Title: Twistor String Theory and Perturbative Yang-Mills 3

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Abstract:

Twistor Space descript.
treats + & − particles in a very different way

Can YM do the same thing?

$$S_{YM} = \frac{1}{g^2_{YM}} \int d^4x \, \mathrm{tr}\, F_{\mu\nu} F^{\mu\nu}$$

$$F = dA + A \wedge A$$

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$$S_{YM} = \frac{1}{g^2_{YM}} \int d^4x \, \mathrm{tr}\, F_{\mu\nu} F^{\mu\nu}$$

$$F = dA + A \wedge A$$

Canon. Norm Kinetic Terms $F = dA + g\, A \wedge A$

$g \to 0 \Rightarrow$ Free No interact vertices.

Trick: $$S = g \int \mathrm{tr}\, F^{+2} + \left(\mathrm{tr}\, F^{-2} - \mathrm{tr}\, F^{+2} \right)$$

$g \to 0 \Rightarrow$ Free No interact vertices.

Trick:

$$S = g \int \mathrm{tr} F^{+2} + \underbrace{\left(\mathrm{tr} F^{-2} - \mathrm{tr} F^{+2} \right)}_{\text{Total derivative}}$$

$$F^{\pm}_{\mu\nu} = \left(F_{\mu\nu} \pm \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \right)$$

$g \to 0 \Rightarrow$ Free   No interact vertices.

Trick: $S = g\int \mathrm{tr}\, F^{+2} + \cancel{\left(\mathrm{tr}\, F^{-2} - \mathrm{tr}\, F^{+2}\right)}$

~~Total derivative~~

Perturb.

$F^{\pm}_{\mu\nu} = \left(F_{\mu\nu} \pm \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}\right)$

$S(A, G)$

$g \to 0 \Rightarrow$ Free   No interact vertices.

Trick: $S = g\int \mathrm{tr}\, F^{+2} + \cancel{\left(\mathrm{tr}\, F^{-2} - \mathrm{tr}\, F^{+2}\right)}$

~~Total derivation~~

Pert.l.

$$F^{\pm}_{\mu\nu} = \left(F_{\mu\nu} \pm \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}\right)$$

$$S(A, G) = \int \mathrm{tr}\left(G F - \frac{g^2_{YM}}{2} G^2\right)$$

Integrate out G recover $\int d^4x\, \mathrm{tr}\, F^{+2}$.

$g \to 0 \Rightarrow$ Free No interact vertices.

Trick: $S = g \int \mathrm{tr} F^{+2} + \cancel{\left(\mathrm{tr} F^{-2} - \mathrm{tr} F^{+2} \right)}$

~~Total derivative~~

Pert'd.

$F^{\pm}_{\mu\nu} = \left(F_{\mu\nu} + \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \right)$

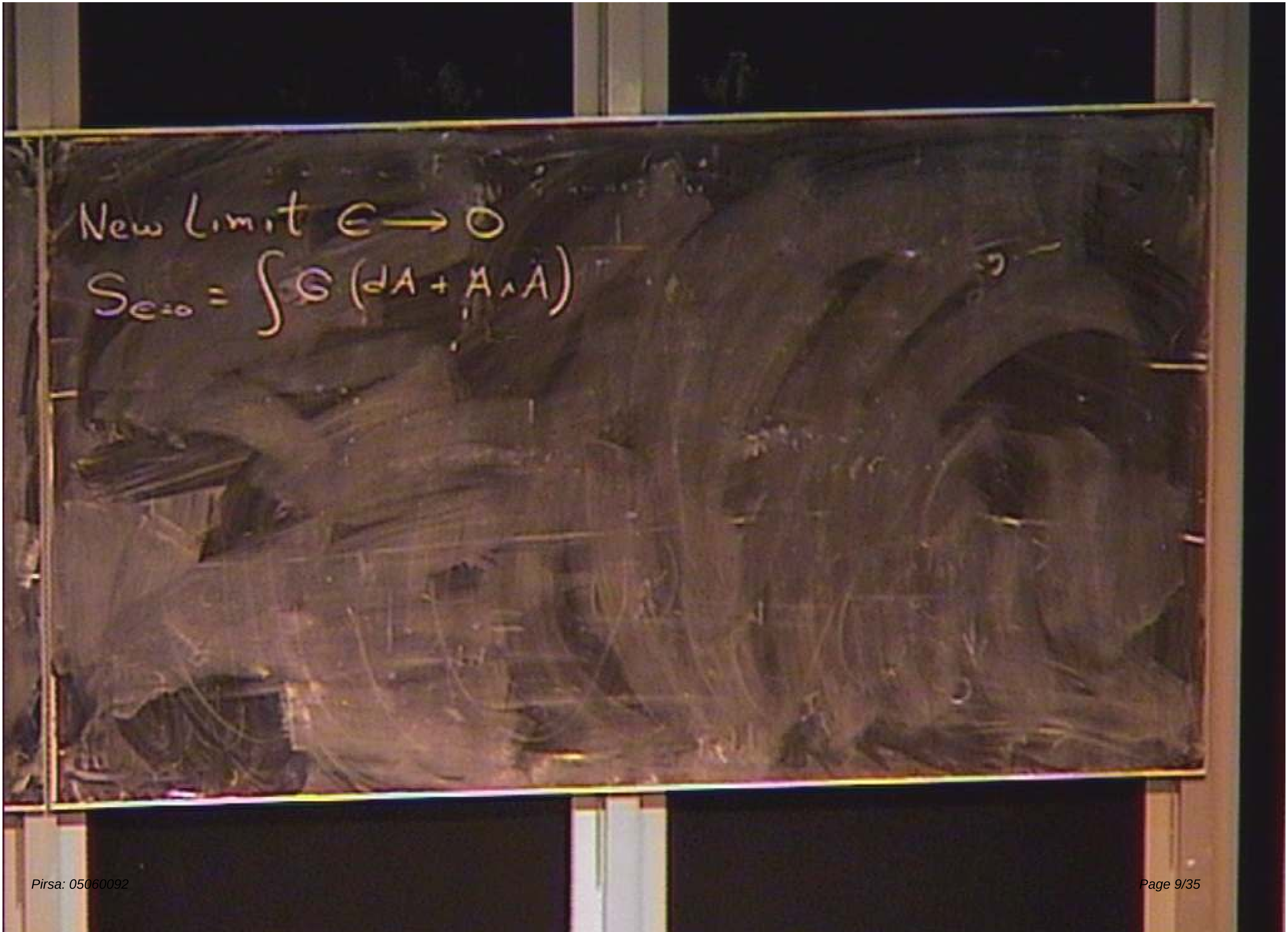
$$S(A, G) = \int \mathrm{tr}\left(G F^{+} - \frac{g_{YM}}{2} G^{2} \right)$$

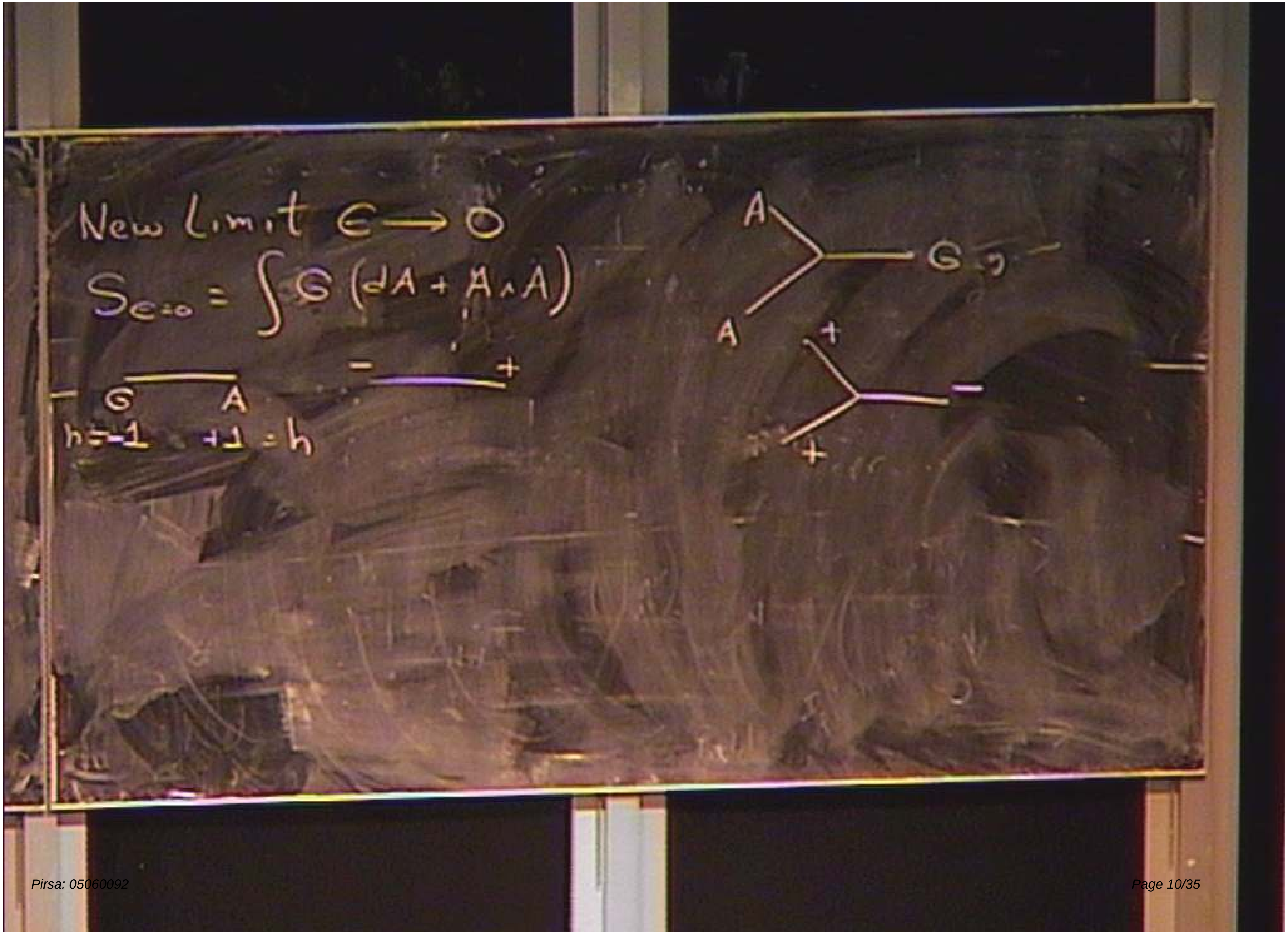
$\epsilon \equiv \frac{g_{YM}}{2}$

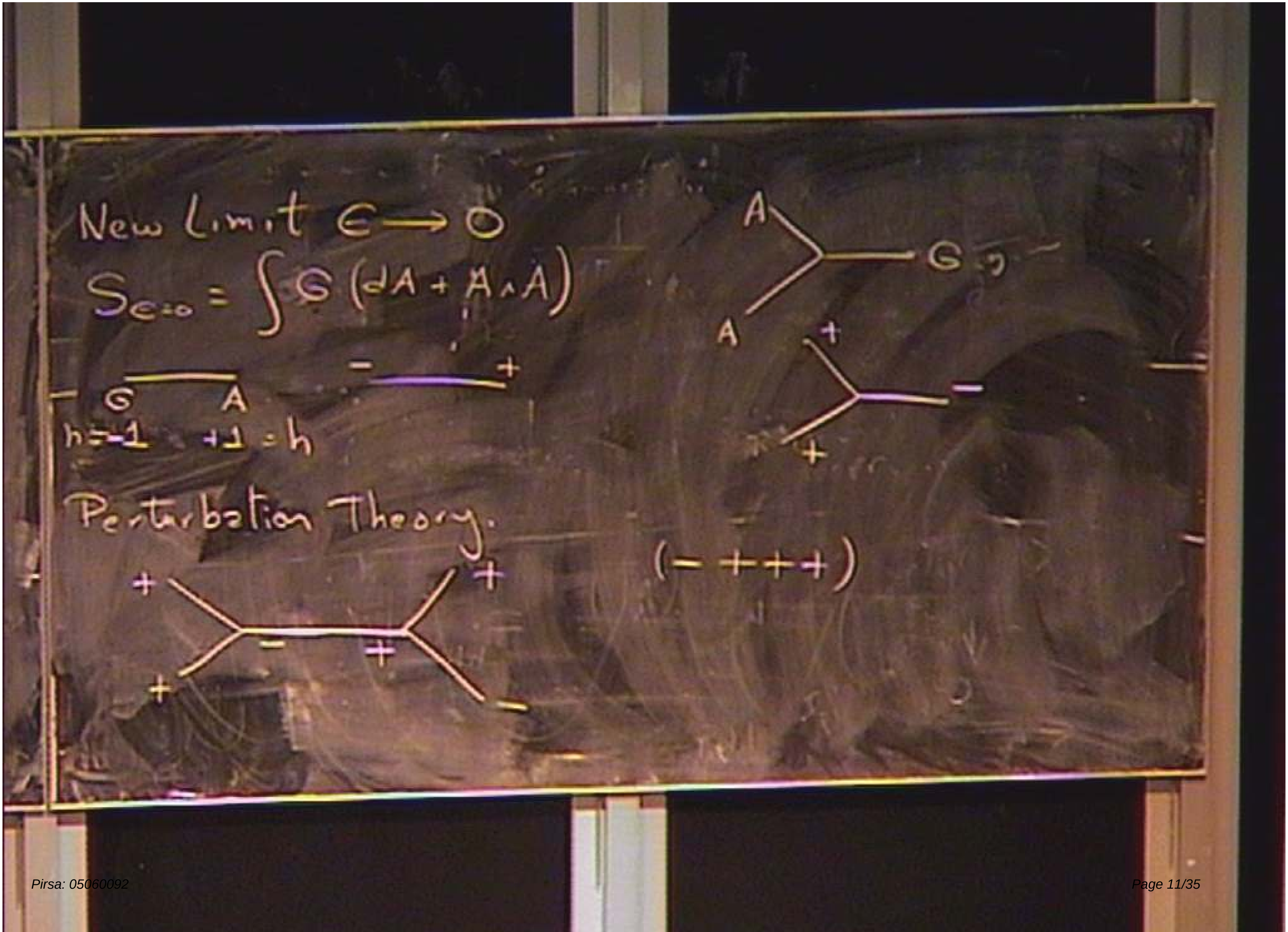
Integrate out G recover $\int d^4x \, \mathrm{tr} F^{+2}$.

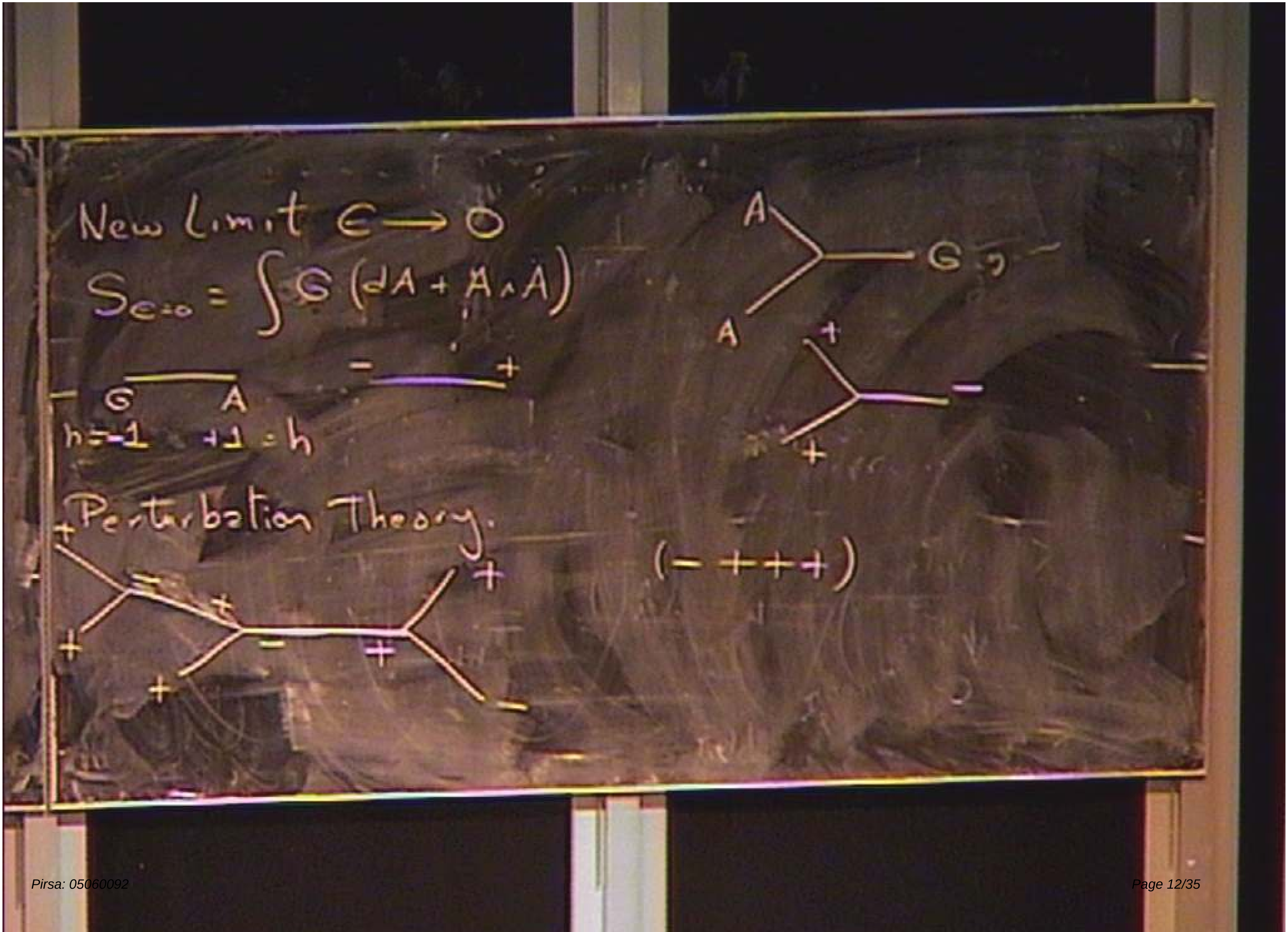
G - self dual part.

$\epsilon \to 0$.

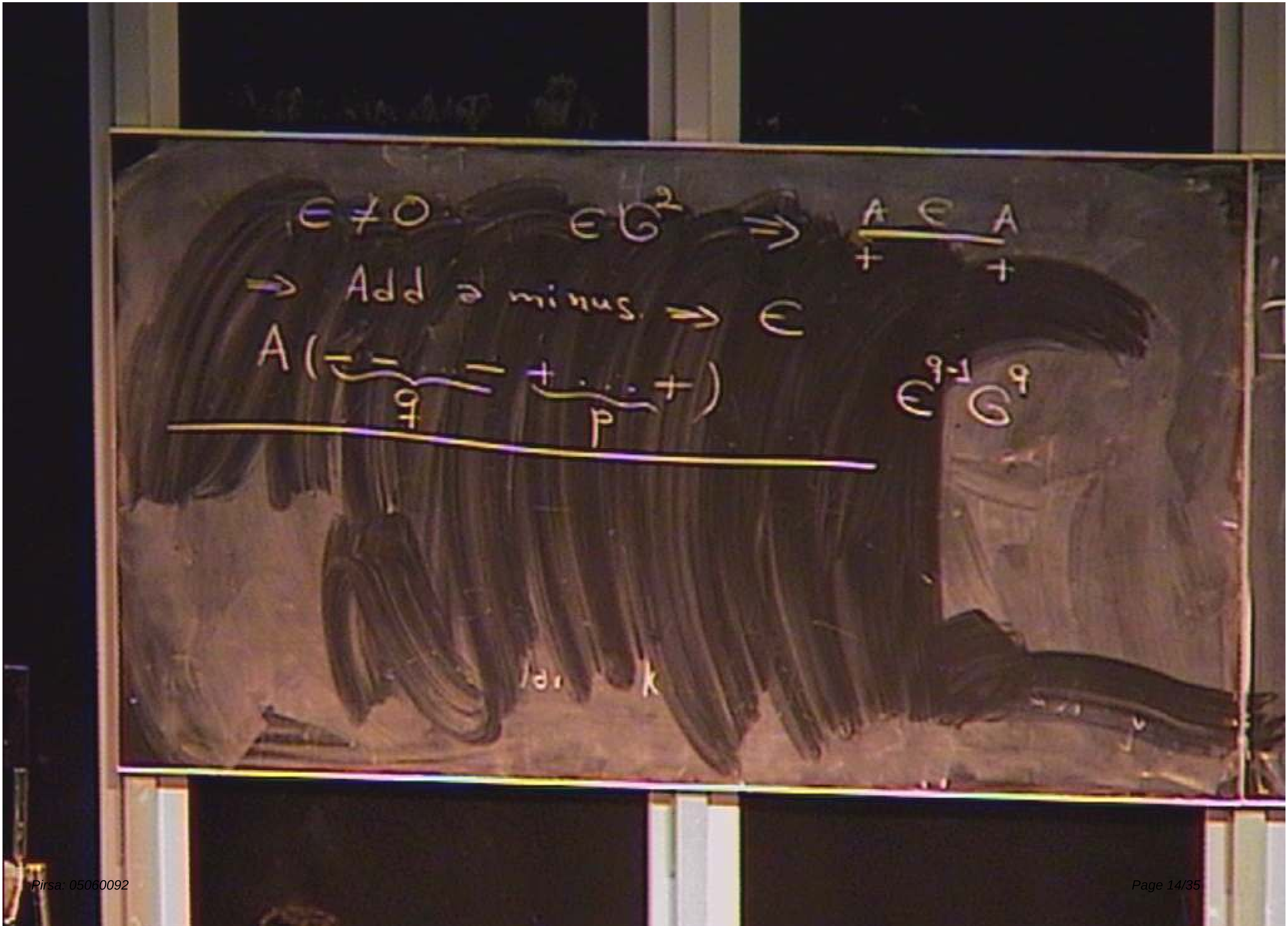

New Limit $\epsilon \longrightarrow 0$
$S_{\epsilon=0} = \int G \, (dA + A \wedge A)$


New Limit ε → 0
S_{ε=0} = ∫ G (dA + A∧A)
G A
h=-1 +1 = h
- +
A
A
G
+
+
-


New Limit ε ⟶ 0
S_{ε=0} = ∫ G (dA + A∧A)
G A
h=−1 +1 = h
Perturbation Theory.
(− + + +)


New Limit $\epsilon \longrightarrow 0$
$S_{\epsilon=0} = \int G\,(dA + A \wedge A)$
G
A
$h=-1$
$+1=h$
Perturbation Theory.
A
A
G
$(-\ +\ +\ +)$


ε≠0


ε ≠ 0   ε G²   ⇒   A ε A
+   +
⇒ Add a minus. ⇒ ε
A(− − ... − + ... +)   ε^{q-1} G^q
q   p

$\epsilon \neq 0$. $\quad \epsilon\, G^2 \Rightarrow \underset{+}{A}\ \epsilon\ \underset{+}{A}$

$\Rightarrow$ Add a minus. $\Rightarrow \epsilon$

$A(\underbrace{- - \cdots -}_{q}\ \underbrace{+ \cdots +}_{p}) \qquad \epsilon^{q-1} G^{q}$

$\epsilon \neq 0$ $\quad \epsilon G^2 \Rightarrow \dfrac{A \quad \epsilon \quad A}{+ \qquad\quad +}$

$\Rightarrow$ Add a minus. $\Rightarrow \epsilon$

$A(\underbrace{- - \cdots -}_{q} \; \underbrace{+ \cdots +}_{p})$ $\quad \epsilon^{q-1} G^{q}$

$U(1)$ Not a symmetry $\quad \epsilon^{d}$ $\quad \boxed{d = q-1}$

$\epsilon \neq 0 \qquad \epsilon G^2 \Rightarrow \dfrac{A \;\; \epsilon \;\; A}{+ \qquad\quad +}$

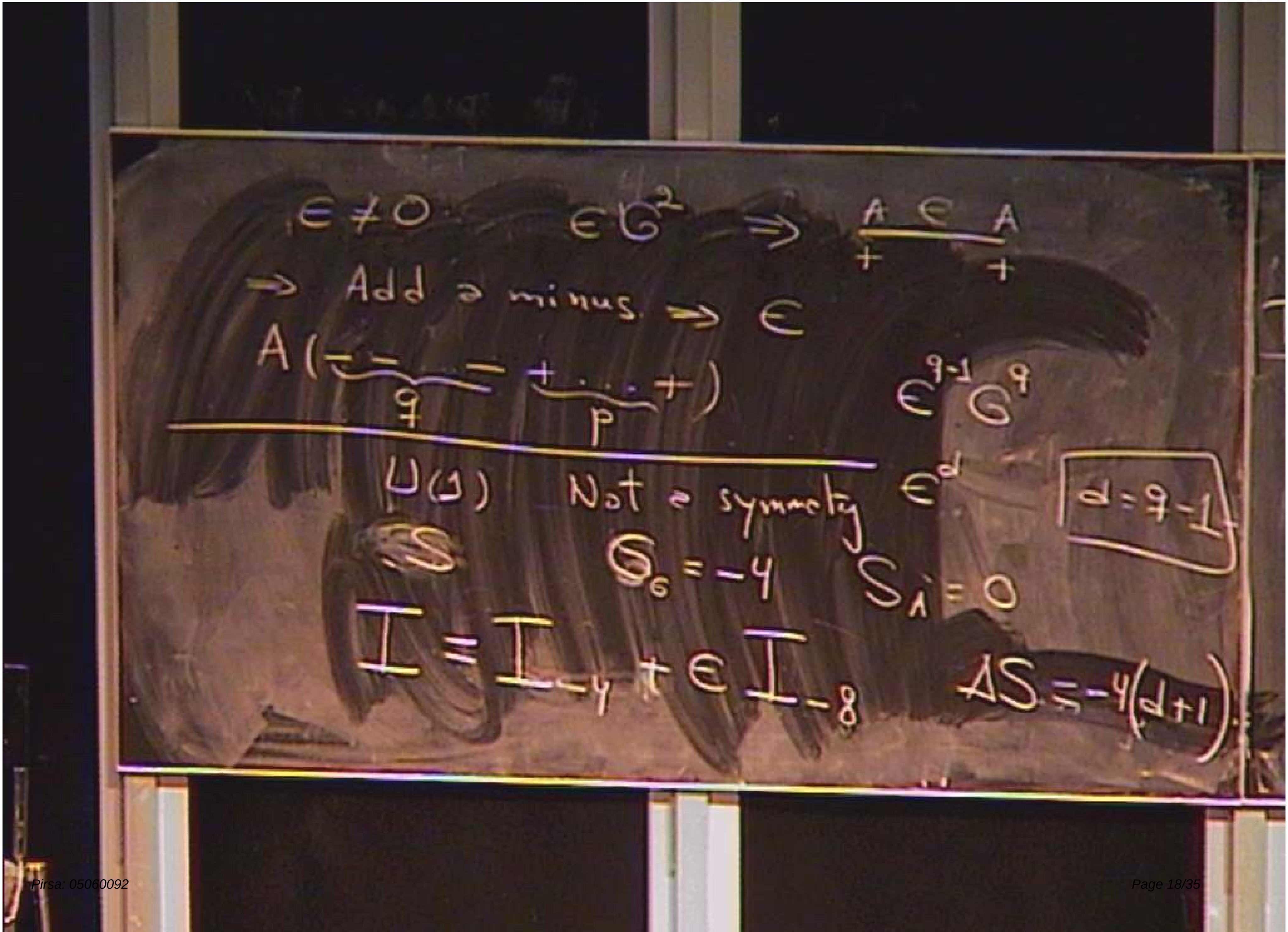
$\Rightarrow$ Add a minus. $\Rightarrow \epsilon$

$$\frac{A(\underbrace{- - \cdots -}_{q}\ \underbrace{+ \cdots +}_{p})}{U(1) \quad \text{Not a symmetry}} \qquad \epsilon^{q-1} G^{q} \qquad \epsilon^{d}$$

$$\boxed{d = q-1}$$

$S_6 = -4 \qquad S_1 = 0$

$$I = I_{-4} + \epsilon I_{-8}$$


ε ≠ 0
ε б²
⇒ Add a minus ⇒ ε
A (− − ... − + ... +)
q
p
ε^(q-1) б^q
U(1)
Not a symmetry
ε^d
d = q − 1
S₆ = −4
S₁ = 0
I = I₋₄ + ε I₋₈
ΔS = −4(d+1)

Twistor String Theory

1) TFT : $\mathcal{N}=2$ in $D=2$: $Q_{\alpha+}$, $Q_{\beta-}$

$SO(2)_{Lor.}$ $SO(2)_{Int.}$

Algebra

$\{Q_{\alpha+}, Q_{\beta-}\} = \gamma^{\mu}_{\alpha\beta} P_{\mu}$

$[J, Q_{\pm\alpha}] = \pm\frac{1}{2} Q_{\pm\alpha}$ $[R, Q_{\alpha\pm}] = \pm\frac{1}{2} Q_{\alpha\pm}$

Obs: $Q = Q_{+1} + i Q_{+2} + Q_{-1} \pm i Q_{-2}$ is nilpotent, i.e. $Q^2 = 0$

$Q \rightarrow$ BRST? No, Q is not a scalar!

Topological Twist:

$J' = J + R \Rightarrow Q$ is a scalar.

$\tilde{T}_{\mu\nu} = T_{\mu\nu} - \partial_{(\mu} J_{\nu)} = \{Q, \Lambda_{\mu\nu}\}$

Physical States : Cohomology of Q

$Q\varphi = 0 \qquad \varphi \sim \varphi' \text{ if } \varphi - \varphi' = Q\chi$

Physical Operators :

$[Q, \mathcal{O}] = 0 \qquad \mathcal{O} \sim \mathcal{O}' \text{ if } \mathcal{O}' - \mathcal{O} = \{Q, \nu\}$

Topological ?

$$\Rightarrow \underbrace{\delta_{g_{\mu\nu}} \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle}_{= 0} = \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \int \delta(\sqrt{g}\, g^{\mu\nu}) \underbrace{\tilde{T}_{\mu\nu}}_{[Q, \Lambda_{\mu\nu}]} \rangle$$

Metric independent!

2) $\mathbb{CP}^{M-1}$ model: $Z^I : \Sigma \to X$

$$I = \int d^2x \sqrt{\gamma} \left(\gamma^{\alpha\beta} g_{I\bar{J}} \frac{DZ^I}{Dx^\alpha} \frac{D\bar{Z}^{\bar{J}}}{Dx^\beta} + D\,\underbrace{\left(g_{I\bar{J}} Z^I \bar{Z}^{\bar{J}} - r \right)} \right)$$

$U(1)$

$\mathbb{CP}^{M-1}$: $Z^I \sim t Z^I \quad t \in \mathbb{C}^*$

$t = \rho e^{i\alpha}$

Gauge equivalence.

3) $\mathbb{CP}^{M-1|P}$ model:

(Z^I, ψ^A) $\quad$ $I: 1, \ldots, M$ $\quad$ $A: 1, \ldots, P$

$U(1): \psi^A \to e^{i\alpha} \psi^A$

Obs: If X is Kähler, then we have $\mathcal{N}=2$ theory.

Let's do the twist !

R-symmetries $(X, \theta, \bar{\theta})$

Axial: $\theta^+ \to e^{i\gamma}\theta^+ \quad \theta^- \to e^{-i\gamma}\theta^-$ → B-model

Vector: $\theta^\alpha \to e^{i\gamma}\theta^\alpha$ → A-model

Q: A or B ? $\quad \mathbb{CP}^{M-1|P}$

A → Any P

B → Only allows P = M

(For M = 4 ⇒ $\mathbb{CP}^{3|4}$)

~ $\mathcal{N}=4$ SYM

B-model :

Obs: Classically R_A is a symmetry.

QM R_A is anomalous unless

X is a $\underbrace{CY}$.

↳ Globally defined Holomorphic top form.

$$\Omega = \epsilon_{I_1 \ldots I_M} \epsilon_{A_1 \ldots A_P} Z^{I_1} dZ^{I_2} \wedge \ldots \wedge dZ^{I_M} \wedge d\psi^{A_1} \ldots \wedge d\psi^{A_P}$$

Invariant under U(1) iff P = M.

Obs: In the B-model $Q \sim \bar{\partial}$

$\Rightarrow$ Physical States $\sim$ Cohomology classes.

$$\cong$$

$$\bigoplus_{p,q} H^p(X, \Lambda^q TX)$$

$(0,p)$-forms on X with values in $\Lambda^q TX$.

A would-be twistor R-symmetry:

$$S : \psi^A \to e^{i\beta} \psi^A$$
$$Z^I \to Z^I$$

Obs: Since $\bar{\psi}$ does not appear

$$\Omega \to e^{-4i\beta} \Omega$$

$$\Rightarrow \quad S_\Omega = -4$$

(Recall YM theory $I = I_{-4} + \epsilon I_{-8}$)

4) Gauge group $U(N)$:

Add N D5-branes

$(Z^I, \bar{Z}^I, \psi^A)$ ~~$\bar{\psi}$~~

Physical fields: $(0,p)$-forms.

(It turns out that $p=1$ is physical & $p>1$ are ghosts)

Effective Action:

- A $(0,1)$-form. ~~$\bar{A}$ $(1,0)$-form~~
- Lagrangian $(3,3)$-form.
- Ω is a $(3,0)$-form.

$$I = \int_X \Omega_{3,0} \wedge \mathrm{tr}\left(A \bar{\partial} A + \frac{2}{3} A \wedge A \wedge A\right)$$

This is called the Holomorphic Chern-Simons action.

5) Write the SUSY version:

$$\mathcal{A} = d\bar{Z}^{\bar{I}}\Big(A_{\bar{I}} + \psi^A \chi_{\bar{I}A} + \psi^A\psi^B \phi_{\bar{I}AB}$$
$$+ \epsilon_{ABCD}\psi^A\psi^B\psi^C \tilde{\chi}^D_{\bar{I}} + \psi^1\psi^2\psi^3\psi^4 G\Big)$$

Action:

$$I = \int \Omega_{SUSY} \wedge \mathrm{tr}\left(\mathcal{A}\,\bar{\partial}\mathcal{A} + \tfrac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)$$

Obs: ψ^A are sections of $\mathcal{O}(1)$

$\mathcal{A}_{\bar{I}} \sim \mathcal{O}(0) \Rightarrow$

	A	χ	ϕ	$\tilde{\chi}$	G
	$\mathcal{O}(0)$	$\mathcal{O}(-1)$	$\mathcal{O}(-2)$	$\mathcal{O}(-3)$	$\mathcal{O}(-4)$
S:	0	-1	-2	-3	-4

Obs: Correlation functions: Ω_{Bos}, $\tilde{A}$, ϑ

Amplitude in Twistor Space   Wavef.

Recall:

$$\left(\lambda^a \frac{\partial}{\partial \lambda^a} + \mu^{\dot{a}}\frac{\partial}{\partial \mu^{\dot{a}}}\right)\tilde{A} = (-2-2h)\tilde{A}$$

$\Omega \sim \mathcal{O}(4)$ $\quad \tilde{A} \sim \mathcal{O}(-2-2h)$ $\quad \vartheta \sim \mathcal{O}(\ell)$

Need: $4 + (-2-2h) + \ell = 0$

Ans: NO! Recall the S-charges.

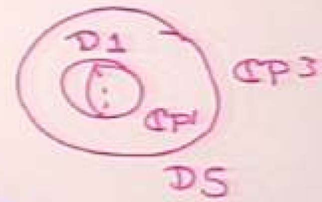
The action $\int \Omega \wedge \mathrm{tr}(\ldots)$ only has $S=-4$.

$\Rightarrow$ It only describes $I = I_{-4} + \epsilon I_{-8}$

(After all this we can only do $(-+++\ldots+)$)

Do not give up!

6) D-Instantons :



Obs: Apart from the $U(1)$,
there are modes that describe the motion of the instanton.

$\Rightarrow$ Parameterize the moduli space of holomorphic curves of a given degree in $\mathbb{CP}^3$.

Genus 0 : $\Phi : \mathbb{C} \longrightarrow \mathbb{CP}^{3|4}$

$(u^1, u^2) \qquad (Z^I, \psi^A)$

$$Z^I = c^I_{i_1 \dots i_d} u^{i_1} \dots u^{i_d}$$

$$\psi^A = \beta^A_{i_1 \dots i_d} u^{i_1} \dots u^{i_d}$$

Measure: $d\Gamma = \prod d c^I_{i_1 \dots i_d} \prod d\beta^A_{i_1 \dots i_d}$

S-charge :

$S_\psi = 1 \quad \Rightarrow S_\beta = 1$

$\Rightarrow \quad S_{d\Gamma} = -4(d+1)$

Great !

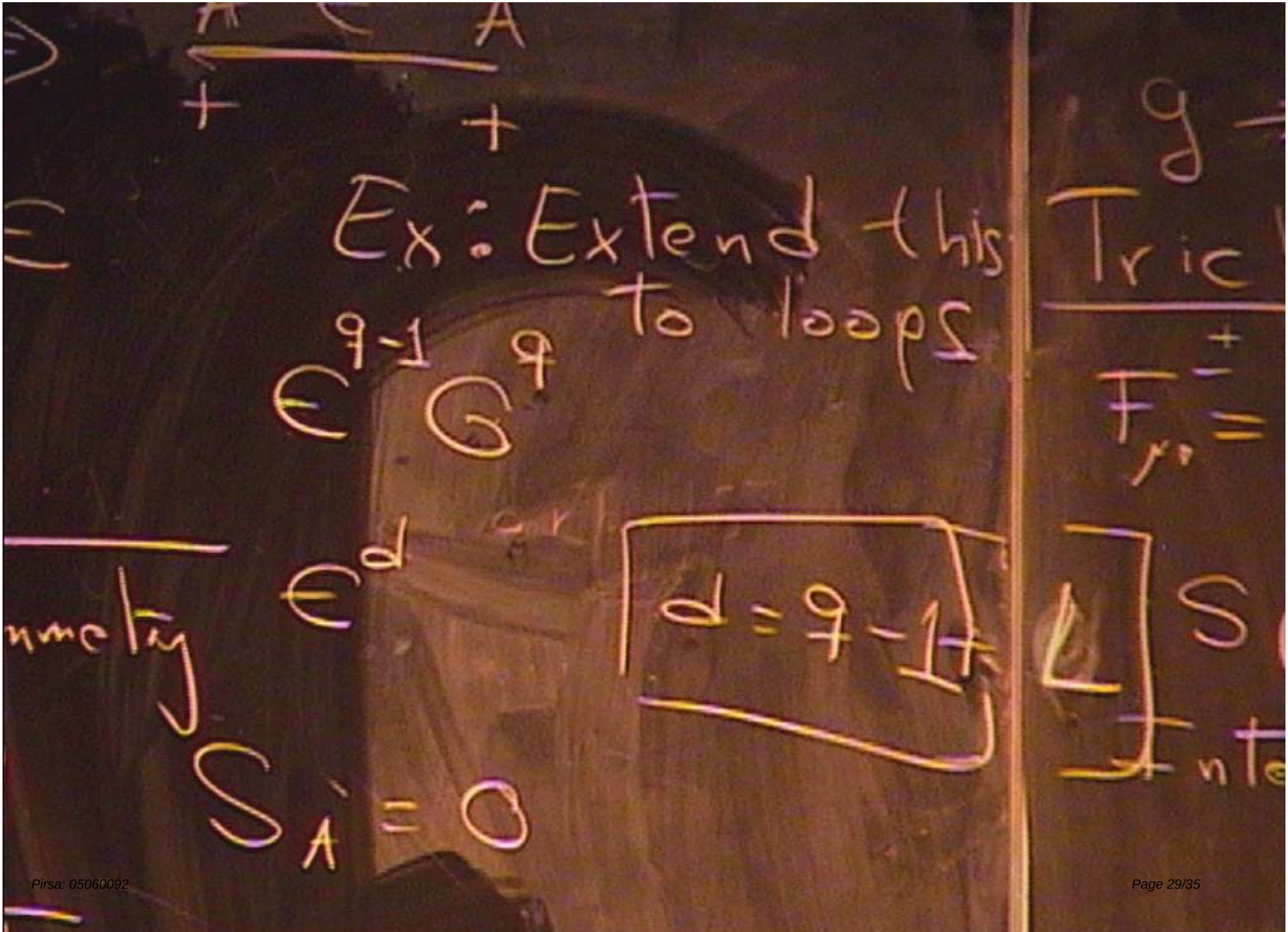
In order to get a non zero answer we need $d+1$ factors of G. $(\psi^1 \ldots \psi^4 G)$

$\Rightarrow q = d+1$ or $d = q-1$

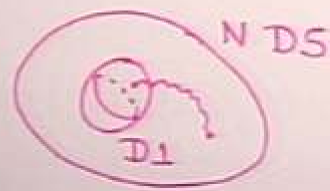
One can also argue that a D-instanton with $g>0$ & degree d has

$\Delta S = -4(d+1-g)$

$\Rightarrow d = q-1+g$


Ex: Extend this
to loops
d = q - 1

MHV amplitudes from Twistor String Theory



D1-D5 Strings

Recall: B-model, zero modes give (0, p)-forms.

$$I_{D1\text{-}D5} = \int \beta\,(\bar\partial + A)\,\alpha$$

$\beta \rightarrow$ D5-D1, $\bar{N}$

$\alpha \rightarrow$ D1-D5, N

$A \rightarrow$ External gauge field. (0,1) form.

$\Rightarrow$ Current $J_x{}^y = \alpha_x \beta^y dz$

Ext. Field $\mathcal{A} = \mathcal{A}_{\bar z}\, d\bar z$

Scattering amplitude:

$$A^{MHV}_{rs}(1,\dots,n) = \int_{\mathcal{M}} \langle V_{\phi_1} \dots V_{\phi_n} \rangle$$

$$V_{\phi_i} = \int_C \underbrace{J_i}_{\text{Current}}\ \underbrace{\phi_i}_{\text{Ext. wavefunct.}}$$

$$\text{MHV}: q=2 \Rightarrow d=1 \qquad (g=0) \quad (u^1, u^2)$$

$$z^I = C_i^I u^i$$

$$\psi^A = \beta_i^A u^i \qquad SL(2,\mathbb{C}) \Rightarrow u^i = \lambda^i \qquad i \leftrightarrow a$$

$$\text{Def}: \quad C_a^{\dot{a}} \equiv -x_a^{\dot{a}} \qquad \beta_a^A \equiv -\theta_a^A$$

$$\Rightarrow \mu_{\dot{a}} + x_{a\dot{a}}\lambda^a = 0 \qquad \psi^A + \theta_a^A \lambda^a = 0$$

$$A_{rs}^{MHV} = \int d^4x \, d^8\theta \, \langle V_{\phi_1} \ldots V_{\phi_n} \rangle$$

Extract color structure of wave functions

$$\phi_i = T_i \, v_i$$

$$\Rightarrow V_{\phi_i} = \int_C \mathrm{tr}\left(T_i \, \alpha \, \beta(\lambda_i)\right) v_i$$

Obs: Single trace contribution comes from contractions of $\beta(\lambda_i)$ with $\alpha(\lambda_{i+1})$

Fermion Correlator on a sphere :

$$\alpha^x(z)\,\beta_y(z') \sim \frac{\delta^x{}_y}{z-z'}$$

Obs : $dz_i \longleftrightarrow \langle \lambda_i\, d\lambda_i \rangle$ $\quad \frac{1}{z_i - z_{i+1}} \longrightarrow \frac{1}{\langle \lambda_i, \lambda_{i+1} \rangle}$

$$A^{MHV}_{rs} = Tr(T_1 \ldots T_n) \int d^4x\, d^8\theta \int \underbrace{\frac{\prod_{i=1}^{n} \langle \lambda_i\, d\lambda_i \rangle\, \vartheta_i}{\prod_{k=1}^{n} \langle \lambda_k, \lambda_{k+1} \rangle}}$$

- Homogeneous of degree 0 in each λ_i
- (1,0)-form in each λ_i
- $SL(2,\mathbb{C})$ invariant
- Simple poles in $\lambda_{i+1} = \lambda_i$

Wave function ϑ_i :

Scattering amplitude of plane waves

$$\phi(x) = e^{ip\cdot x} = e^{i x_{a\dot{a}} \zeta^a \tilde{\zeta}^{\dot{a}}}$$

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The corresponding ϑ_i in Twistor space is:

$$\vartheta_i(\lambda,\mu,\psi) = \bar{\delta}(\langle\lambda,\xi\rangle)\, \underbrace{e^{i[\bar{\xi},\mu]}}_{\substack{\text{Generates} \\ \text{a delta f.}}} \underbrace{g(\psi)}_{\substack{\text{Helicity} \\ \text{Info.}}}$$

Def: $\bar{\delta}(\langle\lambda,\xi\rangle)$ is a $(0,1)$-form such that

$$\int \langle\lambda\, d\lambda\rangle\, \bar{\delta}(\langle\lambda,\xi\rangle)\, B(\lambda) = B(\xi)$$

Helicity: Recall

$$\mathcal{A} = \underbrace{A}_{h=1} + \ldots + \psi^1\psi^2\psi^3\psi^4 \underbrace{G}_{-1}$$

$$g(\psi) = \begin{cases} 1 & h = + \\ \psi^1\psi^2\psi^3\psi^4 & h = - \end{cases}$$

$$\int \frac{\prod_{i=1}^{n} \langle \lambda_i\, d\lambda_i \rangle\, \vartheta_i}{\prod_{k=1}^{n} \langle \lambda_k, \lambda_{k+1} \rangle} = \frac{1}{\prod_{k=1}^{n} \langle \zeta_k, \zeta_{k+1} \rangle} \prod_{i=1}^{n} e^{i[\tilde{\zeta}_i, \mu_i(\zeta_i)]}\, g_i(\psi_{(\zeta_i)})$$

$$A(1^+, 2^+, \ldots, r^-, \ldots, s^-, \ldots, n-1^+, n^+) = A_{rs}^{MHV}$$

$$\int d^8\theta\, g_r(\psi_{(\zeta_r)})\, g_s(\psi_{(\zeta_s)}) = \int d^8\theta \prod_{A=1}^{4} \psi_r^A \prod_{B=1}^{4} \psi_s^B = \langle \zeta_r, \zeta_s \rangle^4$$

$$\psi_r^A = -\theta_a^A \zeta_r^a \qquad \psi_s^B = -\theta_a^B \zeta_s^a$$

$$\int d^4x \prod_{i=1}^{n} e^{i[\tilde{\zeta}_i, \mu_i(\zeta_i)]} = \int d^4x\, e^{i x_{a\dot{a}} \sum_{j=1}^{n} \zeta_j^a \tilde{\zeta}_j^{\dot{a}}} = \delta^{(4)}\Big(\sum_{j=1}^{n} \zeta_j^a \tilde{\zeta}_j^{\dot{a}}\Big)$$

$$\mu_{i\dot{a}} = -x_{a\dot{a}} \zeta_i^a$$

Finally:

$$A_{rs}^{MHV} = \mathrm{tr}(T_1 \ldots T_n)\, \delta^{(4)}\Big(\sum_{j=1}^{n} \zeta_j^a \tilde{\zeta}_j^{\dot{a}}\Big) \frac{\langle \zeta_r, \zeta_s \rangle^4}{\prod_{k=1}^{n} \langle \zeta_k, \zeta_{k+1} \rangle}$$