Title: Complex manifolds

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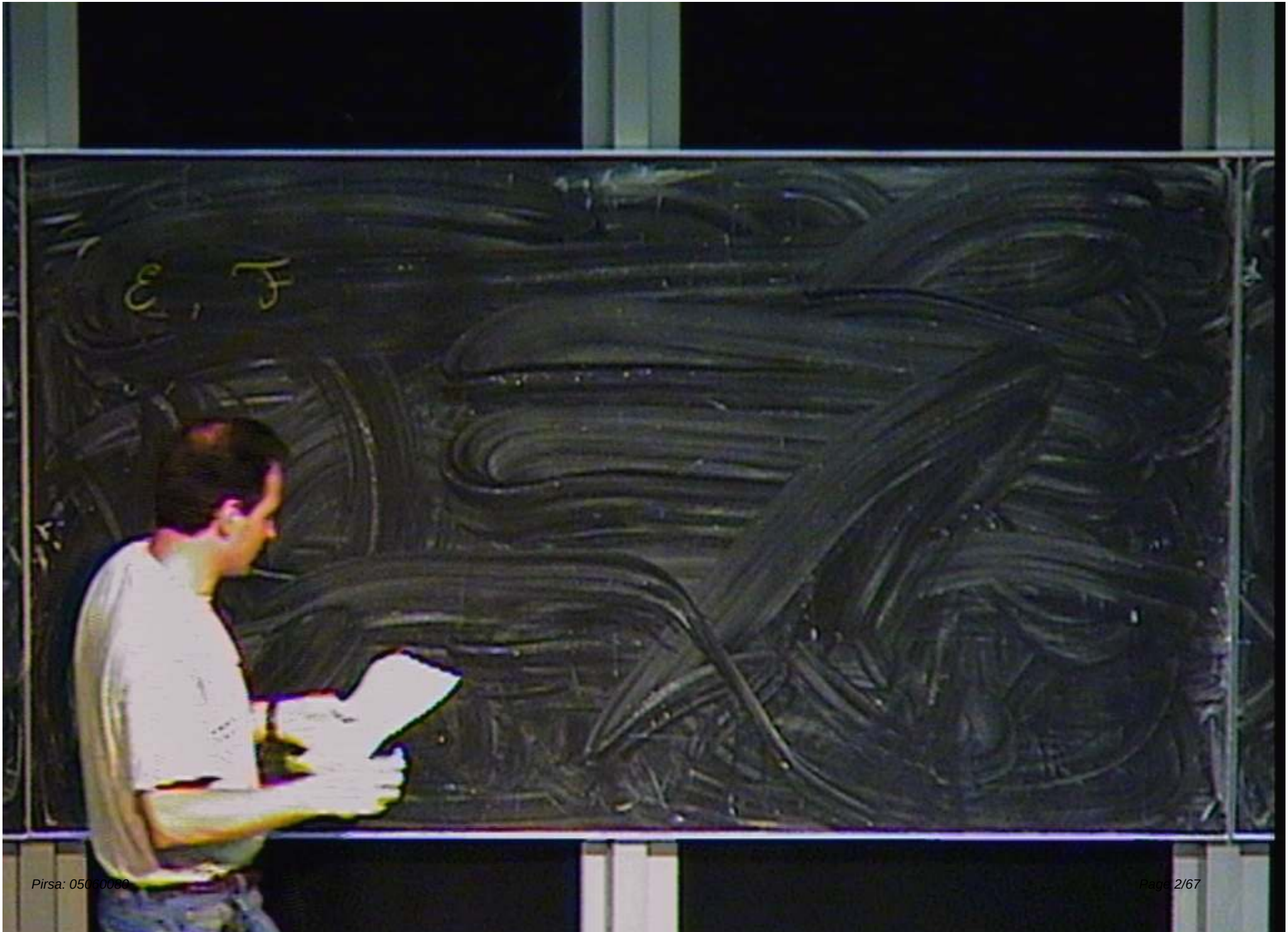
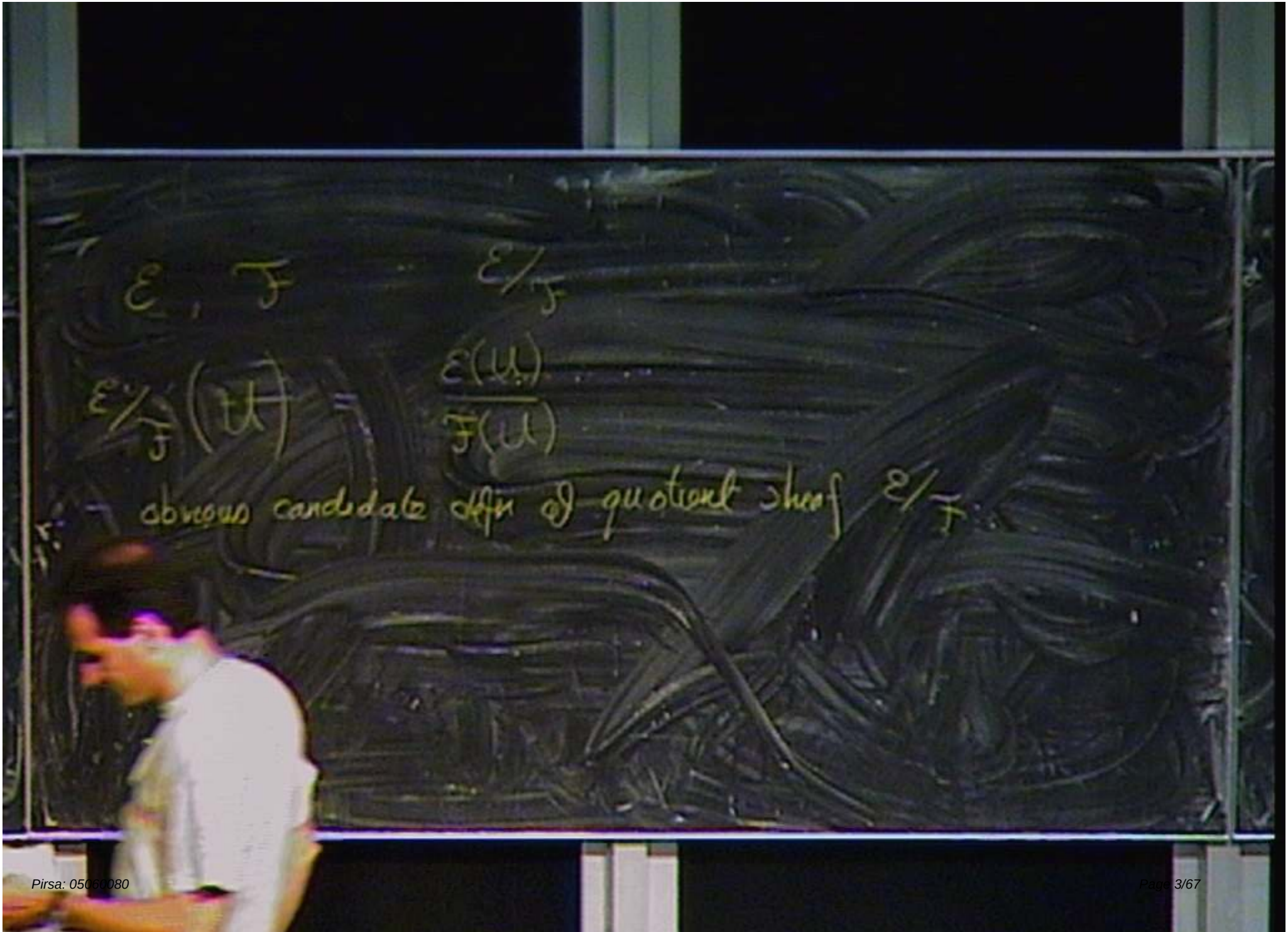
URL: http://pirsa.org/05060080

Abstract:




$\mathcal{E}$, $\mathcal{F}$
$\mathcal{E}/\mathcal{F}$
$\mathcal{E}/\mathcal{F}(U) = \frac{\mathcal{E}(U)}{\mathcal{F}(U)}$
obvious candidate defn of quotient sheaf $\mathcal{E}/\mathcal{F}$

Let $\mathcal{O}$ be the sheaf of hol functions (+)

Let $\mathcal{O}^*$ " " " nowhere zero hol functions (×)

$$\mathcal{O} \xrightarrow{\exp(2\pi i \cdot -)} \mathcal{O}^*$$

This is a sheaf map

The coker of any map $\mathcal{E} \xrightarrow{f} \mathcal{F}$ is defined as $\mathcal{F}/_{\mathrm{Im}(f)}$

Let $X = \mathbb{C}^* = \mathbb{C} - \{0\}$

Let z be the coord of $\mathbb{C}^*$

Clearly z is in $\mathcal{O}^*(X)$

z is NOT in the image of $\exp(2\pi i -)$

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z is NOT in the image of $\exp(2\pi i -)$

$\Rightarrow z$ is a natural element of coker $= \mathcal{O}^*/f(\mathcal{O})(X)$

Suppose $U \subset \mathbb{C}^*$ is a contractible open set

Then $z|_U$ is in the image of $\exp(2\pi i -)$

$\Rightarrow z|_U = 0$ in $\mathcal{F}(U)$

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Clearly z is in $\mathcal{O}^*(X)$

z is NOT in the image of $\exp(2\pi i -)$

$\Rightarrow$ z is a natural element of coker $= \dfrac{\mathcal{O}^*(X)}{f(\mathcal{O})(X)} = \mathcal{F}(X)$

Suppose $U \subset \mathbb{C}^*$ is a contractible open set.

Then $z|_U$ is in the image of $\exp(2\pi i -)$

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But X can be written as a union of two contractible open sets.

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We may fix this problem by considering only "very small" contractible open sets.

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We may "sheafify" any presheaf by focussing just on these small open sets.

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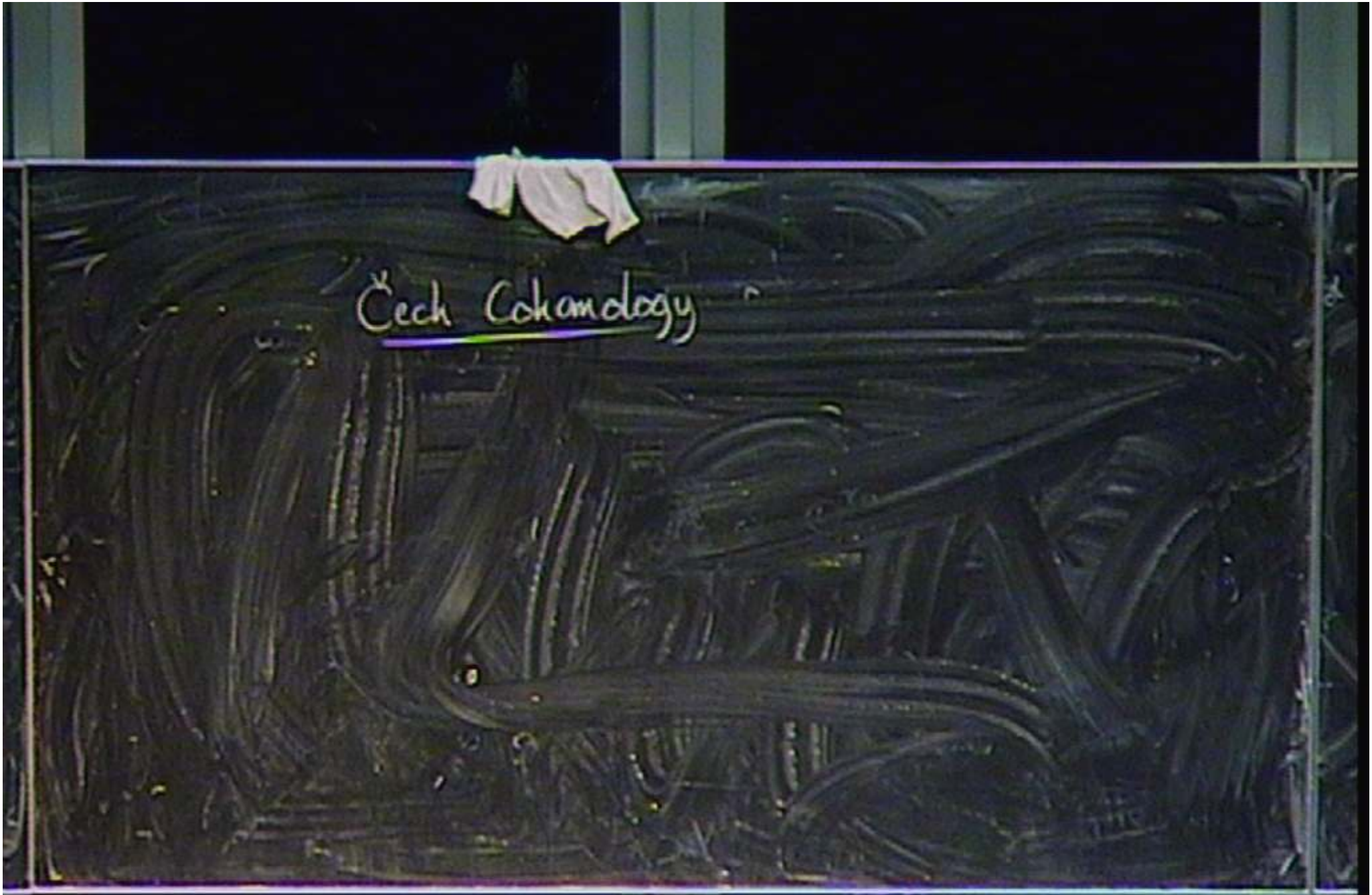
$\mathrm{coker}\,(\mathcal{O} \xrightarrow{\exp} \mathcal{O}^*)$ is trivial


Čech Cohomology

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Let $\underline{\mathcal{U}} = \{\mathcal{U}_\alpha\}$ be an open cover of M

Define $\check{C}^0(\underline{\mathcal{U}}, \mathcal{F}) = \prod_\alpha \mathcal{F}(\mathcal{U}_\alpha)$

$\check{C}^1(\underline{\mathcal{U}}, \mathcal{F}) = \prod_{\alpha<\beta} \mathcal{F}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$

$$\check{C}^p(\underline{\mathcal{U}}, \mathcal{F}) = \prod_{\alpha_0 < \alpha_1 < \cdots < \alpha_p} \mathcal{F}(\mathcal{U}_{\alpha_0} \cap \mathcal{U}_{\alpha_1} \cdots \mathcal{U}_{\alpha_p})$$

- groups of Čech cochain

If $\sigma \in \check{C}^p(\underline{\mathcal{U}}, \mathcal{F})$, let $\sigma_{\alpha_0 \alpha_1 \cdots \alpha_p} \in \mathcal{F}(\mathcal{U}_{\alpha_0} \cap \mathcal{U}_{\alpha_p})$

Coboundary operator:

$$\delta : \check{C}^p(\underline{U}, \mathcal{F}) \longrightarrow \check{C}^{p+1}(\underline{U}, \mathcal{F})$$

$$(\delta\sigma)_{\alpha_0 \alpha_1 \ldots \alpha_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{\alpha_0 \ldots \hat{\alpha}_j \ldots \alpha_{p+1}}$$

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σ is a cocycle if $\delta\sigma = 0$

coboundary if $\sigma = \delta\tau$

It is easy to prove $\delta^2 = 0$

Cochain complex

$$0 \longrightarrow C^0 \xrightarrow{\delta_0} C^1 \xrightarrow{\delta_1} C^2 \xrightarrow{\delta_2}$$

Čech cohomology

$$\check{H}^p(\underline{U}, \mathcal{F}) = \frac{\operatorname{Ker} \delta_p}{\operatorname{Im} \delta_{p-1}}$$

- depends on $\underline{U}$

If the open sets are "very small" $\check{H}^p(\underline{U}, \mathcal{F})$ is independent of $\underline{U}$

- This gives $\check{H}(X, \mathcal{F})$

$\check{H}^1(\underline{U}, \mathcal{F}) = \check{H}^1(X, \mathcal{F})$ for all $\mathcal{F}$

if $\check{H}^k(U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}, \mathcal{F}) = 0$ for all $k > 0$

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Eg $\check{H}^1(\mathbb{P}^1, \mathcal{O}(k))$

In this case we may use $U_0 = \mathbb{P}^1 - \{[0,1]\}$

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$\check{C}^p(\mathcal{U}, \mathcal{O}(k)) = 0$ for $p \geq 2$

So $H^p(\mathbb{P}^2, \mathcal{O}(k)) = 0$ for $p \geq 2$

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= global sections

$\check{H}^0(\mathbb{P}^1, \mathcal{O}(k))$ = functions on U_0 and U_1 which agree (after restriction to overlap)

= global sections (this is always the case)

On U_0, let $z = \frac{Z_1}{Z_0}$ be affine coordinate

and let $z^m \in \mathcal{F}(U_0)$ be a local section

$U_{01} = U_0 \cap U_1$

Clearly $m \geq 0$ for z^m to be hol. in U_0

We may define $\mathcal{O}(k)$ by $\Gamma_{0,01} = 1$

$\Gamma_{1,01} = z^k$

$U_{01} = U_0 \cap U_1$

Clearly $m \geq 0$ for z^m to be hol. in U_0

We may define $\mathcal{O}(k)$ by $r_{0,01} = 1$

$r_{1,01} = z^k$

Let $w = z^{-1} = \frac{Z_0}{Z_1}$ be affine coord on $U_1 = \mathbb{C}$

and let W^ℓ is a local section in $\mathcal{O}(k)(U_1)$

So, the coboundary of this 0-cochain is

$$(\delta\sigma)_{01} = w^l|_{01} - z^m|_{01} = w^l z^k - z^m$$

So we have a cocycle if $l = k - m$

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I.e. m has $k+1$ values

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and let $z^m \in \mathcal{F}(U_0)$ be a local section

$\Rightarrow \dim H^0(\mathbb{P}^1, \mathcal{O}(k)) = k+1 \quad \text{if } k \geq 0$

$= 0 \quad \text{if } k < 0$

We may also write a global section as

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$Z_0^{k-m} Z_1^m$ in hom. coords (homogeneous fn of degree k)

we set $Z_0 = 1$ in U_0

$Z_1 = 1$ in U_1

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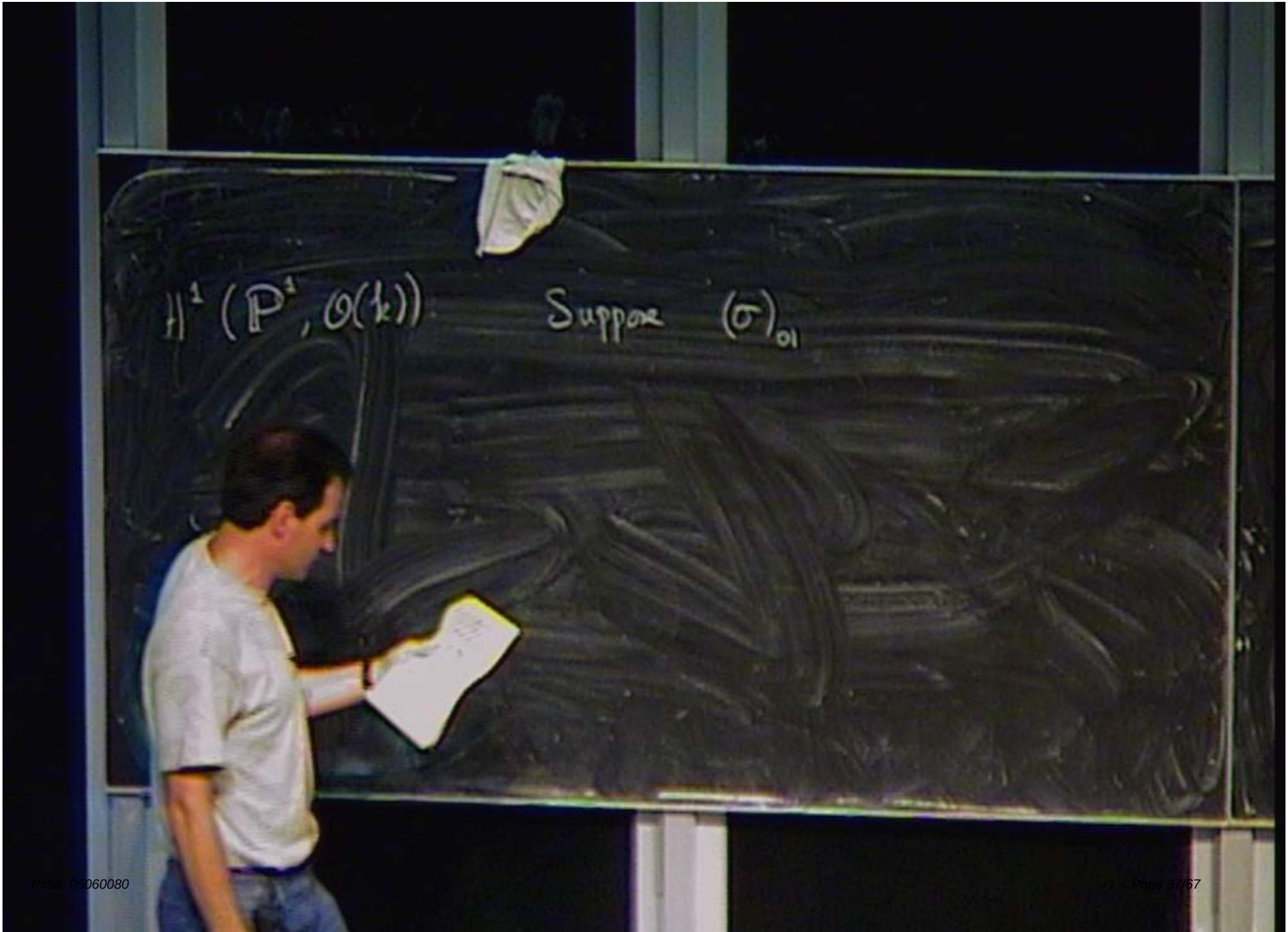
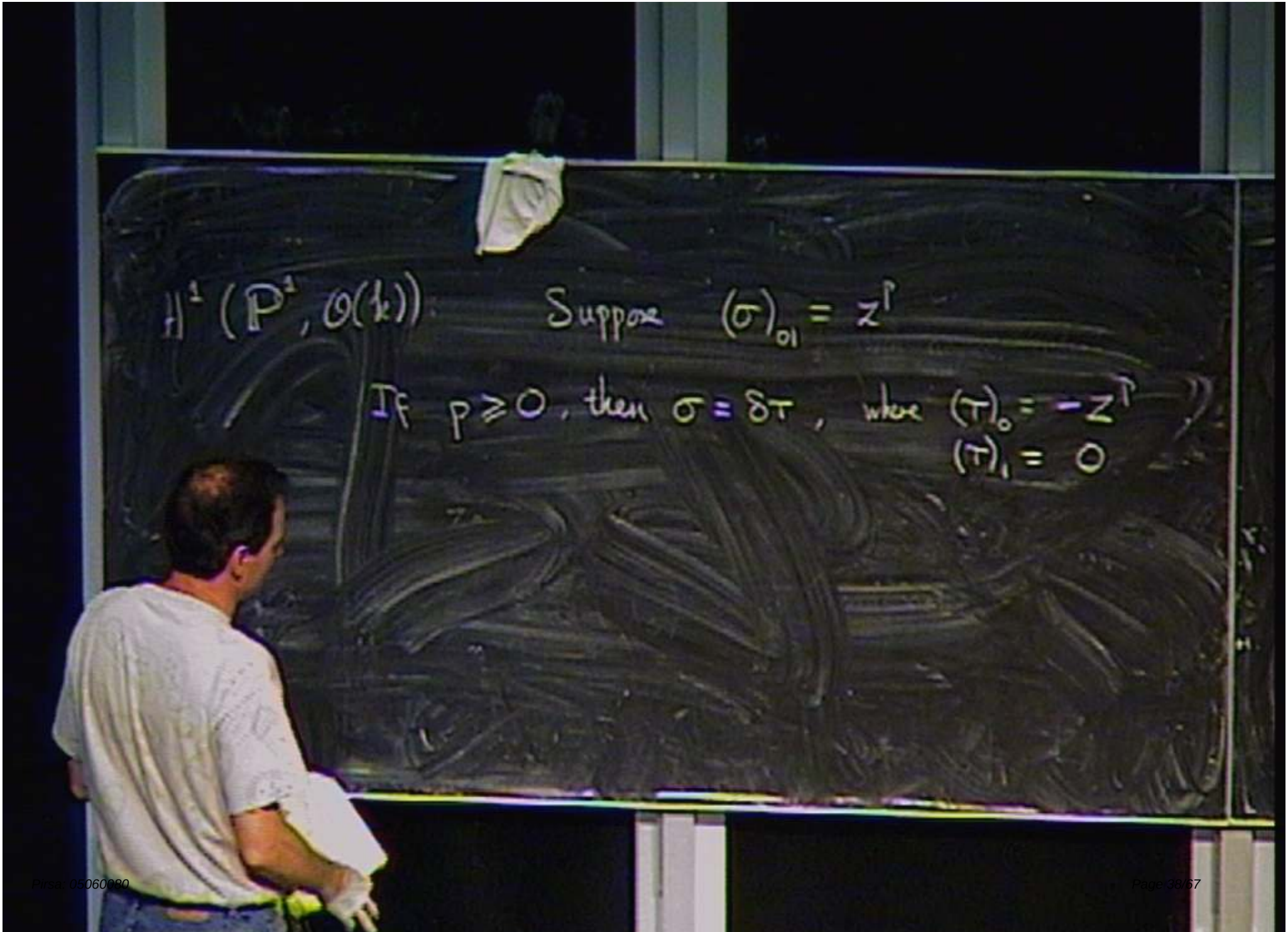

$H^1(\mathbb{P}^1, \mathcal{O}(k))$
Suppose
$(\sigma)_{01}$


$H^1(\mathbb{P}^1, \mathcal{O}(k))$
Suppose $(\sigma)_{01} = z^r$
If $p \geq 0$, then $\sigma = \delta\tau$, where $(\tau)_0 = -z^r$
$(\tau)_1 = 0$

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If $p < k$, then $\sigma = \delta\tau$, where $(\tau)_0 = 0$
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If $p \geq 0$, then $\sigma = \delta\tau$, where $(\tau)_0 = -z^r$
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If $p \leq k$, then $\sigma = \delta\tau$, where $(\tau)_0 = 0$
$(\tau)_1 = w^{k-p}$

So if $p \geq 0$ or $p \leq k$ then σ is trivial in cohom

$k < p < 0$ for a nontrivial element

$\dim H^1(\mathbb{P}^1, \mathcal{O}(k)) = 0$ for $k \geq -1$

$= \mathbb{C}^{-k-1}$ for $k \leq -2$

$k < p < 0$ for a natural element

$$\check{H}^1(\mathbb{P}^1, \mathcal{O}(k)) = 0 \quad \text{for } k \geq -1$$
$$= \mathbb{C}^{-k-1} \quad \text{for } k \leq -2$$

$\dim \check{H}^p(\mathbb{P}^n, \mathcal{O}(k)) = 0$ if $p \neq 0$ and $p \neq n$

$\dim \check{H}^0(\mathbb{P}^n, \mathcal{O}(k)) = \binom{n+m}{m}$

$\dim \check{H}^n(\mathbb{P}^n, \mathcal{O}(k)) = \binom{-k-1}{-n-k-1}$

$k < p < 0$ for a natural element

$$\check{H}^1(\mathbb{P}^1, \mathcal{O}(k)) = 0 \quad \text{for } k \geq -1$$
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$\dim \check{H}^r(\mathbb{P}^n, \mathcal{O}(k)) = 0$ if $p \neq 0$ and $p \neq n$

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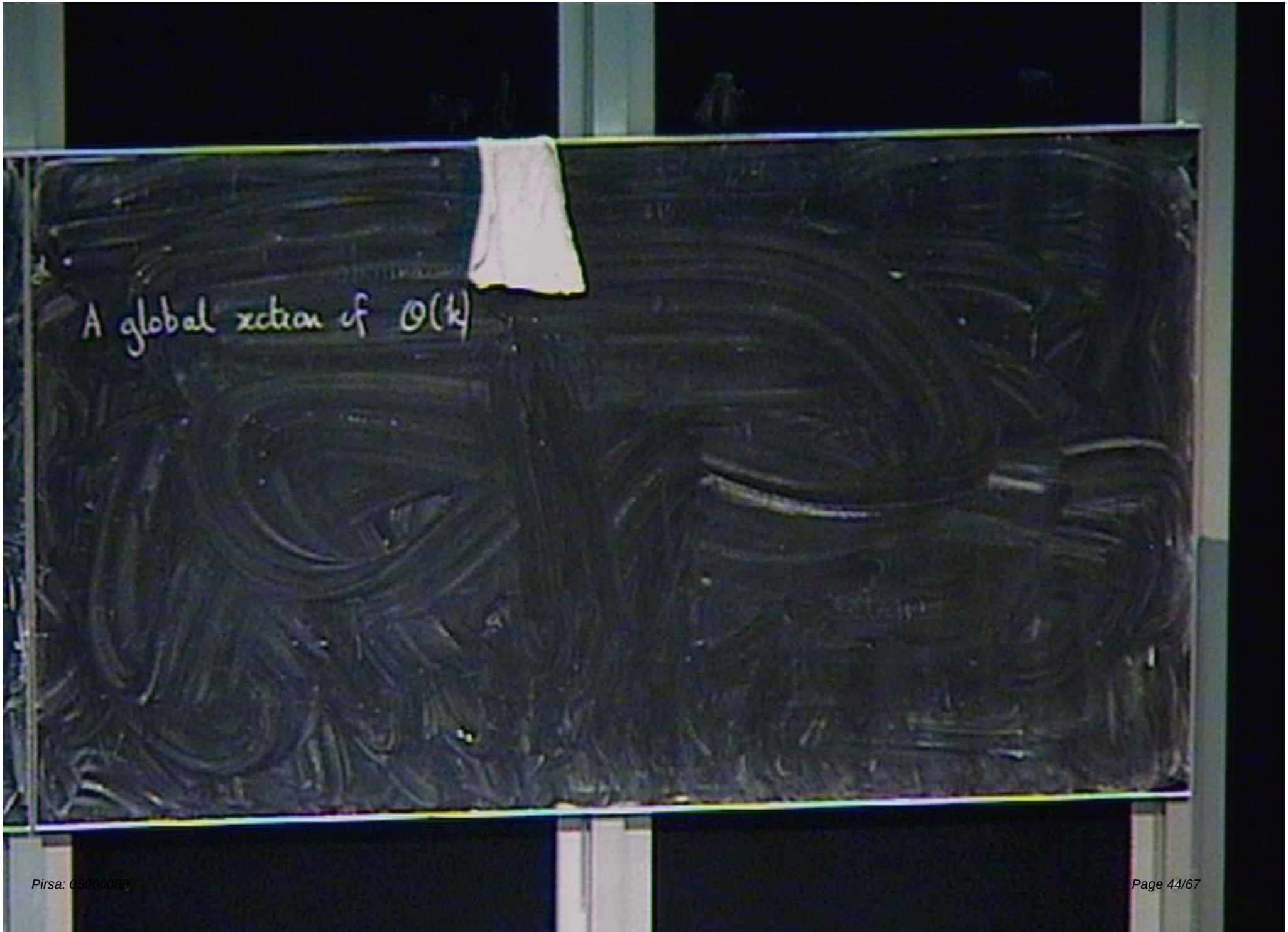

A global section of O(k)

A global section of $\mathcal{O}(k)$ yields a map

$$\mathcal{O} \xrightarrow[\substack{Z_i \text{ of} \\ \deg k}]{\text{hom fn of}} \mathcal{O}(k)$$

sum $\quad \mathcal{O}(p) \longrightarrow \mathcal{O}(k+p)$

De Rham Theorem

$$H^p_{dR}(M) = \check{H}^p(M, \mathbb{R})$$

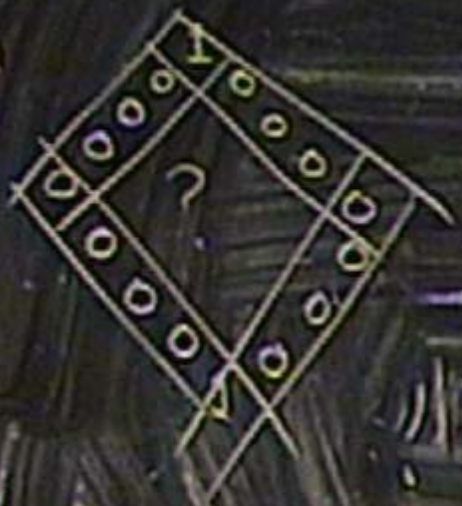
Dolbeault Theorem

$$H^{p,q}_{\bar{\partial}}(M) = \check{H}^q(M, \Omega^p)$$

We know

$H^q(\mathbb{P}^1, \Theta)$

Hodge Diamond of $\mathbb{P}^1$

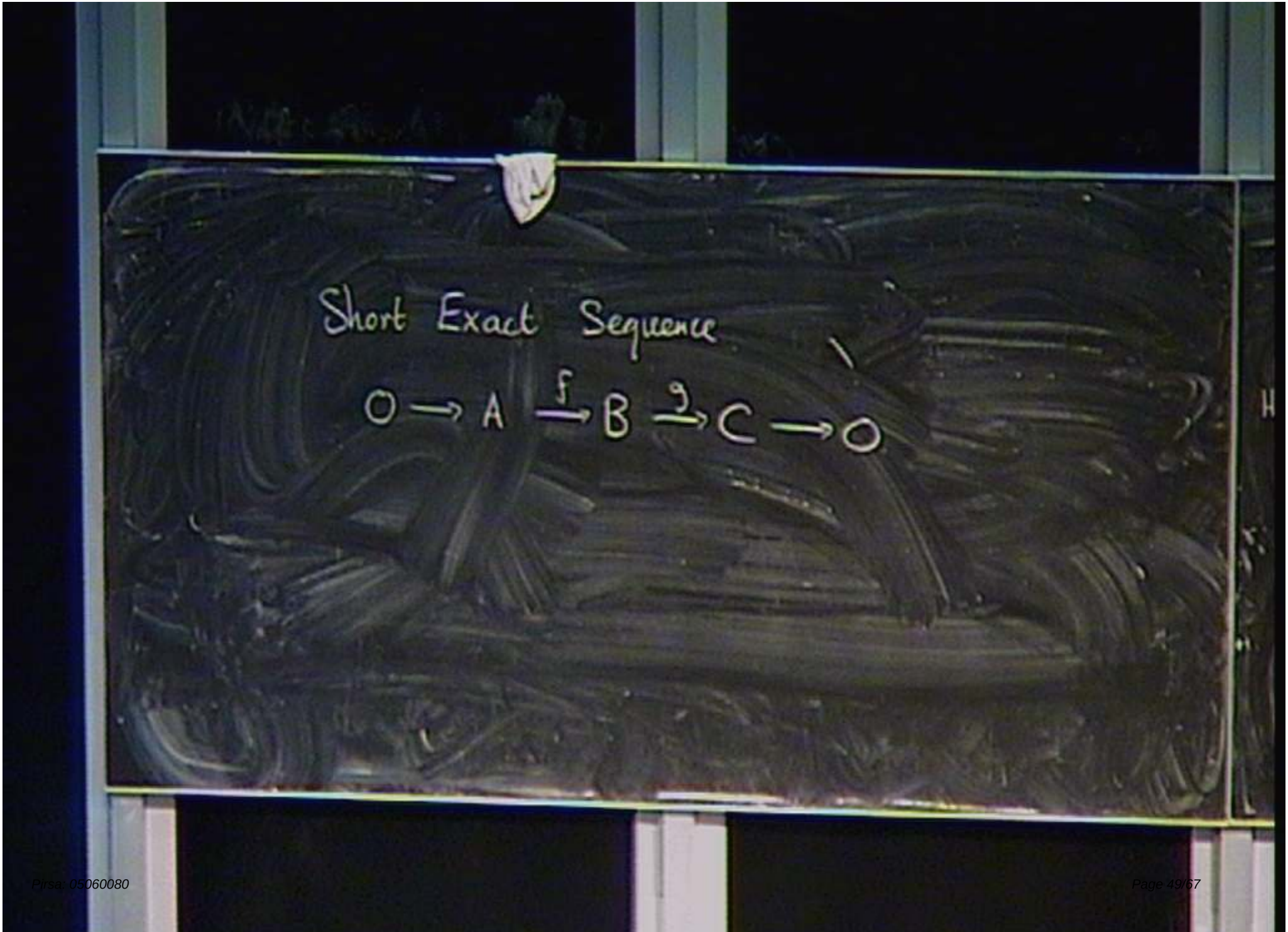


Exact Sequences

$$\to A^{-1} \xrightarrow{d_{-1}} A^0 \xrightarrow{d_0} A^1 \xrightarrow{d_1} A^2$$

is exact if $d_{j+1} d_j = 0$

and $\operatorname{Ker} d_j = \operatorname{Im} d_{j-1}$ $\forall j$.


Short Exact Sequence
$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$

Exact Sequences

$$\cdots \to A^{-1} \xrightarrow{d_{-1}} A^0 \xrightarrow{d_0} A^1 \xrightarrow{d_1} A^2$$

is exact if $d_{j+1} d_j = 0$ (is a complex)

and $\operatorname{Ker} d_j = \operatorname{Im} d_{j-1}$ $\forall j$.

Short Exact Sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

f is injective

g is surjective

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$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

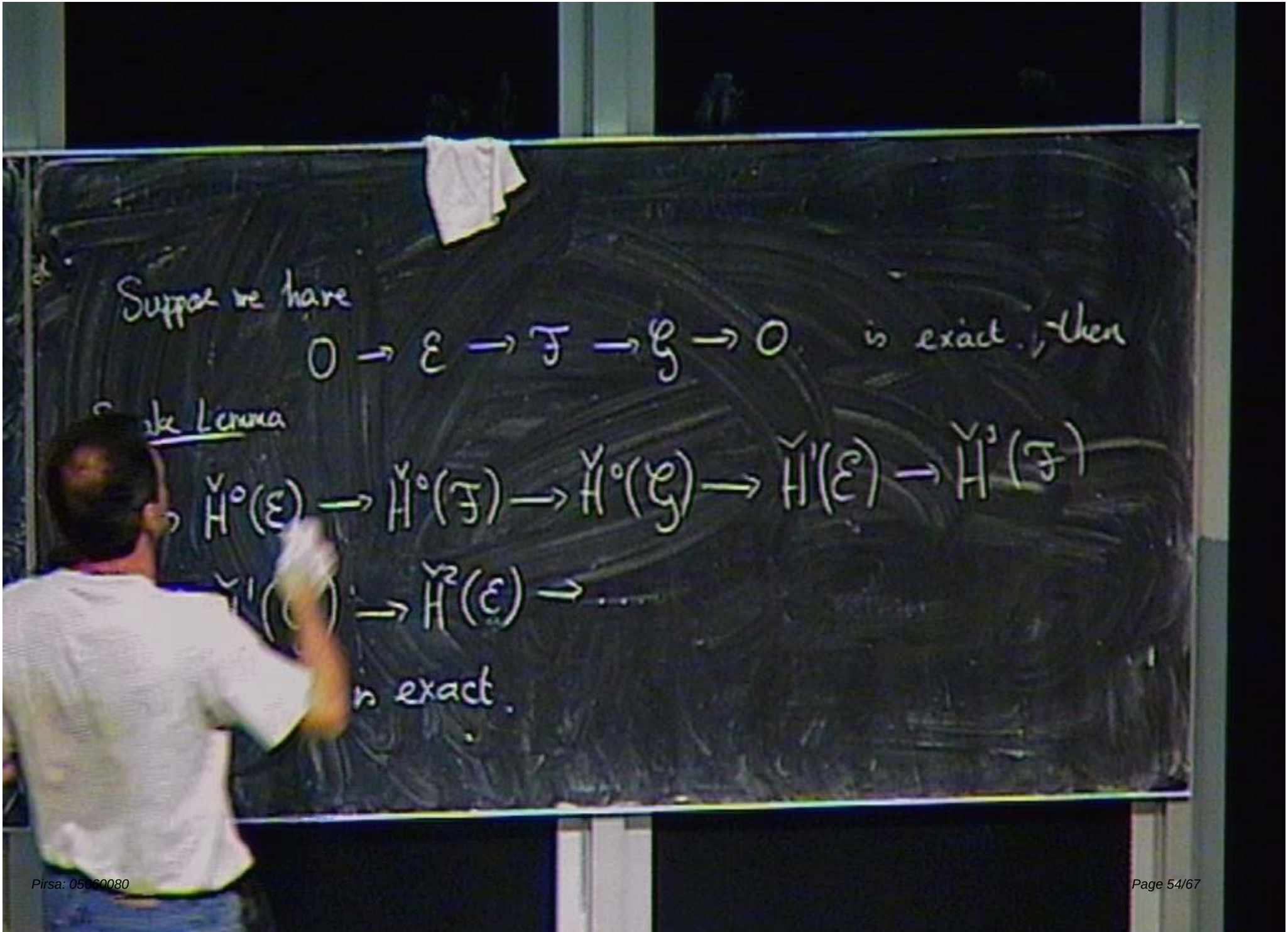
f is injective

g is surjective

$C = B/A$

E.g. $0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp(2\pi i \cdot -)} \mathcal{O}^* \to 0$

is a short exact sequence of sheaves


Suppose we have
$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ is exact, then
ke Lemma
$\check{H}^0(\mathcal{E}) \to \check{H}^0(\mathcal{F}) \to \check{H}^0(\mathcal{G}) \to \check{H}^1(\mathcal{E}) \to \check{H}^1(\mathcal{F})$
$\to \check{H}^2(\mathcal{E}) \to \ldots$
n exact.

Suppose we have

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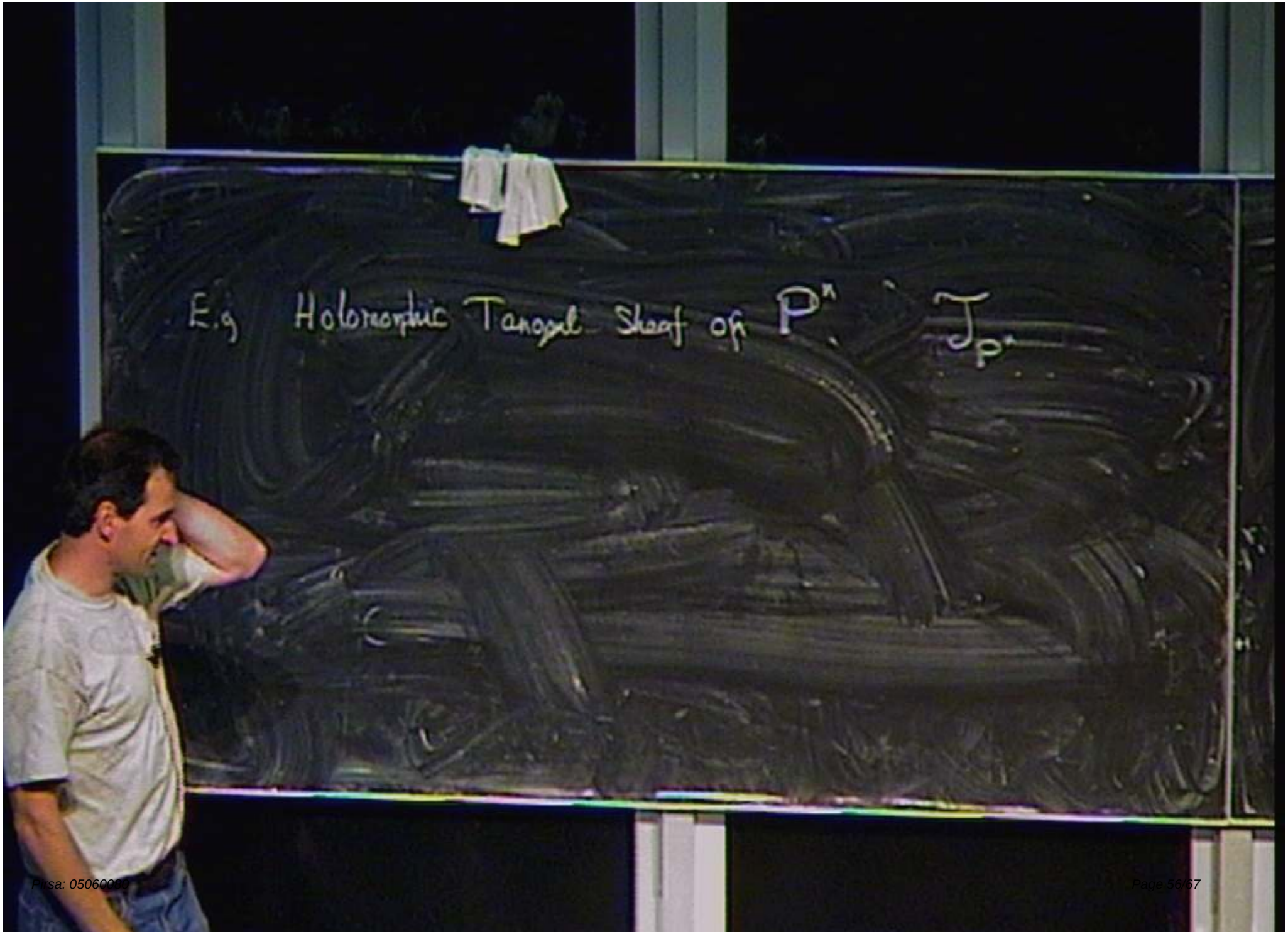
Snake Lemma

$$0 \to \check{H}^0(\mathcal{E}) \to \check{H}^0(\mathcal{F}) \to \check{H}^0(\mathcal{G}) \to \check{H}^1(\mathcal{E}) \to \check{H}^1(\mathcal{F})$$

$$\to \check{H}^1(\mathcal{G}) \to \check{H}^2(\mathcal{E}) \to \cdots$$

is exact.


E.g Holomorphic Tangent Sheaf of $\mathbb{P}^n$ $\mathcal{T}_{\mathbb{P}^n}$

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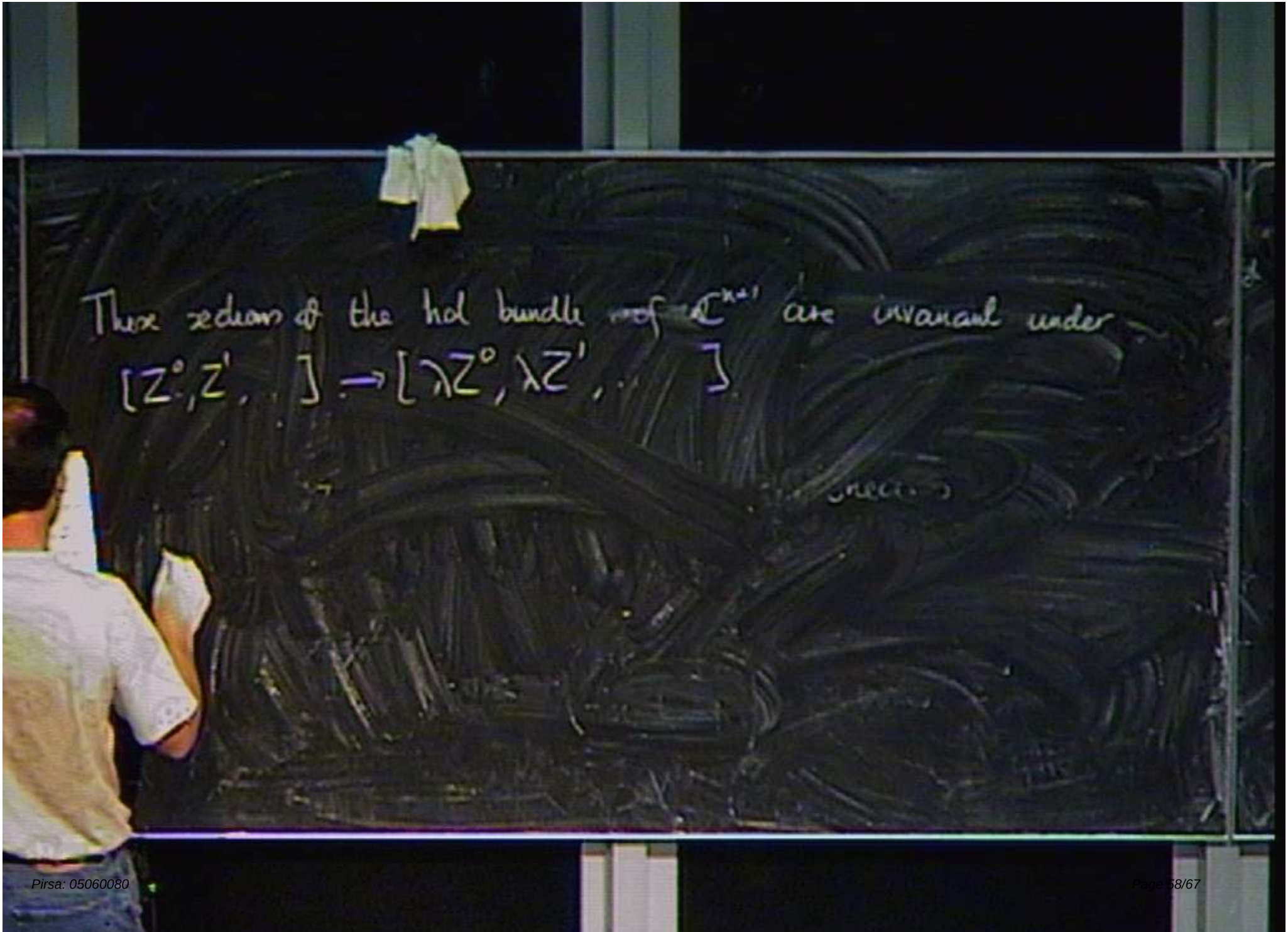
Hol tangent bundle on $\mathbb{C}^{n+1}$ has a basis $\frac{\partial}{\partial z^j}$ for $j=0,\dots,n$

i.e. a vector field is $f^j(z^k)\frac{\partial}{\partial z^j}$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$[z^0, z^1, \ldots] \to [\lambda z^0, \lambda z^1, \ldots]$

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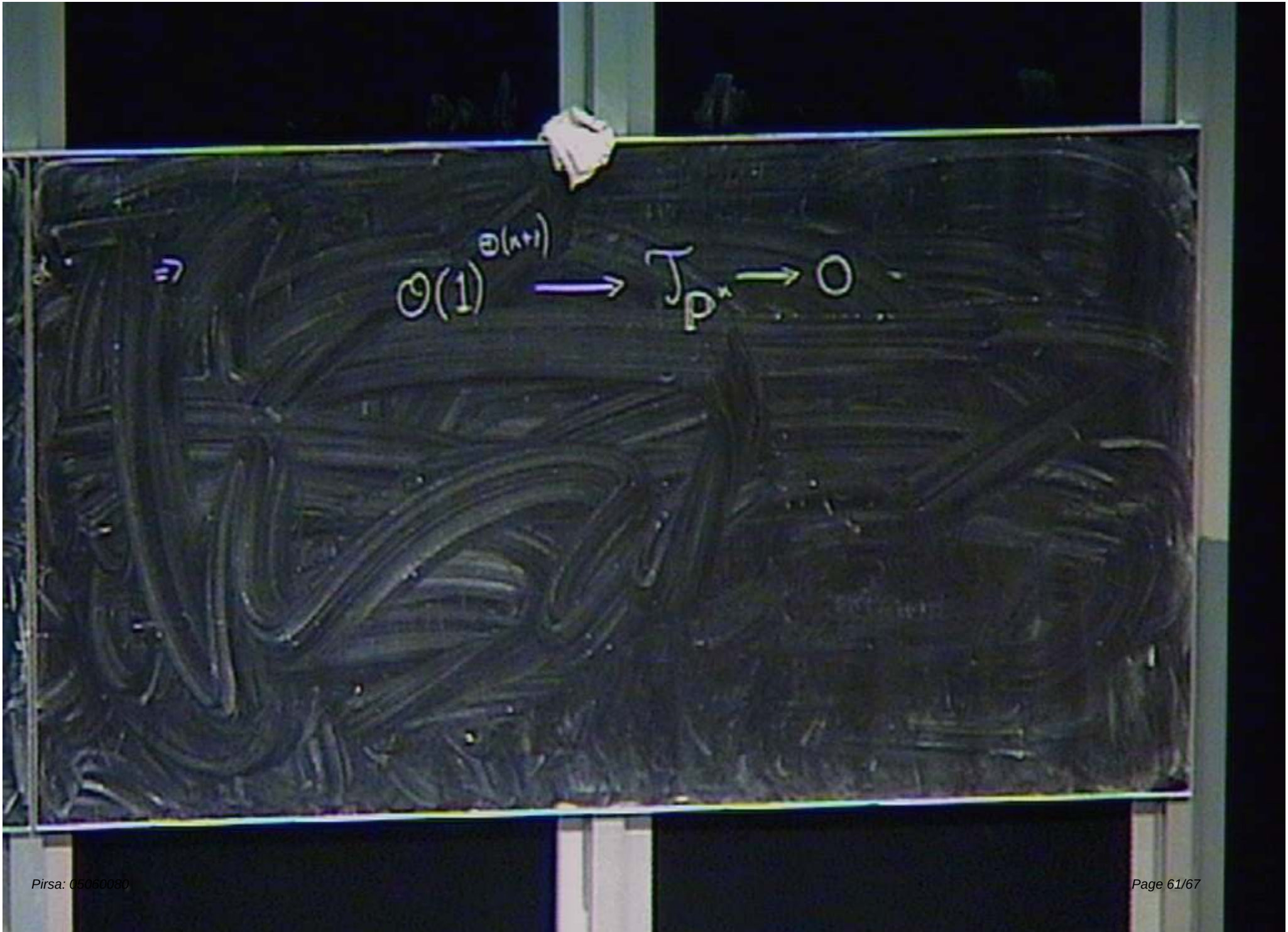
if f^j's are linear in Z^{k_j}.

These sections of the hol bundle of $\mathbb{C}^{n+1}$ are invariant under

$$[Z^0, Z^1, \ldots] \to [\lambda Z^0, \lambda Z^1, \ldots]$$

if f^j's are linear in Z^k's

A linear function in Z^k's is a section of $\mathcal{O}(1)$


$\mathcal{O}(1)^{\oplus(n+1)} \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0$

$$\Rightarrow \quad \mathcal{O}(1)^{\oplus(n+1)} \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0$$

If $x^j = \frac{z^j}{z^0}$ are affine coords on U_0

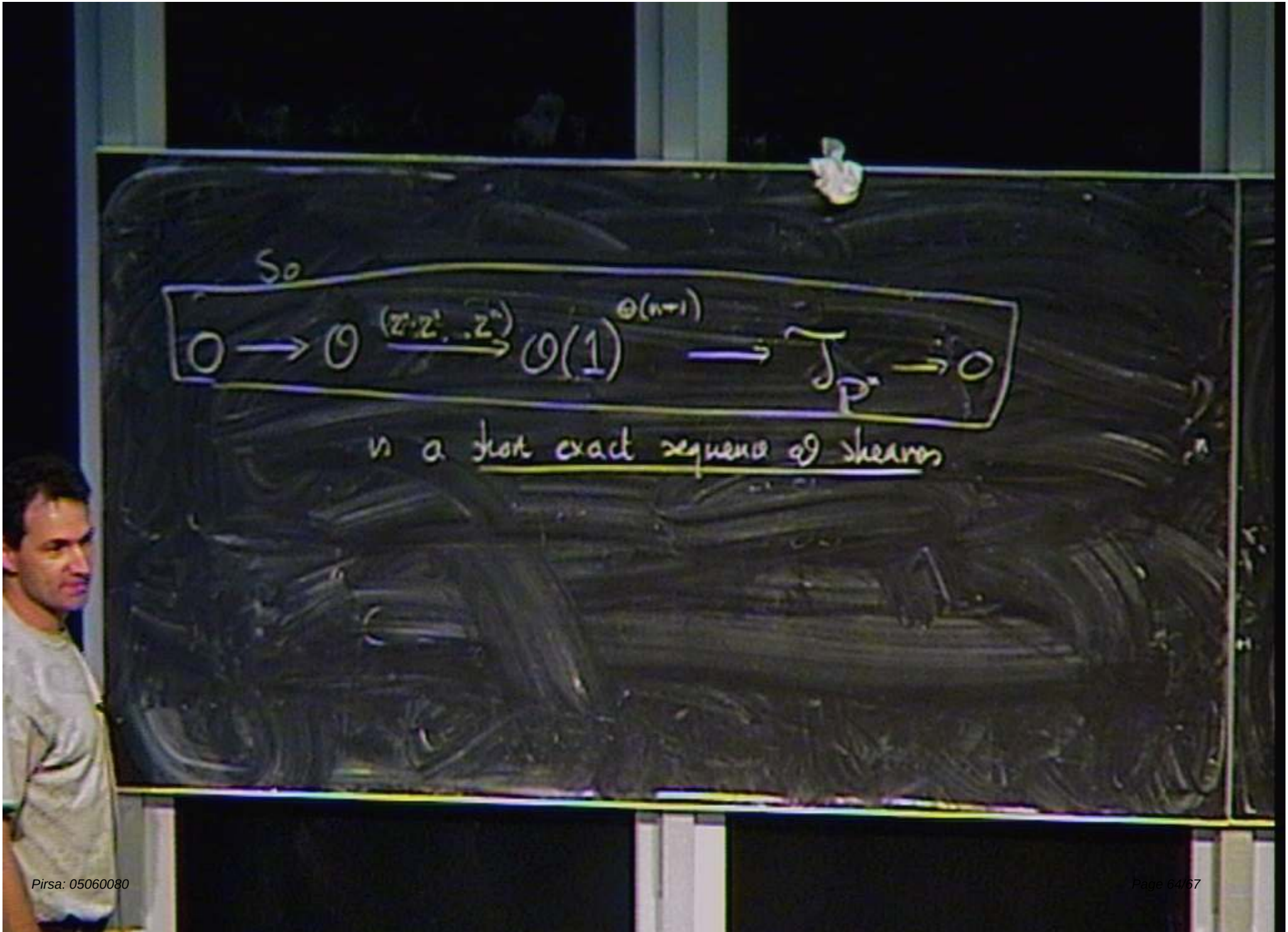
then $\frac{\partial}{\partial z^0} = -\frac{z^j}{(z^0)^2}\frac{\partial}{\partial x^j}$

$\frac{\partial}{\partial z^j} = \frac{1}{z^0}\frac{\partial}{\partial x^j}$

$\Big\} \quad \sum_j z^j \frac{\partial}{\partial z^j} = 0$

So

$$0 \to \mathcal{O} \xrightarrow{(z^0, z^1, \ldots, z^n)} \mathcal{O}(1)^{\oplus(n+1)} \to T_{\mathbb{P}^n} \to 0$$


So
$0 \to \mathcal{O} \xrightarrow{(z^0, z^1, \ldots, z^n)} \mathcal{O}(1)^{\oplus(n+1)} \to \mathcal{T}_{\mathbb{P}^n} \to 0$
is a short exact sequence of sheaves

Define $T_{\mathbb{P}^n}(k) = T_{\mathbb{P}^n} \otimes \mathcal{O}(k)$

$$0 \to \mathcal{O}(k) \to \mathcal{O}(k+1)^{\oplus(n+1)} \to T_{\mathbb{P}^n}(k) \to 0$$

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$$0 \to \mathcal{O}(k) \to \mathcal{O}(k+1)^{\oplus(n+1)} \to T_{\mathbb{P}^n}(k) \to 0$$

Also

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O} \to 0$$

(Euler exact sequence)

$$H^0(\mathcal{J}_{\mathbb{P}^n}) = \mathbb{C}^{n^2+2n}$$

$$H^r(\mathcal{J}_{\mathbb{P}^n}) = 0 \quad \text{for} \quad r \neq 0$$