

Title: Supergravity

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Abstract:

Lecture 2 Outline

- Conditions for unbroken supersymmetry

Killing spinors

Integrability and holonomy

- The Euclidean Taub-NUT solution
- Sasaki-Einstein manifolds

Five-dimensional examples: S^5 , $T^{1,1}$, $Y^{p,q}$ and $L^{p,q,r}$

Supersymmetric configurations

- We are interested in classical solutions of supergravity preserving some or all of the supersymmetries
 - Traditional heterotic string compactified on Calabi-Yau
 - Explore dualities and hidden symmetries of M-theory
 - Classify all supersymmetric vacua;
progress towards solving the vacuum degeneracy problem
 - Black hole entropy and microstates;
starting point for exploring near-extremal black holes
- Sometimes we may wish to make a distinction between
 - Supersymmetric vacua, which are generally regular and do not have horizons
 - BPS objects with horizons such as supergravity p -branes and black holes

Conditions for unbroken supersymmetry

- In field theory, for the vacuum to be invariant under a (super)symmetry generated by Q

$$Q|\text{vac}\rangle = 0 \quad \rightarrow \quad \langle \text{vac} | [Q, \Phi] | \text{vac} \rangle = 0$$

- For a background configuration, this implies that we are seeking a solution to the classical equations of motion such that $\delta\Phi = 0$ for all fields Φ
- Supergravity involves two types of variations

$$\delta(\text{Fermion}) = \partial(\text{Boson})\epsilon, \quad \delta(\text{Boson}) = \bar{\epsilon}(\text{Fermion})$$

For a purely bosonic background, the condition $\delta(\text{Boson}) = 0$ is automatic

- **Goal:** Bosonic field configuration with $\delta(\text{Fermion}) = 0$

Killing spinors

- In a supergravity theory, with graviton and gravitino

$$\delta e_{\mu}^{\alpha} = \frac{1}{4} i \bar{\epsilon} \Gamma^{\alpha} \psi_{\mu}$$

$$\delta \psi_{\mu} = \hat{\nabla}_{\mu} \epsilon \quad \hat{\nabla}_{\mu} = \nabla_{\mu} + \frac{1}{288} (\Gamma_{\mu}^{\nu\rho\sigma\tau} - 8\delta_{\mu}^{\nu} \Gamma^{\rho\sigma\tau}) F_{\nu\rho\sigma\tau}$$

$$\delta \lambda = [\dots] \epsilon$$

we must solve for $\delta \psi_{\mu} = 0$ or $\hat{\nabla}_{\mu} \epsilon = 0$

This is known as the Killing spinor equation

With other fermions, we also solve $\delta \lambda = 0$, etc

- Isometries of the background spacetime \leftrightarrow Killing vector
- Unbroken supersymmetry \leftrightarrow Killing spinor

Pure geometry solutions

- The simplest case to consider is for all fields vanishing except for the metric — pure geometry

In this case we simply demand $\nabla_\mu \epsilon = 0$ on a d -dimensional Riemannian (or Lorentzian signature) manifold

Existence of a covariantly constant spinor

- Relation to supersymmetric string compactifications

If $\mathcal{M}^{10} = \mathbb{M}^{1,3} \times X^6$ where X^6 is a compact manifold admitting a covariantly constant spinor, $\nabla_m \epsilon(y) = 0$, then the resulting system is supersymmetric in four dimensions

The number of $D = 4$ supersymmetries is related to the number of independent Killing spinors on X^6

Integrability of the Killing spinor equation

- The equation $\nabla_\mu \epsilon = 0$ is first order, but involves derivatives on ϵ

We may remove derivatives on ϵ by studying the integrability condition

$$\nabla_\mu \epsilon = 0 \quad \rightarrow \quad [\nabla_\mu, \nabla_\nu] \epsilon = \frac{1}{4} R_{\mu\nu\rho\sigma} \Gamma^{\rho\sigma} \epsilon = 0$$

This expresses conditions on the curvature to ensure the existence of a Killing spinor

- In addition, $R_{\mu\nu\rho\sigma} \Gamma^{\rho\sigma} \epsilon = 0$ implies

$$0 = R_{\mu\nu\rho\sigma} \Gamma^\nu \Gamma^{\rho\sigma} \epsilon = R_{\mu\nu\rho\sigma} (\Gamma^{\nu\rho\sigma} + 2g^{\nu\rho} \Gamma^\sigma) \epsilon = -2R_{\mu\sigma} \Gamma^\sigma \epsilon$$

or $R_{\mu\nu} \Gamma^\nu \epsilon = 0$

This is almost the vacuum Einstein equation $R_{\mu\nu} = 0$

Supersymmetry and the equations of motion

- Completely unbroken supersymmetry ($R_{\mu\nu}\Gamma^\nu\epsilon = 0$ for all ϵ) implies $R_{\mu\nu}\Gamma^\nu = 0$

Using $\text{Tr}(\Gamma^\mu\Gamma^\nu) = 2g^{\mu\nu}\text{Tr} 1$, we find $R_{\mu\nu} = 0$

Thus a background with maximal supersymmetry automatically satisfies the equations of motion

- For partially broken supersymmetry, we need a more refined argument

We would like to remove $\Gamma^\nu\epsilon$ from the condition $R_{\mu\nu}\Gamma^\nu\epsilon = 0$

But we may only have a single ϵ to work with

Partially broken supersymmetry and the eom

- If ϵ is a Killing spinor, then $R_{\mu_0\nu}\Gamma^\nu\epsilon = 0$ for any fixed μ_0

Without summing over μ_0 , note that $(R_{\mu_0\lambda}\Gamma^\lambda)(R_{\mu_0\nu}\Gamma^\nu)\epsilon = 0$

$$\rightarrow R_{\mu_0\lambda}g^{\lambda\nu}R_{\mu_0\nu}\epsilon = 0 \quad \rightarrow \quad R_{\mu_0\lambda}g^{\lambda\nu}R_{\mu_0\nu} = 0 \quad \rightarrow \quad \|R_{\mu_0}\| = 0 \quad \forall \mu_0$$

- For Euclidean signature (positive definite metric) $\|R_{\mu_0}\| = 0$ implies all components must vanish

Since this is true for all μ_0 , we are guaranteed that $R_{\mu\nu} = 0$

- But for Lorentzian signature the best we can do is conclude the spacetime is Ricci-null

A background with partial supersymmetry satisfies some, but not necessarily all, of the equations of motion

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Holonomy and supersymmetry

- Another way to understand unbroken supersymmetry

$[\nabla_\mu, \nabla_\nu]\epsilon$ represents the effect of parallel transport of a spinor around an infinitesimal loop

$[\nabla_\mu, \nabla_\nu]\epsilon = \frac{1}{2}R_{\mu\nu}{}^{ab}\Sigma_{ab}\epsilon$ where Σ_{ab} is a $SO(n)$ rotation generator

- The group of rotations acting on ϵ is known as the holonomy group

$$H \subseteq SO(n) \quad \epsilon = \text{spinor of } SO(n)$$

- The number of preserved supersymmetries is then the number of singlets in the decomposition of ϵ under H

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Holonomy classification

- M. Berger in *Bull. Soc. Math. France* **83**, 225 (1955) classified the possible holonomy groups in Euclidean signature

For an irreducible non-symmetric Riemannian metric, the possible holonomy groups are

dim	H	Ricci flat?	
n	$SO(n)$		generic metric
$2n$	$U(n)$		complex Kähler
$2n$	$SU(n)$	yes	Calabi-Yau
$4n$	$Sp(2n)$	yes	hyperkähler
$4n$	$Sp(2n) \times Sp(2)$		quaternionic Kähler
7	G_2	yes	exceptional holonomy
8	$Spin(7)$	yes	exceptional holonomy

- In Lorentzian signature, the classification is based on either a timelike or a null Killing vector $\xi^\mu = \bar{\epsilon} \Gamma^\mu \epsilon$ [see e.g. R.L. Bryant, [math.DG/0004073](https://math.dg/0004073)]

Some examples: flat space

- Perhaps the most trivial example of a supersymmetric background is simply flat space itself

Maximally symmetric and maximally supersymmetric

$$\nabla_{\mu}\epsilon = 0 \quad \rightarrow \quad \partial_{\mu}\epsilon = 0 \quad \rightarrow \quad \epsilon(x) = \epsilon_0$$

- The fraction of preserved supersymmetries is 1

We can choose a basis

$$\epsilon^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \epsilon^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \quad \epsilon^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix} \quad \text{etc.}$$

The details depend on the dimension and type of spinor

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The details depend on the dimension and type of spinor

Some examples: $SU(2)$ holonomy

- In four Euclidean dimensions, the tangent space group is

$$SO(4) = SU(2)_+ \times SU(2)_-$$

- This allows us to decompose

$$R_{\mu\nu}{}^{ab}\Gamma_{ab}\epsilon = R_{\mu\nu}{}^{ab}\left(\frac{1}{2}(1 + \Gamma^5)\Gamma_{ab} + \frac{1}{2}(1 - \Gamma^5)\Gamma_{ab}\right)\epsilon$$

Note the projections $P_{\pm} = \frac{1}{2}(1 \pm \Gamma^5)$

Also, since $\Gamma^5\Gamma_{ab} = -\epsilon_{abcd}\Gamma^{cd}$, we have

$$R_{\mu\nu}{}^{ab}\Gamma_{ab}\epsilon = (R_{\mu\nu}{}^{ab}\Gamma_{ab}P_- + R_{\mu\nu}{}^{ab}\Gamma_{ab}P_+)\epsilon$$

where

$$R_{\mu\nu}{}^{\pm ab} = \frac{1}{2}(\delta_a^c\delta_b^d \pm \frac{1}{2}\epsilon_{ab}{}^{cd})R_{\mu\nu cd}$$

Some examples: $SU(2)$ holonomy

- We may solve for a Killing spinor by picking a self-dual connection, i.e. by setting $R_{\mu\nu}^{-ab} = 0$

The resulting condition is $R_{\mu\nu}^{+ab}\Gamma_{ab}P_-\epsilon = 0$ where $\Gamma_{ab}P_-$ clearly generates the $SU(2)_-$ subgroup of $SO(4)$

- **Holonomy:** The spinor of $SO(4)$ is $4 = (2, 1) + (1, 2)$ which decomposes as $4 \rightarrow 2 + 1 + 1$ under $SO(4) \supset SU(2)_-$ (two singlets)
- **Killing spinor:** An arbitrary spinor ϵ decomposes into positive and negative chirality spinors $\epsilon_{\pm} = P_{\pm}\epsilon$
The ϵ_+ spinors are Killing spinors since they are annihilated by $R_{\mu\nu}^{+ab}\Gamma_{ab}P_-$ (the ϵ_- spinors are fermion zero modes)
- In either case, we conclude that this is a $\frac{1}{2}$ -BPS background

Some examples: $SU(2)$ holonomy

- A compact manifold of $SU(2)$ holonomy is known as a K3 surface
See e.g. P.S. Aspinwall, *K3 Surfaces and String Duality*, hep-th/9611137
- Non-compact examples include

Euclidean Taub-NUT asymptotically locally flat ($\mathbb{R}^3 \times S^1$)

$$ds_4^2 = H[dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)] + H^{-1}(dy - q \cos \theta d\phi)^2$$

$$H = 1 + q/r$$

Eguchi-Hanson asymptotically locally Euclidean (\mathbb{R}^4/Z_2)

$$ds_4^2 = h^{-1}dr^2 + \frac{1}{4}r^2[d\theta^2 + \sin^2 \theta d\phi^2 + h(d\psi + \cos \theta d\phi)^2]$$

$$h = 1 - a^4/r^4$$

Some examples: $SU(3)$ holonomy

- In six Euclidean dimensions, the tangent space group is

$$SO(6) = SU(4)$$

- Consider the $SU(3)$ subgroup

$$SO(6) \supset U(3) \supset SU(3)$$

The complex spinor decomposes as

$$4 \rightarrow 3_{-1} + 1_3 \rightarrow 3 + 1$$

so this is a $\frac{1}{4}$ -BPS background: Calabi-Yau (complex) 3-fold

Note the presence of a covariantly constant $(3, 0)$ -form $\Omega_{\mu\nu\rho} = \bar{\epsilon}\Gamma_{\mu\nu\rho}\epsilon$

Supersymmetry of Taub-NUT

- The $D6$ -brane of IIA theory lifts to pure geometry in $D = 11$

$U(1)$ bundled over \mathbb{R}^3

- In general, this is a multi-centered Taub-NUT configuration

$$ds_4^2 = H(\vec{x})d\vec{x}^2 + H^{-1}(\vec{x})(dy + A_{(1)})^2$$

where $F_{(2)} = dA_{(1)}$ and $F_{(2)} = *_3 dH$ (which requires $d *_3 dH = 0$)

- To examine the supersymmetry, we may examine the Killing spinor equation $\nabla_\mu \epsilon = 0$ where

$$\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab}$$

Computing the spin connection

- Introduce a vielbein basis

$$g_{\mu\nu} = e_{\mu}^a \delta_{ab} e_{\nu}^b \quad \text{or} \quad ds^2 = e^a \otimes e^a$$

- The spin connection may be computed by demanding the torsion free condition $de + \omega \wedge e = 0$

The Cartan structure equation $de^a = \frac{1}{2} c_{bc}^a e^b \wedge e^c$

In terms of the anholonomy coefficients $\omega_{abc} = \frac{1}{2}(c_{abc} - c_{bca} + c_{cab})$

- Note that there is a shortcut for diagonal metrics

If $e_{\mu}^a = f_{\mu}^a \delta_{\mu}^a$ so that $g_{\mu\nu} = f_{\mu}^2 \delta_{\mu\nu}$ then

$$\omega_{\mu\nu\lambda} = g_{\mu[\nu,\lambda]} \equiv \frac{1}{2}(\partial_{\lambda} g_{\mu\nu} - \partial_{\nu} g_{\mu\lambda})$$

The spin connection for Taub-NUT

- Given

$$ds_4^2 = H(\vec{x})d\vec{x}^2 + H^{-1}(\vec{x})(dy + A_{(1)})^2$$

we choose a natural vielbein basis

$$e^a = H^{1/2}dx^a \quad e^4 = H^{-1/2}(dy + A_{(1)})$$

- Then we obtain

$$\omega_{abc} = H^{-3/2}\delta_{a[b}\partial_{c]}H$$

$$\omega_{ab4} = \frac{1}{2}H^{-3/2}F_{ab}$$

$$\omega_{4ab} = -\frac{1}{2}H^{-3/2}F_{ab}$$

$$\omega_{4a4} = \frac{1}{2}H^{-3/2}\partial_a H$$

This allows us to write down the covariant derivative

The Killing spinor projection

- For the above metric, the Killing spinor equation becomes

$$0 = \nabla_m \epsilon = [\partial_m + X_m + A_m X_4] \epsilon$$

$$0 = \nabla_y \epsilon = [\partial_y + X_4] \epsilon$$

where

$$X_a = \frac{1}{4} H^{-1} (\Gamma_a^b \partial_b H + F_{ab} \Gamma^{b4}) \quad X_4 = \frac{1}{4} H^{-2} (\Gamma^{a4} \partial_a H - \frac{1}{2} F_{ab} \Gamma^{ab})$$

With some manipulation, we obtain

$$X_a = \frac{1}{4} H^{-1} \Gamma^{b4} (F_{ab} + \epsilon_{ab}{}^c \partial_c H \Gamma^5) \quad X_4 = \frac{1}{4} H^{-2} \Gamma^{a4} (\partial_a H + \frac{1}{2} \epsilon_a{}^{bc} F_{bc} \Gamma^5)$$

where $\Gamma^5 = \Gamma^{1234}$

The Killing spinor projection

- These matrices admit zero eigenvalues when $F_{(2)} = *_3 dH$

$$\nabla_m \epsilon = [\partial_m + \frac{1}{2} H^{-2} (H \epsilon_{\bar{m}a}{}^b + A_m \delta_a^b) \partial_b H \Gamma^{a4} P_+] \epsilon$$

$$\nabla_y \epsilon = [\partial_y + \frac{1}{2} H^{-2} \partial_a H \Gamma^{a4} P_+] \epsilon$$

(and similarly for $F_{(2)} = - *_3 dH$, but with P_-) where

$$P_{\pm} = \frac{1}{2} (1 \pm \Gamma^5)$$

- This is a $\frac{1}{2}$ -BPS projection, with Killing spinors

$$\epsilon = P_- \epsilon_0$$

- Note that we must satisfy the Bianchi identity

$$dF_{(2)} = 0 \quad \rightarrow \quad d *_3 dH = 0 \quad \rightarrow \quad H = 1 + \sum \frac{q_i}{|\vec{x} - \vec{x}_i|}$$

Relation to $SU(2)$ holonomy

- This provides an explicit realization of $SU(2)_+ \times SU(2)_- \rightarrow SU(2)_+$

$$SU(2)_+ : J_+^{ab} = \frac{1}{2}i\Gamma^{ab}P_+ \quad SU(2)_- : J_-^{ab} = \frac{1}{2}i\Gamma^{ab}P_-$$

- In this case, the 4-component spinor ϵ has $SU(2)_+$ eigenvalues

$$\{\frac{1}{2}, -\frac{1}{2}, 0, 0\} \rightarrow 4 \rightarrow 2 + 1 + 1$$

In general, each compatible projection reduces the supersymmetry by half

$$\frac{1}{2}\text{-BPS}, \frac{1}{4}\text{-BPS}, \frac{1}{8}\text{-BPS}, \text{ etc.}$$

Taub-NUT as a cone over S^3

- Another look at the $q = 1$ Taub-NUT metric

$$\begin{aligned} ds_4^2 &= H[dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)] + H^{-1}(d\psi + \cos \theta d\phi)^2 \\ &= Hdr^2 + Hr^2(\sigma_1^2 + \sigma_2^2) + H^{-1}\sigma_3^2 \end{aligned}$$

Where the σ^a are left-invariant 1-forms on S^3 satisfying $d\sigma^a = -\frac{1}{2}\epsilon_{abc}\sigma^b \wedge \sigma^c$

$$\sigma^1 = -\sin \psi d\theta + \cos \psi \sin \theta d\phi$$

$$\sigma^2 = \cos \psi d\theta + \sin \psi \sin \theta d\phi$$

$$\sigma^3 = -d\psi - \cos \theta d\phi \quad \theta \in [0, \pi], \phi \in [0, 2\pi], \psi \in [0, 4\pi]$$

- In general, we may obtain Kähler metrics as cones

$$d\hat{s}_d^2 = dr^2 + r^2 ds_{d-1}^2(X)$$

Taub-NUT as a cone over S^3

- Another look at the $q = 1$ Taub-NUT metric

$$\text{This results in } \hat{\nabla}_r = \partial_r \quad \hat{\nabla}_m = \nabla_m + \frac{1}{2}\Gamma_m\Gamma^{\bar{r}}$$

This yields a modified $(d - 1)$ -dimensional Killing spinor equation

$$\hat{\epsilon} = \epsilon(x^m) \quad \text{with} \quad \mathcal{D}_m\epsilon = 0 \quad \text{where} \quad \mathcal{D}_m = \nabla_m + \frac{1}{2}\Gamma_m\Gamma^{\bar{r}}$$

- Study the integrability

$$[\mathcal{D}_m, \mathcal{D}_n] = [\nabla_m, \nabla_n] + \frac{1}{4}[\Gamma_m\Gamma^{\bar{r}}, \Gamma_n\Gamma^{\bar{r}}] = \frac{1}{4}[R_{mnpq} - (g_{mp}g_{nq} - g_{mq}g_{np})]\Gamma^{\bar{p}\bar{q}}$$

Take a Γ -trace to obtain

$$\Gamma^n[\mathcal{D}_m, \mathcal{D}_n] = -\frac{1}{2}[R_{m\bar{n}} - (d - 2)g_{m\bar{n}}]\Gamma^{\bar{n}}$$

Sasaki-Einstein manifolds

- For the metric $d\hat{s}_d^2 = dr^2 + r^2 ds_{d-1}^2(X)$

Compute $\hat{\omega}_m^{ab} = \omega_m^{ab} \quad \hat{\omega}_m^{a\bar{r}} = e_m^a$

This results in $\hat{\nabla}_r = \partial_r \quad \hat{\nabla}_m = \nabla_m + \frac{1}{2}\Gamma_m\Gamma^{\bar{r}}$

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Sasaki-Einstein manifolds

- As we have seen before, this tensor must vanish (in Euclidean signature) given a Killing spinor
- We conclude
 - The space X^{d-1} is Einstein, $R_{mn} = \Lambda g_{mn}$ with $\Lambda = d - 2$
 - Integrability demands $[\mathcal{D}_m, \mathcal{D}_n]\epsilon = \frac{1}{4}C_{mnpq}\Gamma^{pq}\epsilon = 0$
- Existence of a Killing spinor on X^{d-1} is related to Killing spinors on \mathcal{M}^d

\mathcal{M}^d is Ricci-flat and Kähler (Calabi-Yau) if and only if X^{d-1} is Einstein and preserves a Killing spinor (Sasaki-Einstein)

Sasaki-Einstein manifolds

- For the metric $d\hat{s}_d^2 = dr^2 + r^2 ds_{d-1}^2(X)$

Compute $\hat{\omega}_m^{ab} = \omega_m^{ab}$ $\hat{\omega}_m^{a\bar{r}} = e_m^a$

This results in $\hat{\nabla}_r = \partial_r$ $\hat{\nabla}_m = \nabla_m + \frac{1}{2}\Gamma_m\Gamma^{\bar{r}}$

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$$[\mathcal{D}_m, \mathcal{D}_n] = [\nabla_m, \nabla_n] + \frac{1}{4}[\Gamma_m\Gamma^{\bar{r}}, \Gamma_n\Gamma^{\bar{r}}] = \frac{1}{4}[R_{mnpq} - (g_{mp}g_{nq} - g_{mq}g_{np})]\Gamma^{\bar{p}\bar{q}}$$

Take a Γ -trace to obtain

$$\Gamma^n[\mathcal{D}_m, \mathcal{D}_n] = -\frac{1}{2}[R_{m,n} - (d-2)g_{m,n}]\Gamma^n$$

Sasaki-Einstein manifolds

- As we have seen before, this tensor must vanish (in Euclidean signature) given a Killing spinor
- We conclude
 - The space X^{d-1} is Einstein, $R_{mn} = \Lambda g_{mn}$ with $\Lambda = d - 2$
 - Integrability demands $[\mathcal{D}_m, \mathcal{D}_n]\epsilon = \frac{1}{4}C_{mnpq}\Gamma^{pq}\epsilon = 0$
- Existence of a Killing spinor on X^{d-1} is related to Killing spinors on \mathcal{M}^d

\mathcal{M}^d is Ricci-flat and Kähler (Calabi-Yau) if and only if X^{d-1} is Einstein and preserves a Killing spinor (Sasaki-Einstein)

Examples: Cones over S^{d-1}

- Perhaps the simplest Einstein space is a sphere $X^{d-1} = S^{d-1}$

The cone over round S^{d-1} is trivially flat space $ds_d^2 = dr^2 + r^2 d\Omega_{d-1}^2$

- For anything other than a round sphere, the tip of the cone ($r = 0$) is singular
- More generally, Sasaki-Einstein metrics always admit a constant norm Killing vector, which may be taken as $K = c^{-1} \partial / \partial \psi$

This allows us to write (at least locally)

$$ds_{d-1}^2 = ds_{d-2}^2 + c^2 (d\psi + A)^2$$

where ds_{d-2}^2 is Kähler-Einstein (and dA is related to the Kähler form)

Examples: $T^{1,1}$ space

- The spaces $T^{p,q}$ are obtained by fibering $U(1)$ over $S^2 \times S^2$

$$ds_5^2 = \Lambda_1^{-1}(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + \Lambda_2^{-1}(d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) + c^2(d\psi + p \cos \theta_1 d\phi_1 + q \cos \theta_2 d\phi_2)^2$$

Note $F = dA = -pd\Omega_2 - qd\tilde{\Omega}_2$

- With tangent indices, we find

$$R_{ab} = \Lambda_1(1 - \frac{1}{2}c^2 p^2 \Lambda_1)\delta_{ab} \quad R_{\bar{a}\bar{b}} = \Lambda_2(1 - \frac{1}{2}c^2 q^2 \Lambda_2)\delta_{\bar{a}\bar{b}}$$

$$R_{55} = \frac{1}{2}c^2(p^2 \Lambda_1^2 + q^2 \Lambda_2^2)$$

Λ_1 , Λ_2 and c can be chosen to make the space Einstein ($R_{mn} = 4g_{mn}$)

Examples: $T^{1,1}$ space

- Only $T^{1,1}$ is supersymmetric

$$ds_5^2 = \frac{1}{6}(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + \frac{1}{6}(d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) + \frac{1}{9}(d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2$$

homogeneous space $SU(2) \times SU(2)/U(1)$ with $SU(2) \times U(1) \times U(1)$ isometry (cohomogeneity 1)

- The cone over $T^{1,1}$ is known as the conifold

Since $T^{1,1}$ is topologically $S^2 \times S^3$, the tip is singular with shrinking 2 and 3 cycles

- small resolution: blowing up the S^2
- deformation: blowing up the S^3

Examples: $Y^{p,q}$ space

- Recently constructed by Gauntlett, Martelli, Sparks and Waldram, [hep-th/0403002](https://arxiv.org/abs/hep-th/0403002)

$$ds_5^2 = \frac{1}{6}(1 - cy)(d\theta^2 + \sin^2 \theta d\phi^2) + (wv)^{-1}dy^2 + \frac{1}{36}wv(d\beta - c \cos \theta d\phi)^2 + \frac{1}{9}(d\psi + \cos \theta d\phi + y(d\beta - c \cos \theta d\phi))^2$$

where

$$w = \frac{2(a - y^2)}{1 - cy} \quad v = \frac{a - 3y^2 + 2cy^3}{a - y^2} \quad (y_1 < y < y_2)$$

Note that $T^{1,1}$ is obtained by setting $a = 3$, $c = 0$ and taking $y = \cos \xi$

a and c must be chosen to obtain integral periods p and q over $S^2 \times S^2$

Examples: $L^{p,q,r}$ space

- Additional non-singular generalization to $L^{p,q,r}$ by Cvetič, Lü, Page and Pope, [hep-th/0504225](#)

generically cohomogeneity 2, with isometry $U(1) \times U(1) \times U(1)$

Obtained by analytically continuing the Kerr-AdS₅ black hole with two independent rotation parameters

- Also many examples of 7-manifolds with G_2 holonomy

Next time

- Backgrounds with non-trivial charges or fluxes
 - Freund-Rubin compactification to $AdS \times S$
 - p -brane solutions in supergravity
- Methods for obtaining new supersymmetric solutions