take the clue from usual (super-)gravity.

Begin with the $D = 4$ case.

Recall the geometry of Reissner-Nordstrom solution:

$$\begin{aligned} ds^2 &= -(1-\rho_+/\rho)(1-\rho_-/\rho)dt^2 \\ &\quad + \frac{d\rho^2}{(1-\rho_+/\rho)(1-\rho_-/\rho)} \\ &\quad + \rho^2(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned}$$

Extremal limit:

$$\rho_+ = \rho_-$$

Define

$$\tau = t/\rho_+^2, \qquad r = \rho - \rho_+,$$

Then

$$ds^2 \simeq \rho_+^2(-r^2 d\tau^2 + \frac{dr^2}{r^2}) + \rho_+^2(d\theta^2 + \sin^2\theta d\phi^2)$$

→ near horizon geometry $AdS_2 \times S^2$

All known extremal black holes in four dimensions with non-singular near horizon geometry have near horizon geometry $AdS_2 \times S^2$.

(These include some solutions in the presence of certain higher derivative terms.)

We shall take this as the definition of extremal black holes.

Take a general gauge and general coordinate invariant theory with metric $g_{\mu\nu}$, gauge fields $\{A_\mu^{(i)}\}$, neutral scalar fields $\{\phi_s\}$.

Definition: Extremal black holes in this theory have

1. near horizon geometry $AdS_2 \times S^2$

2. all other background fields respect the $SO(2,1)\times SO(3)$ symmetry of $AdS_2 \times S^2$.

General form of the near horizon geometry of an extremal black hole:

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = v_1\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + v_2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$$\phi_s = u_s$$

$$F^{(i)}_{rt} = e_i, \qquad F^{(i)}_{\theta\phi} = \frac{p_i}{4\pi}\sin\theta\,,$$

v_1, v_2: sizes of AdS_2 and S^2

u_s: scalar field values at the horizon.

$p_i/4\pi$: radial magnetic fields at the horizon

e_i: radial electric fields at the horizon

General form of the near horizon geometry of an extremal black hole:

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = v_1\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + v_2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$$\phi_s = u_s$$

$$F^{(i)}_{rt} = e_i, \qquad F^{(i)}_{\theta\phi} = \frac{p_i}{4\pi}\sin\theta\,,$$

v_1, v_2: sizes of AdS_2 and S^2

u_s: scalar field values at the horizon.

$p_i/4\pi$: radial magnetic fields at the horizon

e_i: radial electric fields at the horizon

General form of the near horizon geometry of an extremal black hole:

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right)$$

$$\phi_s = u_s$$

$$F^{(i)}_{rt} = e_i, \qquad F^{(i)}_{\theta\phi} = \frac{p_i}{4\pi} \sin\theta \,,$$

v_1, v_2: sizes of AdS_2 and S^2

u_s: scalar field values at the horizon.

$p_i/4\pi$: radial magnetic fields at the horizon

e_i: radial electric fields at the horizon

$$ds^2 = v_1\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + v_2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$$\phi_s = u_s$$

$$F^{(i)}_{rt} = e_i, \qquad F^{(i)}_{\theta\phi} = \frac{p_i}{4\pi}\sin\theta\,,$$

For this background:

$$R_{\alpha\beta\gamma\delta} = -v_1^{-1}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}), \qquad \alpha,\beta,\gamma,\delta = r,t$$

$$R_{mnpq} = v_2^{-1}(g_{mp}g_{nq} - g_{mq}g_{np}), \qquad m,n,p,q = \theta,\phi$$

$F^{(i)}_{\alpha\beta} \propto$ volume form on AdS_2

$F^{(i)}_{\theta\phi} \propto$ volume form on S^2

$\phi_s =$ constant

$$\rightarrow \quad D_\mu R_{\nu\rho\sigma\tau} = 0, \quad D_\mu F^{(i)}_{\nu\rho} = 0, \quad D_\mu \phi_s = 0$$

Only terms in $\mathcal{L}$ which do not involve explicit covariant derivatives affect the background.

$$ds^2 = v_1\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + v_2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$$\phi_s = u_s$$

$$F^{(i)}_{rt} = e_i, \qquad F^{(i)}_{\theta\phi} = \frac{p_i}{4\pi}\sin\theta\,,$$

For this background:

$$R_{\alpha\beta\gamma\delta} = -v_1^{-1}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}), \qquad \alpha,\beta,\gamma,\delta = r,t$$

$$R_{mnpq} = v_2^{-1}(g_{mp}g_{nq} - g_{mq}g_{np}), \qquad m,n,p,q = \theta,\phi$$

$F^{(i)}_{\alpha\beta}$ ∝ volume form on AdS_2

$F^{(i)}_{\theta\phi}$ ∝ volume form on S^2

ϕ_s = constant

→ $D_\mu R_{\nu\rho\sigma\tau} = 0, \quad D_\mu F^{(i)}_{\nu\rho} = 0, \quad D_\mu \phi_s = 0$

Only terms in $\mathcal{L}$ which do not involve explicit covariant derivatives affect the background.

$$ds^2 = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2}\right) + v_2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$$\phi_s = u_s$$

$$F^{(i)}_{rt} = e_i, \qquad F^{(i)}_{\theta\phi} = \frac{p_i}{4\pi} \sin\theta\,,$$

For this background:

$$R_{\alpha\beta\gamma\delta} = -v_1^{-1}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}), \qquad \alpha,\beta,\gamma,\delta = r,t$$

$$R_{mnpq} = v_2^{-1}(g_{mp}g_{nq} - g_{mq}g_{np}), \qquad m,n,p,q = \theta,\phi$$

$F^{(i)}_{\alpha\beta} \propto$ volume form on AdS_2

$F^{(i)}_{\theta\phi} \propto$ volume form on S^2

$\phi_s =$ constant

$$\rightarrow \quad D_\mu R_{\nu\rho\sigma\tau} = 0, \quad D_\mu F^{(i)}_{\nu\rho} = 0, \quad D_\mu \phi_s = 0$$

Only terms in $\mathcal{L}$ which do not involve explicit covariant derivatives affect the background.

$$ds^2 = v_1\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + v_2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$$\phi_s = u_s$$

$$F^{(i)}_{rt} = e_i, \qquad F^{(i)}_{\theta\phi} = \frac{p_i}{4\pi}\sin\theta\,,$$

Let $\sqrt{-\det g}\,\mathcal{L}$ be the Lagrangian density.

Define:

$$f(\vec{u},\vec{v},\vec{e},\vec{p}) \equiv \int d\theta\, d\phi\, \sqrt{-\det g}\,\mathcal{L}$$

$$q_i \equiv \frac{\partial f}{\partial e_i}, \qquad F(\vec{u},\vec{v},\vec{q},\vec{p}) \equiv 2\pi(e_i\, q_i - f(\vec{u},\vec{v},\vec{e},\vec{p}))$$

Thus $F/2\pi$ is the Legendre transform of f with respect to the variables e_i.

Results:

For an extremal black hole of electric charge $\vec{q}$ and magnetic charge $\vec{p}$,

1. the values of $\{u_s\}$, v_1 and v_2 are obtained by extremizing $F(\vec{u},\vec{v},\vec{q},\vec{p})$ with respect to these variables.

$$\frac{\partial F}{\partial u_s} = 0, \qquad \frac{\partial F}{\partial v_1} = 0, \qquad \frac{\partial F}{\partial v_2} = 0$$

2. the near horizon electric field variables e_i are given by:

$$e_i = \frac{1}{2\pi} \frac{\partial F(\vec{u},\vec{v},\vec{q},\vec{p})}{\partial q_i}$$

3. the black hole entropy S_{BH} is given by

$$S_{BH} = F(\vec{u},\vec{v},\vec{q},\vec{p})$$

at the extremum of F with respect to $\vec{u}$, $\vec{v}$.

Significance

1. If F has no flat direction, then the complete near horizon solution is determined from these equations.

→ the near horizon solution and hence the entropy $S_{BH} = F$ is independent of the asymptotic values of the moduli fields.

2. If F has flat directions then some parameters are not determined from F and could depend on the asymptotic values of the moduli fields.

But F is flat along those directions

→ $S_{BH} = F$ still does not depend on these asymptotic moduli.

→ a generalized attractor mechanism.

Ferrara, Kallosh, Strominger; ...

$$f(\vec{u},\vec{v},\vec{e},\vec{p}) = \int d\theta\, d\phi \sqrt{-\det g}\; \mathcal{L}$$

$$q_i = \frac{\partial f}{\partial e_i}$$

$$F(\vec{u},\vec{v},\vec{q},\vec{p}) = 2\pi\left(q_i e_i - f(\vec{u},\vec{v},\vec{e},\vec{p})\right)$$

① $\frac{\partial F}{\partial u_A} = \frac{\partial F}{\partial v_1} = \frac{\partial F}{\partial v_2} = 0$ ② $\frac{\partial F}{\partial q_i} = 2\pi e_i$

③ $S_{BH} = F$

Significance

1. If F has no flat direction, then the complete near horizon solution is determined from these equations.

→ the near horizon solution and hence the entropy $S_{BH} = F$ is independent of the asymptotic values of the moduli fields.

2. If F has flat directions then some parameters are not determined from F and could depend on the asymptotic values of the moduli fields.

But F is flat along those directions

→ $S_{BH} = F$ still does not depend on these asymptotic moduli.

→ a generalized attractor mechanism.

Ferrara, Kallosh, Strominger; ...

In deriving these results we

– do not require the theory or the solution to be supersymmetric,

do not
– require any special structure of the higher derivative terms except general coordinate invariance and gauge invariance.

Single entropy function $F(\vec{u},\vec{v},\vec{q},\vec{p})$ determines:

– near horizon values $\{u_s\}$ of the scalar fields,

– sizes v_1, v_2 of AdS_2 and S^2

– gauge field strengths $\{e_i\}$

– entropy S_{BH}

Similar results hold in higher dimensions for near horizon geometry $AdS_2 \times S^{D-2}$.

In deriving these results we

– do not require the theory or the solution to be supersymmetric,

– do not require any special structure of the higher derivative terms except general coordinate invariance and gauge invariance.

Single entropy function $F(\vec{u}, \vec{v}, \vec{q}, \vec{p})$ determines:

– near horizon values $\{u_s\}$ of the scalar fields,

– sizes v_1, v_2 of AdS_2 and S^2

– gauge field strengths $\{e_i\}$

– entropy S_{BH}

Similar results hold in higher dimensions for near horizon geometry $AdS_2 \times S^{D-2}$.

Derivation of the results:

1. $f = \int d\theta d\phi \sqrt{-\det g}\, \mathcal{L}$

→ requiring the action to be stationary along the directions of v_1, v_2 and u_s deformations gives

$$\frac{\partial f}{\partial u_s} = 0, \qquad \frac{\partial f}{\partial v_1} = 0, \qquad \frac{\partial f}{\partial v_2} = 0.$$

The $SO(2,1) \times SO(3)$ invariance of the background

→ these are the only independent components of the metric and the scalar field equations.

2. Non-trivial part of the gauge field equations and Bianchi identities:

$$\partial_r \left(\frac{\delta S}{\delta F_{rt}^{(i)}} \right) = 0, \qquad \partial_r F_{\theta\phi}^{(i)} = 0 \ .$$

Evaluate the integration constants at $r \to \infty$

→ they are proportional to the electric charges q_i and the magnetic charges p_i.

When evaluated on the near horizon geometry, this gives

$$\frac{\partial f}{\partial e_i} = q_i$$

and

$$\int d\theta \, d\phi F_{\theta\phi}^{(i)} = p_i$$

↓

already anticipated in the choice of parametrization of the background fields ($F_{\theta\phi}^{(i)} = p_i \sin\theta / 4\pi$).

Thus for given $\vec{q}$ and $\vec{p}$ the equations determining the background are:

$$\frac{\partial f}{\partial u_s} = 0, \quad \frac{\partial f}{\partial v_1} = 0, \quad \frac{\partial f}{\partial v_2} = 0, \quad \frac{\partial f}{\partial e_i} = q_i$$

f is a function of $\vec{u}$, $\vec{v}$, $\vec{e}$ and $\vec{p}$.

Now recall the definition of F:

$$F(\vec{u},\vec{v},\vec{q},\vec{p}) = 2\pi\,(\vec{q}\cdot\vec{e} - f(\vec{u},\vec{v},\vec{e},\vec{p}))$$

Properties of the Legendre transform → these equations are equivalent to:

$$\frac{\partial F}{\partial u_s} = 0, \quad \frac{\partial F}{\partial v_1} = 0, \quad \frac{\partial F}{\partial v_2} = 0, \quad \frac{\partial F}{\partial q_i} = 2\pi\, e_i$$

Thus for given $\vec{q}$ and $\vec{p}$ the equations determining the background are:

$$\frac{\partial f}{\partial u_s} = 0, \quad \frac{\partial f}{\partial v_1} = 0, \quad \frac{\partial f}{\partial v_2} = 0, \quad \frac{\partial f}{\partial e_i} = q_i$$

f is a function of $\vec{u}$, $\vec{v}$, $\vec{e}$ and $\vec{p}$.

Now recall the definition of F:

$$F(\vec{u}, \vec{v}, \vec{q}, \vec{p}) = 2\pi \left(\vec{q} \cdot \vec{e} - f(\vec{u}, \vec{v}, \vec{e}, \vec{p})\right)$$

Properties of the Legendre transform → these equations are equivalent to:

$$\frac{\partial F}{\partial u_s} = 0, \quad \frac{\partial F}{\partial v_1} = 0, \quad \frac{\partial F}{\partial v_2} = 0, \quad \frac{\partial F}{\partial q_i} = 2\pi\, e_i$$

$$f(\vec{u}, \vec{v}, \vec{e}, \vec{p}) = \int d\theta \, d\phi \sqrt{-\det g} \; \mathcal{L}$$

$$q_i = \frac{\partial f}{\partial e_i}$$

$$F(\vec{u}, \vec{v}, \vec{q}, \vec{p}) = 2\pi \left(q_i e_i - f(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \right)$$

① $\frac{\partial F}{\partial u_A} = \frac{\partial F}{\partial v_1} = \frac{\partial F}{\partial v_2} = 0$ ② $\frac{\partial F}{\partial q_i} = 2\pi e_i$

③ $S_{BH} = F$

$$f(\vec{u},\vec{v},\vec{e},\vec{p}) = \int d\theta\, d\phi \sqrt{-\det g}\; \mathcal{L}$$

$$q_i = \frac{\partial f}{\partial e_i}$$

$$F(\vec{u},\vec{v},\vec{q},\vec{p}) = 2\pi\left(q_i e_i - f(\vec{u},\vec{v},\vec{e},\vec{p})\right)$$

① $\frac{\partial F}{\partial u_n} = \frac{\partial F}{\partial v_1} = \frac{\partial F}{\partial v_2} = 0$ ② $\frac{\partial F}{\partial q_i} = 2\pi e_i$

③ $S_{BH} = F$

Thus for given $\vec{q}$ and $\vec{p}$ the equations determining the background are:

$$\frac{\partial f}{\partial u_s} = 0, \quad \frac{\partial f}{\partial v_1} = 0, \quad \frac{\partial f}{\partial v_2} = 0, \quad \frac{\partial f}{\partial e_i} = q_i$$

f is a function of $\vec{u}$, $\vec{v}$, $\vec{e}$ and $\vec{p}$.

Now recall the definition of F:

$$F(\vec{u}, \vec{v}, \vec{q}, \vec{p}) = 2\pi \left(\vec{q} \cdot \vec{e} - f(\vec{u}, \vec{v}, \vec{e}, \vec{p})\right)$$

Properties of the Legendre transform → these equations are equivalent to:

$$\frac{\partial F}{\partial u_s} = 0, \quad \frac{\partial F}{\partial v_1} = 0, \quad \frac{\partial F}{\partial v_2} = 0, \quad \frac{\partial F}{\partial q_i} = 2\pi\, e_i$$

Alternatively we can define:

$$F(\vec{u},\vec{v},\vec{e},\vec{q},\vec{p}) \equiv 2\pi(e_i q_i - f(\vec{u},\vec{v},\vec{e},\vec{p}))$$

The $\vec{u}$, $\vec{v}$ and $\vec{e}$ are determined from:

$$\frac{\partial F}{\partial u_s} = 0, \quad \frac{\partial F}{\partial v_1} = 0, \quad \frac{\partial F}{\partial v_2} = 0, \quad \frac{\partial F}{\partial e_i} = 0$$

$$S_{BH} = F$$

at the extremum.

(17)

Computation of black hole entropy:

Higher derivative terms → the entropy is no longer given by the area law.

Wald; Iyer and Wald; Jacobson, Kang, Myers

For spherically symmetric black holes:

$$S_{BH} = -8\pi \int_H d\theta\, d\phi \frac{\delta S}{\delta R_{rtrt}} \sqrt{-g_{rr}\, g_{tt}}\,,$$

In computing $\delta S/\delta R_{\mu\nu\rho\sigma}$

1. express the action S in terms of symmetrized covariant derivatives of fields

2. treat $R_{\mu\nu\rho\sigma}$ as independent variables.

Computation of black hole entropy:

Higher derivative terms → the entropy is no longer given by the area law.

Wald; Iyer and Wald; Jacobson, Kang, Myers

For spherically symmetric black holes:

$$S_{BH} = -8\pi \int_H d\theta\, d\phi\, \frac{\delta S}{\delta R_{rtrt}} \sqrt{-g_{rr}\, g_{tt}}\,,$$

In computing $\delta S / \delta R_{\mu\nu\rho\sigma}$

1. express the action S in terms of symmetrized covariant derivatives of fields

2. treat $R_{\mu\nu\rho\sigma}$ as independent variables.

Computation of black hole entropy:

Higher derivative terms → the entropy is no longer given by the area law.

Wald; Iyer and Wald; Jacobson, Kang, Myers

For spherically symmetric black holes:

$$S_{BH} = 8\pi \int_H d\theta\, d\phi\, \frac{\delta S}{\delta R_{rtrt}} \sqrt{-g_{rr}\, g_{tt}}\,,$$

In computing $\delta S / \delta R_{\mu\nu\rho\sigma}$

1. express the action S in terms of symmetrized covariant derivatives of fields

2. treat $R_{\mu\nu\rho\sigma}$ as independent variables.

$$S_{BH} = -8\pi \int_H d\theta\, d\phi\, \frac{\delta S}{\delta R_{rtrt}} \sqrt{-g_{rr}\, g_{tt}}\,,$$

For our background $D_\mu \phi_s$, $D_\rho F^{(i)}_{\mu\nu}$ and $D_\tau R_{\mu\nu\rho\sigma}$ all vanish.

→ can ignore all terms in $\mathcal{L}$ which involve covariant derivatives of ϕ_s, $F^{(i)}_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}$.

Thus:

$$S_{BH} = -8\pi \int_H d\theta\, d\phi\, \sqrt{-\det g}\, \frac{\partial \mathcal{L}}{\partial R_{rtrt}} \sqrt{-g_{rr}\, g_{tt}}\,,$$

Define:

1. $\mathcal{L}_\lambda = \mathcal{L}$ with each factor of $R_{\alpha\beta\gamma\delta}$ in $\mathcal{L}$ multiplied by λ.

$\alpha, \beta, \gamma, \delta = r, t$

2.

$$f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \equiv \int d\theta\, d\phi \sqrt{-\det g}\, \mathcal{L}_\lambda$$

Then $f_{\lambda=1} = f$.

$$\lambda \frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial \lambda} = 4 \int d\theta\, d\phi \sqrt{-\det g}\, \frac{\partial \mathcal{L}_\lambda}{\partial R_{rtrt}} R_{rtrt}$$

For our background $R_{rtrt} = +\sqrt{-g_{rr}\, g_{tt}}$

This gives

$$S_{BH} = -2\pi \left. \frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial \lambda} \right|_{\lambda=1}$$

We shall now try to write $\partial f_\lambda / \partial \lambda$ in terms of derivatives of f_λ with respect to v_1 and e_i.

(19)

Define:

1. $\mathcal{L}_\lambda = \mathcal{L}$ with each factor of $R_{\alpha\beta\gamma\delta}$ in $\mathcal{L}$ multiplied by λ.

$\alpha, \beta, \gamma, \delta = r, t$

2.

$$f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \equiv \int d\theta\, d\phi \sqrt{-\det g}\, \mathcal{L}_\lambda$$

Then $f_{\lambda=1} = f$.

Define:

1. $\mathcal{L}_\lambda = \mathcal{L}$ with each factor of $R_{\alpha\beta\gamma\delta}$ in $\mathcal{L}$ multiplied by λ.

$\alpha, \beta, \gamma, \delta = r, t$

2.
$$f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \equiv \int d\theta \, d\phi \sqrt{-\det g}\, \mathcal{L}_\lambda$$

Then $f_{\lambda=1} = f$.

Define:

1. $\mathcal{L}_\lambda = \mathcal{L}$ with each factor of $R_{\alpha\beta\gamma\delta}$ in $\mathcal{L}$ multiplied by λ.

$\alpha, \beta, \gamma, \delta = r, t$

2.

$$f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \equiv \int d\theta\, d\phi \sqrt{-\det g}\, \mathcal{L}_\lambda$$

Then $f_{\lambda=1} = f$.

Define:

1. $\mathcal{L}_\lambda = \mathcal{L}$ with each factor of $R_{\alpha\beta\gamma\delta}$ in $\mathcal{L}$ multiplied by λ.

$\alpha, \beta, \gamma, \delta = r, t$

2.

$$f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \equiv \int d\theta\, d\phi \sqrt{-\det g}\, \mathcal{L}_\lambda$$

Then $f_{\lambda=1} = f$.

$$\lambda \frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial \lambda} = 4 \int d\theta\, d\phi \sqrt{-\det g}\, \frac{\partial \mathcal{L}_\lambda}{\partial R_{rtrt}} R_{rtrt}$$

Define:

1. $\mathcal{L}_\lambda = \mathcal{L}$ with each factor of $R_{\alpha\beta\gamma\delta}$ in $\mathcal{L}$ multiplied by λ.

$\alpha, \beta, \gamma, \delta = r, t$

2.

$$f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \equiv \int d\theta \, d\phi \sqrt{-\det g} \, \mathcal{L}_\lambda$$

Then $f_{\lambda=1} = f$.

$$\lambda \frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial \lambda} = 4 \int d\theta \, d\phi \sqrt{-\det g} \, \frac{\partial \mathcal{L}_\lambda}{\partial R_{rtrt}} R_{rtrt}$$

For our background $R_{rtrt} = +\sqrt{-g_{rr} \, g_{tt}}$

Define:

1. $\mathcal{L}_\lambda = \mathcal{L}$ with each factor of $R_{\alpha\beta\gamma\delta}$ in $\mathcal{L}$ multiplied by λ.

$\alpha, \beta, \gamma, \delta = r, t$

2.

$$f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \equiv \int d\theta\, d\phi \sqrt{-\det g}\, \mathcal{L}_\lambda$$

Then $f_{\lambda=1} = f$.

$$\lambda \frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial \lambda} = 4 \int d\theta\, d\phi \sqrt{-\det g}\, \frac{\partial \mathcal{L}_\lambda}{\partial R_{rtrt}} R_{rtrt}$$

Define:

1. $\mathcal{L}_\lambda = \mathcal{L}$ with each factor of $R_{\alpha\beta\gamma\delta}$ in $\mathcal{L}$ multiplied by λ.

$\alpha, \beta, \gamma, \delta = r, t$

2.

$$f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \equiv \int d\theta\, d\phi\, \sqrt{-\det g}\, \mathcal{L}_\lambda$$

Then $f_{\lambda=1} = f$.

$$\lambda \frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial \lambda} = 4 \int d\theta\, d\phi\, \sqrt{-\det g}\, \frac{\partial \mathcal{L}_\lambda}{\partial R_{rtrt}} R_{rtrt}$$

Define:

1. $\mathcal{L}_\lambda = \mathcal{L}$ with each factor of $R_{\alpha\beta\gamma\delta}$ in $\mathcal{L}$ multiplied by λ.

$\alpha, \beta, \gamma, \delta = r, t$

2.

$$f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \equiv \int d\theta\, d\phi \sqrt{-\det g}\, \mathcal{L}_\lambda$$

Then $f_{\lambda=1} = f$.

$$\lambda \frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial \lambda} = 4 \int d\theta\, d\phi \sqrt{-\det g}\, \frac{\partial \mathcal{L}_\lambda}{\partial R_{rtrt}} R_{rtrt}$$

For our background $R_{rtrt} = +\sqrt{-g_{rr}\, g_{tt}}$

Define:

1. $\mathcal{L}_\lambda = \mathcal{L}$ with each factor of $R_{\alpha\beta\gamma\delta}$ in $\mathcal{L}$ multiplied by λ.

$\alpha, \beta, \gamma, \delta = r, t$

2.

$$f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \equiv \int d\theta\, d\phi \sqrt{-\det g}\, \mathcal{L}_\lambda$$

Then $f_{\lambda=1} = f$.

$$\lambda \frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial \lambda} = 4 \int d\theta\, d\phi \sqrt{-\det g}\, \frac{\partial \mathcal{L}_\lambda}{\partial R_{rtrt}} R_{rtrt}$$

For our background $R_{rtrt} = +\sqrt{-g_{rr}\, g_{tt}}$

Define:

1. $\mathcal{L}_\lambda = \mathcal{L}$ with each factor of $R_{\alpha\beta\gamma\delta}$ in $\mathcal{L}$ multiplied by λ.

$\alpha, \beta, \gamma, \delta = r, t$

2.

$$f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \equiv \int d\theta\, d\phi \sqrt{-\det g}\, \mathcal{L}_\lambda$$

Then $f_{\lambda=1} = f$.

$$\lambda \frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial \lambda} = 4 \int d\theta\, d\phi \sqrt{-\det g}\, \frac{\partial \mathcal{L}_\lambda}{\partial R_{rtrt}} R_{rtrt}$$

For our background $R_{rtrt} = +\sqrt{-g_{rr}\, g_{tt}}$

This gives

$$S_{BH} = -2\pi \left. \frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial \lambda} \right|_{\lambda=1}$$

19

Define:

1. $\mathcal{L}_\lambda = \mathcal{L}$ with each factor of $R_{\alpha\beta\gamma\delta}$ in $\mathcal{L}$ multiplied by λ.

2. $\alpha, \beta, \gamma, \delta = r, t$

$$f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \equiv \int d\theta\, d\phi \sqrt{-\det g}\, \mathcal{L}_\lambda$$

Then $f_{\lambda=1} = f$.

$$\lambda \frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial \lambda} = 4 \int d\theta\, d\phi \sqrt{-\det g}\, \frac{\partial \mathcal{L}_\lambda}{\partial R_{rtrt}} R_{rtrt}$$

For our background $R_{rtrt} = +\sqrt{-g_{rr}\, g_{tt}}$

This gives

$$S_{BH} = -2\pi \left. \frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial \lambda} \right|_{\lambda=1}$$

We shall now try to write $\partial f_\lambda / \partial \lambda$ in terms of derivatives of f_λ with respect to v_1 and e_i.

Define:

1. $\mathcal{L}_\lambda = \mathcal{L}$ with each factor of $R_{\alpha\beta\gamma\delta}$ in $\mathcal{L}$ multiplied by λ.

$\alpha, \beta, \gamma, \delta = r, t$

2.

$$f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \equiv \int d\theta\, d\phi \sqrt{-\det g}\, \mathcal{L}_\lambda$$

Then $f_{\lambda=1} = f$.

$$\lambda \frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial \lambda} = 4 \int d\theta\, d\phi \sqrt{-\det g}\, \frac{\partial \mathcal{L}_\lambda}{\partial R_{rtrt}} R_{rtrt}$$

For our background $R_{rtrt} = +\sqrt{-g_{rr}\, g_{tt}}$

This gives

$$S_{BH} = -2\pi \left. \frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial \lambda} \right|_{\lambda=1}.$$

We shall now try to write $\partial f_\lambda / \partial \lambda$ in terms of derivatives of f_λ with respect to v_1 and e_i.

1. Every factor of λ in $\mathcal{L}$ must appear in the combination

$$\lambda g^{rr} g^{tt} R_{rtrt} = \lambda v_1^{-1}$$

2. Every $F^{(i)}_{rt}$ in $\mathcal{L}$ must appear in the combination:

$$\sqrt{-g^{rr} g^{tt}}\, F^{(i)}_{rt} = e_i v_1^{-1}$$

3. $R_{\theta\phi\theta\phi}$, $F^{(i)}_{\theta\phi}$ and u_s do not have any accompanying factor of v_1 (size of AdS_2).

4. There are no covariant derivatives contracting with the metric.

→ the only other v_1 dependence of f_λ comes from $\sqrt{-\det g}$ multiplying $\mathcal{L}$.

Result:

$$f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p}) = v_1 g(\vec{u}, v_2, \vec{p}, \lambda v_1^{-1}, e_i v_1^{-1})$$

for some function g.

$$f_\lambda(\vec{u},\vec{v},\vec{e},\vec{p}) = v_1 g(\vec{u}, v_2, \vec{p}, \lambda v_1^{-1}, e_i v_1^{-1})$$

$$\downarrow$$

$$\lambda\frac{\partial f_\lambda(\vec{u},\vec{v},\vec{e},\vec{p})}{\partial\lambda} + v_1\frac{\partial f_\lambda(\vec{u},\vec{v},\vec{e},\vec{p})}{\partial v_1} + e_i\frac{\partial f_\lambda(\vec{u},\vec{v},\vec{e},\vec{p})}{\partial e_i} - f_\lambda(\vec{u},\vec{v},\vec{e},\vec{p}) = 0 .$$

For $\lambda = 1$, $f_\lambda \to f$.

Equation of motion $\to$ $\partial f/\partial v_1 = 0$.

Thus

$$\begin{aligned} S_{BH} &= -2\pi \left.\frac{\partial f_\lambda(\vec{u},\vec{v},\vec{e},\vec{p})}{\partial\lambda}\right|_{\lambda=1} \\ &= 2\pi\left(e_i\frac{\partial f}{\partial e_i} - f\right) \\ &= F(\vec{u},\vec{v},\vec{q},\vec{p}) . \end{aligned}$$

$\to$ the desired result.

Test of the final formula

Take Einstein-Maxwell theory in $D=4$:

$$\mathcal{L} = \frac{1}{16\pi G_N} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Consider an extremal black hole solution with near horizon geometry:

$$ds^2 = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right)$$

$$F_{rt} = e, \qquad F_{\theta\phi} = (p/4\pi)\, \sin\theta$$

Then

$$\begin{aligned} f(v_1, v_2, e, p) &= \int d\theta d\phi \sqrt{-\det g}\, \mathcal{L} \\ &= v_1 v_2 \left[\frac{1}{16\pi G_N} \left(-\frac{2}{v_1} + \frac{2}{v_2} \right) \right. \\ &\qquad \left. + \frac{1}{2} v_1^{-2} e^2 - \frac{1}{2} v_2^{-2} \left(\frac{p}{4\pi} \right)^2 \right] \end{aligned}$$

$$f(v_1, v_2, e, p) = 4\pi v_1 v_2 \left[\frac{1}{16\pi G_N}\left(-\frac{2}{v_1} + \frac{2}{v_2}\right) + \frac{1}{2} v_1^{-2} e^2 - \frac{1}{2} v_2^{-2}\left(\frac{p}{4\pi}\right)^2\right].$$

Then

$$\begin{aligned} F(v_1, v_2, e, q, p) &= 2\pi(q\,e - f) \\ &= 2\pi\left[q\,e + \frac{1}{4G_N}(-2v_1 + 2v_2) - 2\pi\, v_2 v_1^{-1} e^2 + 2\pi\, v_1 v_2^{-1}\left(\frac{p}{4\pi}\right)^2\right] \end{aligned}$$

$\partial F/\partial e = 0$, $\partial F/\partial v_1 = 0$, $\partial F/\partial v_2 = 0$ gives

$$q = 4\pi\, v_2 v_1^{-1} e, \qquad v_1 = v_2 = G_N \frac{q^2 + p^2}{4\pi}.$$

$$S_{BH} = F = \frac{1}{4}(q^2 + p^2)$$

Test of the final formula

Take Einstein-Maxwell theory in $D = 4$:

$$\mathcal{L} = \frac{1}{16\pi G_N} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Consider an extremal black hole solution with near horizon geometry:

$$ds^2 = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right)$$

$$F_{rt} = e, \qquad F_{\theta\phi} = (p/4\pi) \sin\theta$$

Then

$$\begin{aligned} f(v_1, v_2, e, p) &= \int d\theta d\phi \sqrt{-\det g}\, \mathcal{L} \\ &= v_1 v_2 \left[\frac{1}{16\pi G_N} \left(-\frac{2}{v_1} + \frac{2}{v_2} \right) \right. \\ &\qquad \left. + \frac{1}{2} v_1^{-2} e^2 - \frac{1}{2} v_2^{-2} \left(\frac{p}{4\pi} \right)^2 \right] \end{aligned}$$

$$f(v_1, v_2, e, p) \;=\; 4\pi\, v_1 v_2 \left[\frac{1}{16\pi G_N}\left(-\frac{2}{v_1} + \frac{2}{v_2}\right) + \frac{1}{2} v_1^{-2} e^2 - \frac{1}{2} v_2^{-2}\left(\frac{p}{4\pi}\right)^2\right].$$

Then

$$\begin{aligned} F(v_1, v_2, e, q, p) \;&=\; 2\pi(q\,e - f) \\ &=\; 2\pi\left[q\,e + \frac{1}{4G_N}(-2v_1 + 2v_2) - 2\pi\, v_2\, v_1^{-1}\, e^2 + 2\pi\, v_1\, v_2^{-1}\left(\frac{p}{4\pi}\right)^2\right] \end{aligned}$$

$\partial F/\partial e = 0$, $\partial F/\partial v_1 = 0$, $\partial F/\partial v_2 = 0$ gives

$$q = 4\pi\, v_2 v_1^{-1} e, \qquad v_1 = v_2 = G_N\,\frac{q^2 + p^2}{4\pi}.$$

$$S_{BH} = F = \frac{1}{4}(q^2 + p^2)$$