

Title: Complex Manifolds

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Abstract:

Hypersurfaces in P^N

Let $f(Z)$ be a hom. function of degree d in P^N

$f=0$ defines a smooth variety X

if $f=df=0$ has no solutions

Hodge Diamond of the Quintic CY 3-fold

$$\mathcal{O}_{P^3} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}$$

Hodge Diamond of the Quintic $\mathbb{C}P^3$ fold

$$\longrightarrow \mathcal{O}_{\mathbb{P}^3} \xrightarrow[\text{to } X]{\text{restrict}} \mathcal{O}_X \longrightarrow \mathcal{O}$$

Hodge Diamond of the Quintic $\mathbb{C}Y$ 3-fold

$$\xrightarrow{\times f} \mathcal{O}_{\mathbb{P}^N} \xrightarrow[\text{to } X]{\text{restrict}} \mathcal{O}_X \rightarrow \mathcal{O}$$

Hodge Diamond of the Quintic CY 3-fold

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-d) \xrightarrow{\times f} \mathcal{O}_{\mathbb{P}^n} \xrightarrow[\text{to } X]{\text{restrict}} \mathcal{O}_X \rightarrow 0$$

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If X is a Calabi-Yau manifold of complex dim n ,
then $H^n(X, \mathcal{O}_X) = \mathbb{C}$

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$$-n = N - 1$$

$$H^n(X, \mathcal{O}_X) = H^n(\mathbb{P}^N, \mathcal{O}_X)$$

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-d) \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_X \rightarrow 0$$

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-d) \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_X \rightarrow 0$$

H^0

$$0 \rightarrow 0 \rightarrow \mathbb{C} \rightarrow \mathbb{C}$$

H^2

$$\rightarrow 0 \rightarrow 0 \rightarrow 0$$

$\vdots$

$$\rightarrow 0 \rightarrow \mathbb{C}^{(d-1) \atop (n+1)}$$

$$0 \rightarrow 0$$

$$\mathbb{C}^{(d-1) \atop (n+1)}$$

$$H^*(X, \mathcal{O}_X)$$

$(d \atop n)$

$$= \mathbb{C}$$

H^n
 H^{n+1}

$$0 \rightarrow \mathcal{O}_{P^n}(-d) \rightarrow \mathcal{O}_{P^n} \rightarrow \mathcal{O}_X \rightarrow 0$$

 H^0

$$0 \rightarrow 0 \rightarrow \mathbb{C} \rightarrow \mathbb{C} \rightarrow 0$$

 H^1

$$0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0$$

 H^2

$$0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0$$

 H^{n+1}

$$0 \rightarrow \mathbb{C}^{(d-1)}_{n+1} \rightarrow 0 \rightarrow 0 \rightarrow 0$$

 $H^*(X, \mathcal{O}_X)$
 $(d)_n$

$$X \xrightarrow{s} \mathbb{C}^n \xrightarrow{d} \mathbb{C}^n$$

$$d = n+2$$

$\bar{n}$	d	
1	3	Elliptic Curve
2	4	K3 Surface
3	5	Quintic 3-fold

$$h^{p,q} = \dim H^q(X, \Omega_X^p)$$

$T_X^?$

Adjunction

$$0 \rightarrow T_X \rightarrow T_W$$

IC

$$W = \mathbb{P}^n$$

$$h^{p,q} = \dim H^q(X, \Omega_X^p)$$

T_X ?

Adjunction

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$$W = \mathbb{P}^n$$

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$$0 \rightarrow T_X \rightarrow T_{W|X} \rightarrow \mathcal{N} \rightarrow 0$$

$\mathcal{N}$ Normal bundle of $X \subset W$

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$\mathcal{N}$ Normal bundle of $X \subset W$

$$0 \rightarrow T_X \rightarrow T_{W|X} \rightarrow \mathcal{O}_W(-1)|_X \rightarrow 0$$

$$W = \mathbb{P}^n$$

$$h^{p,q} = \dim H^q(X, \Omega_X^p)$$

T_X ?

Adjunction

$$0 \rightarrow T_X \rightarrow T_{W|X} \rightarrow \mathcal{N} \rightarrow 0$$

$\mathcal{N}$ Normal bundle of $X \subset W$

$$0 \rightarrow T_X \rightarrow T_{W|X} \rightarrow \mathcal{O}_W(s)|_X \rightarrow 0$$

On a Calabi-Yau manifold of dim n

$\Omega^n = \wedge^n T_x^*$ is a trivial line bundle

$$\Rightarrow \wedge^p T_x \cong \wedge^{n-d} T_x^* = \Omega_x^{n-d}$$

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$\Omega^n = \wedge^n T_x^*$ is a trivial line bundle

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$$m-H^q(X, \mathcal{O}_X) = \mathbb{C}^{0,q}$$

Now consider $H^q(X, \mathcal{T}_X) = H^q(X, \Omega_X^2)$

Cohomology of $\mathcal{O}_W(5)|_X = \mathcal{O}_X(5)$

$$H^*(X, \mathcal{O}_X(5)) = \mathbb{C}^{125}, 0, 0,$$

$$\begin{array}{ccccccc}
 & 0 & \longrightarrow & \mathcal{O}_W & \longrightarrow & \mathcal{O}_W(5) & \longrightarrow \mathcal{O}_X(5) \longrightarrow 0 \\
 H^0 & 0 & \longrightarrow & \mathbb{C} & \xrightarrow{\quad} & \mathbb{C}^{126} & \longrightarrow \mathbb{C}^{125} \\
 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
 \end{array}$$

$$0 \rightarrow \mathcal{O}_W(-s) \rightarrow \mathcal{O}_W \rightarrow \mathcal{O}_X \rightarrow 0$$

"Tensor by $\mathcal{I}_W$ "

$$0 \rightarrow \mathcal{I}_W(-s) \rightarrow \mathcal{I}_W \rightarrow \mathcal{I}_W|_X \rightarrow 0$$

$$H^*(W, \mathcal{I}_W) = \mathbb{C}^{24}, 0, 0, 0, \dots$$

$$0 \rightarrow \mathcal{O}_W(-5) \rightarrow \mathcal{O}_W(-4)^{\oplus 5} \rightarrow \mathcal{I}_W(-5) \rightarrow 0$$

$$\begin{array}{ccccccc} 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\ 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\ 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\ 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\ \mathbb{C} & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \end{array}$$

H^3

H^4

$$H^*(W, \mathcal{I}_w) = \mathbb{C}^{2n}, 0, 0, 0, \dots$$

$$H^*(W, \mathcal{I}_w(-5)) = 0, 0, 0, \mathbb{C}$$

$$0 \longrightarrow \mathcal{O}_w(-5) \longrightarrow \mathcal{O}_w(-4)^{\oplus 5} \longrightarrow \mathcal{I}_w(-5) \longrightarrow 0$$

 H^0
 0
 $\longrightarrow$
 0
 $\longrightarrow$
 0
 H^1
 0
 $\longrightarrow$
 0
 $\longrightarrow$
 0
 H^2
 0
 $\longrightarrow$
 0
 $\longrightarrow$
 0
 H^3
 0
 $\longrightarrow$
 0
 $\longrightarrow$
 $\mathbb{C}$
 H^4
 $\mathbb{C}$
 $\longrightarrow$
 0
 $\longrightarrow$
 0

$$H^*(X, \mathcal{T}_w|_X) = \mathbb{C}^{24}, 0, \mathbb{C}, 0,$$

$$0 \rightarrow \mathcal{T}_w(-5) \xrightarrow{\quad} \mathcal{T}_w \xrightarrow{\quad} \mathcal{T}_w|_X \rightarrow 0$$

$$H^0$$

$$0$$

$$\mathbb{C}^{24}$$

$$\mathbb{C}^{24}$$

$$H^1$$

$$0$$

$$0$$

$$0$$

$$H^2$$

$$0$$

$$0$$

$$\mathbb{C}$$

$$H^3$$

$$\mathbb{C}$$

$$0$$

$$0$$

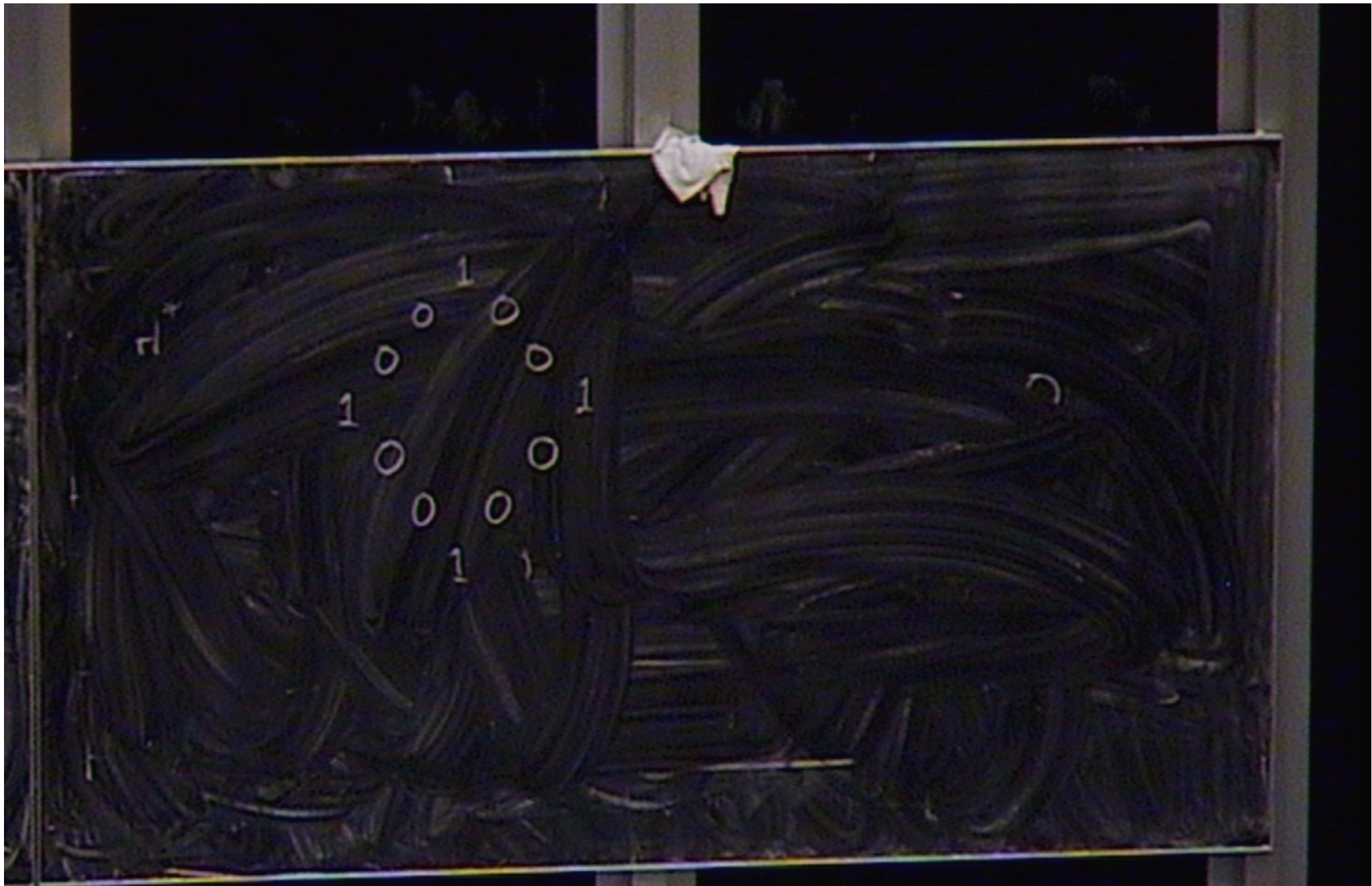
$$H^4$$

$$0$$

$$0$$

$$0$$

$$\dim H^0(X, \mathcal{T}_X) = \dim H^0(X, \Omega_X^2) = h^{2,0} \\ = h^{0,2} = \dim H^2(X, \mathcal{O}_X) = 0$$



Hodge Numbers
 H^* of Quadric

		1		
	0		0	
	0	1	0	
1	101		101	1
	0	1	0	
	0		0	
	1		1	

Deformations of complex structure

Complex structure J^μ_ν , satisfies $J^\mu_\nu J^\nu_\rho = -\delta^\mu_\rho$

and $N^\mu_{\nu\bar{\rho}} = 0$

In cx coords $J^j_k = -i\delta^j_k, \bar{J}^{\bar{j}}_{\bar{k}} = -i\delta^{\bar{j}}_{\bar{k}}, J^j_{\bar{k}} = \bar{J}^{\bar{j}}_k = 0$

$$J^\mu_\nu \rightarrow J^\mu_\nu + T^\mu_\nu$$

$$J^2 = -1 \Rightarrow T^j_k = T^{\bar{j}}_{\bar{k}} = 0$$

leaving $T^j_{\bar{k}}$

$$\text{Let } T = T^j_{\bar{k}} dz^{\bar{k}} \frac{\partial}{\partial z^j}$$

Then $N^\mu_{\nu 3} = 0 \Rightarrow \bar{\partial} T = 0$

If $T = \bar{\partial} v$ for some v

Then $N''_{\alpha\beta} = 0 \Rightarrow \bar{\partial}T = 0$

If $T = \bar{\partial}v$ for some v , then this change in "cr str." is equivalent to a change of coords.

$$Z^j \rightarrow Z^j + v^j.$$

(e) a trivial change in cr str.

Deformation of cx str to 1st order given by $H^1(X, \mathcal{T}_X)$

Deformation of cx str to 1st order given by $H^1(X, \mathcal{T}_X)$
For a CY manifold, all 1st order defs are true defs

H^0
 H^1

$$0 \rightarrow \mathcal{T}_X \rightarrow \mathcal{T}_{W|X} \rightarrow \mathcal{O}_X(5) \rightarrow 0$$

$$\mathbb{C}^{101}$$

There are 101 def's of ex str

and
in $\mathcal{T}_X$

$$Z_0^5 + Z_1^5 + Z_2^5 +$$

$$+ 56.5 Z_0^3 Z_4 Z_5 + \dots Z_0 Z_1 Z_2 Z_3 Z_4 +$$

$$Z_0^5 + Z_1^5 + Z_2^5 +$$

$$+ 56 \cdot 5 Z_0^3 Z_4 Z_5 + \dots Z_0 Z_1 Z_2 Z_3 Z_4 +$$

$$126 - 1 = 125$$

$$Z_0^5 + Z_1^5 + Z_2^5 +$$

$$+ 56 \cdot 5 Z_0^3 Z_4 Z_5 + \dots Z_0 Z_1 Z_2 Z_3 Z_4$$

$$126 - 1 = 125$$

$$\dim \mathbb{PGL}(5, \mathbb{C}) = 24$$

$$\hline 101$$

H^0
 H^1

$$0 \rightarrow \mathcal{T}_X \rightarrow \mathcal{T}_{W|X} \rightarrow \mathcal{O}_X(5) \rightarrow 0$$

$$\begin{array}{c} 0 \\ \boxed{\mathbb{C}^{101}} \end{array}$$

$$\boxed{\mathbb{C}^{24}}$$

$$\boxed{\mathbb{C}^{125}}$$

$\leftarrow$ There are 101 def's of ex str
Polynomial deformations

and
 in $\mathcal{T}_X$

$W = \mathbb{P}^3$ $X = K3$ surface, $d=4$

$$0 \rightarrow T_X \rightarrow T_{W|X} \rightarrow \mathcal{O}_X(4) \rightarrow 0$$

$W = \mathbb{P}^3$ $X = K3$ surface, $d=4$

$$0 \rightarrow T_X \rightarrow T_{W|X} \rightarrow \mathcal{O}_X(4) \rightarrow 0$$

$$\begin{array}{ccccc} 0 & \rightarrow & \mathbb{C}^{15} & \rightarrow & \mathbb{C}^{34} \\ \boxed{\mathbb{C}^{20}} & \rightarrow & \mathbb{C} & \rightarrow & 0 \\ 0 & \rightarrow & 0 & \rightarrow & 0 \end{array}$$

$\dim H^1(X) = 20.$

$W = \mathbb{P}^3$ $X = K3$ surface, $d=4$

$$0 \rightarrow T_X \rightarrow T_{W|X} \rightarrow \mathcal{O}_X(4) \rightarrow 0$$

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$\dim H^1(X, T_X) = 20$. but only 19 defs of Quartic Eqn