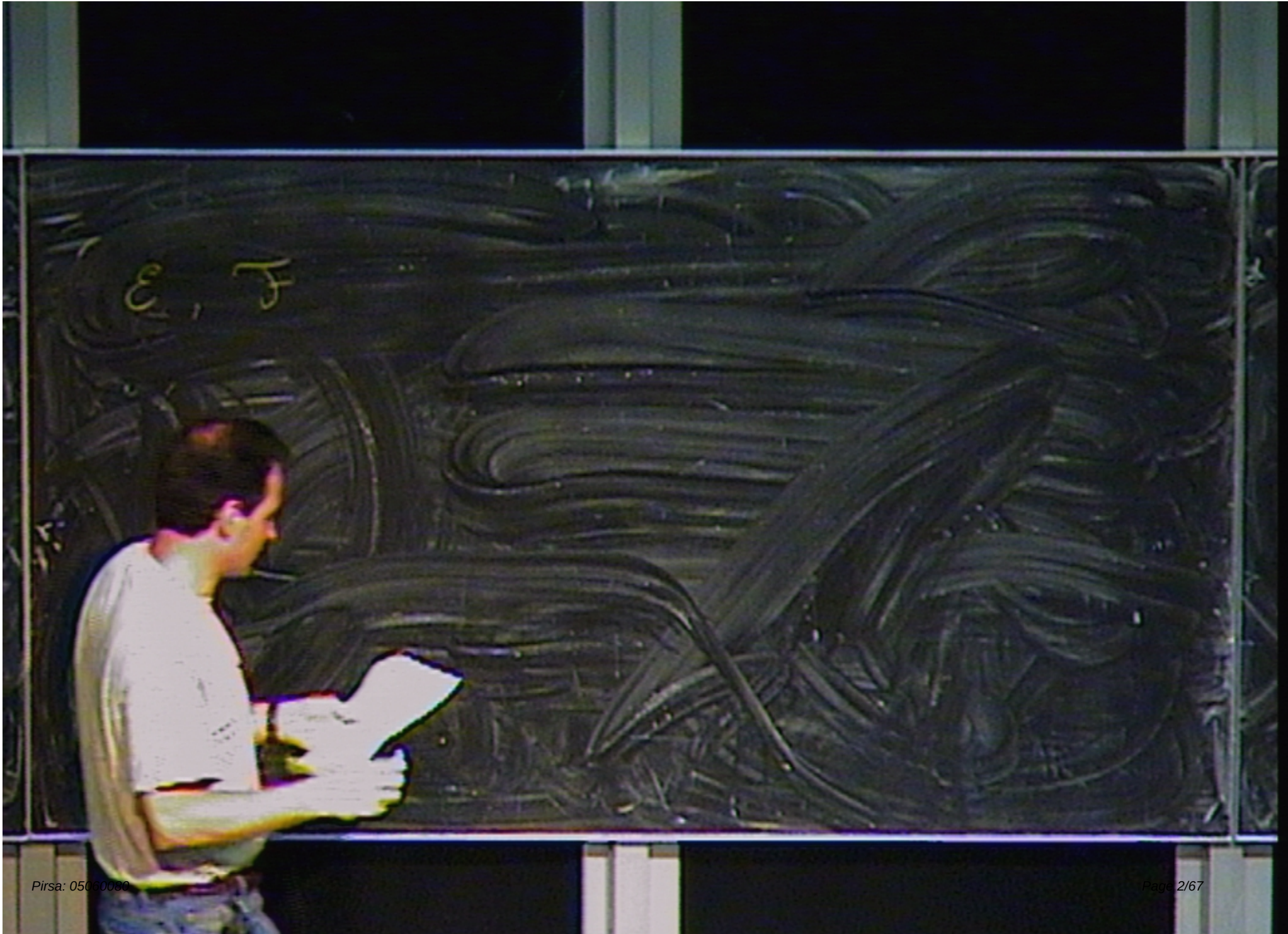


Title: Complex manifolds

Date: Jun 22, 2005 04:30 PM

URL: <http://pirsa.org/05060080>

Abstract:



$\mathcal{E}, \mathcal{F} \quad \mathcal{E}/\mathcal{F}$

$$\mathcal{E}/\mathcal{F}(U) = \frac{\mathcal{E}(U)}{\mathcal{F}(U)}$$

obvious candidate defn of quotient sheaf $\mathcal{E}/\mathcal{F}$

Let $\mathcal{O}$ be the sheaf of hol functions $(+)$

Let $\mathcal{O}^*$ " " " nowhere zero hol functions $(\times)$

$$\mathcal{O} \xrightarrow{\exp(2\pi i \cdot -)} \mathcal{O}^*$$

This is a sheaf map

The coker of any map $E \xrightarrow{f} F$ is defined as $F / \text{Im}(f)$

Let $X = \mathbb{C}^* = \mathbb{C} - \{0\}$

Let z be the coord of $\mathbb{C}^*$

Clearly z is in $\mathcal{O}^*(X)$

z is NOT in the image of $\exp(2\pi i -)$

The coker of any map $E \xrightarrow{f} F$ is defined as $F / \text{Im}(f)$

Let $X = \mathbb{C}^* = \mathbb{C} - \{0\}$

Let z be the coord of $\mathbb{C}^*$

Clearly z is in $\mathcal{O}^*(X)$

z is NOT in the image of $\exp(2\pi i -)$

$\rightarrow z$ is a natural element of coker

$$= \mathcal{O}^* / f(\mathcal{O}^*)(X)$$

Suppose $U \subset \mathbb{C}^*$ is a contractible open set

Then $z|_U$ is in the image of $\exp(2\pi i \cdot)$

$$\Rightarrow z|_U = 0 \Rightarrow \text{in } \mathbb{F}(U)$$



The coker of any map $E \xrightarrow{f} F$ is defined as $F / \text{Im}(f)$

Let $X = \mathbb{C}^* = \mathbb{C} - \{0\}$

Let z be the coord of $\mathbb{C}^*$

Clearly z is in $\mathcal{O}^*(X)$

z is NOT in the image of $\exp(2\pi i -)$

$\rightarrow z$ is a natural element of coker

$$\begin{array}{c} \mathcal{O}^*(X) \\ \parallel \\ \mathcal{O}^* / f(\mathcal{O}) \end{array} (X)$$

Suppose $U \subset \mathbb{C}^*$ is a contractible open set

Then $z|_U$ is in the image of $\exp(2\pi i \cdot)$

$$\Rightarrow z|_U = 0 \text{ in } \mathbb{F}(U)$$



But X can be written as a union of two contractible open sets.

This quotient is a presheaf but not a sheaf.

This quotient is a presheaf but not a sheaf.
We may fix this problem by considering only "very small"
contractible open sets.

This quotient is a presheaf but not a sheaf.

We may fix this problem by considering only "very small" contractible open sets.

We may "sheafify" any presheaf by focussing just on these small open sets.

This quotient is a presheaf but not a sheaf.

We may fix this problem by considering only "very small" contractible open sets.

We may "sheafify" any presheaf by focussing just on
coker $(\mathcal{O}^{\text{tr}} \rightarrow \mathcal{O}^*)$ these small open sets.
is trivial.

Čech Cohomology

Čech Cohomology

Let $\underline{U} = \{U_\alpha\}$ be an open cover of M

Define $\check{C}^0(\underline{U}, \mathcal{F}) = \prod_{\alpha} \mathcal{F}(U_\alpha)$

$$\check{C}^1(\underline{U}, \mathcal{F}) = \prod_{\alpha < \beta} \mathcal{F}(U_\alpha \cap U_\beta)$$

$$\check{C}^p(\underline{U}, \mathcal{F}) = \prod_{\alpha_0 < \alpha_1 < \dots < \alpha_p} \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1} \cap \dots \cap U_{\alpha_p})$$

= groups of Čech cochains

If $\sigma \in \check{C}^p(\underline{U}, \mathcal{F})$, let $\sigma_{\alpha_0, \alpha_1, \dots, \alpha_p} \in \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1} \cap \dots \cap U_{\alpha_p})$

Coboundary operator:

$$\delta: \check{C}^p(\underline{U}, \mathcal{F}) \rightarrow \check{C}^{p+1}(\underline{U}, \mathcal{F})$$

$$(\delta\sigma)_{\alpha_0\alpha_1\ldots\alpha_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{\alpha_0\alpha_1\ldots\hat{\alpha}_j\ldots\alpha_{p+1}} |$$

Coboundary operator:

$$\delta: \check{C}^p(\underline{U}, \mathcal{F}) \rightarrow \check{C}^{p+1}(\underline{U}, \mathcal{F})$$

$$(\delta\sigma)_{\alpha_0\alpha_1\alpha_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{\alpha_0\alpha_1\ldots\hat{\alpha}_j\ldots\alpha_{p+1}} \Big|_{U_0 \cap U_1 \cap \dots \cap U_{p+1}}$$

σ is a 'cocycle' if $\delta\sigma = 0$
coboundary if $\sigma = \delta\tau$

It is easy to prove $\delta^2 = 0$

Cochain complex

$$0 \rightarrow C^0 \xrightarrow{\delta_0} C^1 \xrightarrow{\delta_1} C^2 \xrightarrow{\delta_2} \dots$$

Čech cohomology

$$\check{H}^p(\underline{U}, \mathcal{F}) = \frac{\text{Ker } \delta_p}{\text{Im } \delta_{p-1}}$$

$\check{H}^p_{\underline{U}}$ depends on $\underline{U}$

If the open sets are "very small" $\check{H}^p(\underline{U}, \mathcal{F})$ is independent of $\underline{U}$

- This gives $\check{H}^p(X, \mathcal{F})$

$$\check{H}^1(\underline{U}, \mathbb{F}) = \check{H}^1(X, \mathbb{F}) \text{ for all } p$$

$$\text{if } \check{H}^k(U_{d_0} \cap U_{d_1} \cap \dots \cap U_{d_k}, \mathbb{F}) = 0 \text{ for all } k > 0$$

$$\text{E.g. } H^1(\mathbb{P}^1, \mathcal{O}(k))$$

In this case we may use $U_0 = \mathbb{P}^1 - \{[0, 1]\}$
 $U_1 = \mathbb{P}^1 - \{[1, 0]\}$
 as an open cover $\mathcal{U}$

$$\text{E.g. } H^1(\mathbb{P}^1, \mathcal{O}(k))$$

In this case we may use $U_0 = \mathbb{P}^1 - \{[0, 1]\}$
 $U_1 = \mathbb{P}^1 - \{[1, 0]\}$
 as an open cover $\underline{U}$

$$\sum P(\underline{A}, O(k)) = 0 \text{ for } p \geq 2$$

$$\text{So } \sum P(P^k, O(k)) = 0 \text{ for } p \geq 2$$

$\check{H}^0(P^1, \mathcal{O}(k)) =$ functions on U_0 and U_1 which agree (after restriction to overlap.)
= global sections

$\check{H}^0(\mathbb{P}^1, \mathcal{O}(k)) =$ functions on U_0 and U_1 which agree (after restriction to overlap)
 $=$ global sections (this is always the case)

On U_0 , let $z = \frac{z_1}{z_0}$ be affine coordinate
 and let $z^m \in \mathcal{F}(U_0)$ be a local section

$$U_0 = U_0 \cap U_1$$

Clearly $m \geq 0$ for z^m to be hol. in U_0

We may define $\mathcal{O}(k)$ by $\Gamma_{0,01} = 1$
 $\Gamma_{1,01} = z^k$

$$U_0 = U_0 \cap U_1$$

Clearly $m \geq 0$ for z^m to be hol. in U_0

We may define $\mathcal{O}(k)$ by $\Gamma_{0,01} = 1$
 $\Gamma_{1,01} = z^k$

Let $w = z^{-1} = \frac{z_0}{z_1}$ be affine coord on $U_1 = \mathbb{C}$

and let w^k is a local section in $\mathcal{O}(k)(U_1)$

So, the coboundary of this 0-cochain is
 $(\delta\sigma)_{01} = W^l|_{01} - Z^m|_{01} = W^l Z^k - Z^m$

So we have a cocycle if $l = k - m$

So, the coboundary of this 0-cochain is
 $(\delta\sigma)_{01} = W^l|_{01} - Z^m|_{01} = W^l Z^k - Z^m$

So we have a cocycle if $l = k - m$.

So $0 \leq m \leq k$ for a global hol. section
I.e. m has $k+1$ values

So, the coboundary of this 0-cochain is
 $(\delta\sigma)_{01} = W^l|_{01} - Z^m|_{01} = W^l Z^k - Z^m$

So we have a cocycle if $l = k - m$.

So $0 \leq m \leq k$ for a global hol. section

I.e. m has $k+1$ values

These solutions form a basis for all global sections

So, the coboundary of this 0-cochain is

$$(\delta\sigma)_{01} = W^l|_{01} - Z^m|_{01} = W^l Z^k - Z^m$$

So we have a cocycle if $l = k - m$.

So $0 \leq m \leq k$ for a global hol. section

I.e. m has $k+1$ values

These solutions form a basis for all global sections

$\check{H}^0(\mathbb{P}^1, \mathcal{O}(k)) =$ functions on U_0 and U_1 which agree (after restriction to overlap.)
 $=$ global sections (this is always the case)

On U_0 , let $z = \frac{z_1}{z_0}$ be affine coordinate
 and let $z^m \in \mathcal{F}(U_0)$ be a local section

$$\Rightarrow \dim H^0(\mathbb{P}^1, \mathcal{O}(k)) = \begin{cases} k+1 & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases}$$

We may also write a global section as

$$\Rightarrow \dim H^0(\mathbb{P}^1, \mathcal{O}(k)) = \begin{cases} k+1 & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases}$$

We may also write a global section as

$$\sum_0^{k-m} z_0^{k-m} z_1^m \quad \text{in hom. coords}$$

(homogeneous fn.
of degree k)

we set $z_0 = 1$ in U_0

$z_1 = 1$ in U_1

$$\Rightarrow \dim H^0(\mathbb{P}^1, \mathcal{O}(k)) = \begin{cases} k+1 & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases}$$

We may also write a global section as

$$\sum_0^{k-m} z_0^{k-m} z_1^m \quad \text{in hom. coords}$$

(homogeneous fn.
of degree k)

we set $z_0 = 1$ in U_0

$z_1 = 1$ in U_1

$$H^1(\mathbb{P}^1, \mathcal{O}(k))$$

Suppose $(\sigma)_{01}$

$$H^1(P^1, \mathcal{O}(k))$$

Suppose $(\sigma)_0 = z^p$

If $p \geq 0$, then $\sigma = \delta\tau$, where $(\tau)_0 = -z^p$
 $(\tau)_1 = 0$

$$H^1(P^1, \mathcal{O}(k))$$

Suppose $(\sigma)_{01} = z^p$

If $p \geq 0$, then $\sigma = \delta\tau$, where $(\tau)_0 = -z^p$
 $(\tau)_1 = 0$

If $p < 0$, then $\sigma = \delta\tau$, where $(\tau)_0 = 0$
 $(\tau)_1 = w^{k-p}$

$$H^1(P^1, \mathcal{O}(k))$$

Suppose $(\sigma)_{01} = z^p$

If $p \geq 0$, then $\sigma = \delta\tau$, where $(\tau)_0 = -z^p$
 $(\tau)_1 = 0$

If $p \leq k$, then $\sigma = \delta\tau$, where $(\tau)_0 = 0$
 $(\tau)_1 = w^{k-p}$

So if $p \geq 0$ or $p \leq k$ then σ is trivial in cohom

$k < p < 0$ for a nontrivial element

$$\dim H^1(\mathbb{P}^1, \mathcal{O}(k)) = \begin{cases} 0 & \text{for } k \geq -1 \\ \infty & \text{for } k \leq -2 \end{cases}$$

$k < p < 0$ for a nontrivial element

$$H^k(\mathbb{P}^1, \mathcal{O}(k)) = 0 \text{ for } k \geq -1$$

$$= \mathbb{C}^{-k-1} \text{ for } k \leq -2$$

$$\dim H^p(\mathbb{P}^n, \mathcal{O}(k)) = 0 \text{ if } p \neq 0 \text{ and } p \neq n$$

$$\dim H^0(\mathbb{P}^n, \mathcal{O}(k)) = \binom{n+k}{n}$$

$$\dim H^n(\mathbb{P}^n, \mathcal{O}(k)) = \binom{-k-1}{-n-k-1}$$

$k < p < 0$ for a nontrivial element

$$H^k(\mathbb{P}^n, \mathcal{O}(k)) = 0 \text{ for } k \geq -1$$

$$= \mathbb{C}^{-k-1} \text{ for } k \leq -2$$

$$\dim H^p(\mathbb{P}^n, \mathcal{O}(k)) = 0 \text{ if } p \neq 0 \text{ and } p \neq n$$

$$\dim H^0(\mathbb{P}^n, \mathcal{O}(k)) = \binom{n+k}{k}$$

$$\dim H^n(\mathbb{P}^n, \mathcal{O}(k)) = \binom{-k-1}{-n-k-1}$$

A global action of $O(k)$

A global section of $\mathcal{O}(k)$ yields a map

$$\mathcal{O} \xrightarrow[\substack{\text{sum of} \\ \text{deg } k}]{\text{sum of}} \mathcal{O}(k)$$

sum $\mathcal{O}(p) \longrightarrow \mathcal{O}(k+p)$

De Rham Theorem

$$H_{dR}^p(M) = \check{H}^p(M, \mathbb{R})$$

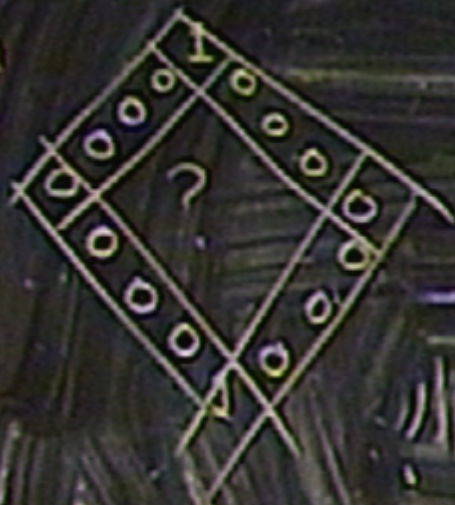
Dolbeault Theorem

$$H_{\bar{\partial}}^{p,q}(M) = \check{H}^q(M, \Omega^p)$$

We know

$$H^1(\mathbb{P}^n, \mathcal{O})$$

Hodge Diamond
of $\mathbb{P}^n$



Exact Sequences

$$\rightarrow A^{-1} \xrightarrow{d_{-1}} A^0 \xrightarrow{d_0} A^1 \xrightarrow{d_1} A^2$$

is exact if $d_{j+1} d_j = 0$

and $\text{Ker } d_j = \text{Im } d_{j-1} \quad \forall j.$

Short Exact Sequence

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

Exact Sequences

$$\rightarrow A^{-1} \xrightarrow{d_{-1}} A^0 \xrightarrow{d_0} A^1 \xrightarrow{d_1} A^2$$

is exact if $d_{j+1} d_j = 0$ (is a complex)

and $\text{Ker } d_j = \text{Im } d_{j-1} \quad \forall j.$

Short Exact Sequence

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

f is injective

g is surjective

Short Exact Sequence

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

f is injective

g is surjective

$$C = B/A$$

E.g. $0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{\exp(2\pi i \cdot -)} \mathcal{O}^* \rightarrow 0$

is a short exact sequence of sheaves

Suppose we have

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0 \text{ is exact, then}$$

the Lemma

$$\check{H}^0(\mathcal{E}) \rightarrow \check{H}^0(\mathcal{F}) \rightarrow \check{H}^0(\mathcal{G}) \rightarrow \check{H}^1(\mathcal{E}) \rightarrow \check{H}^1(\mathcal{F})$$

$$\check{H}^1(\mathcal{G}) \rightarrow \check{H}^2(\mathcal{E}) \rightarrow \dots$$

is exact.

Suppose we have

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0 \quad \text{is exact, then}$$

Snake Lemma

$$\begin{aligned} 0 \rightarrow \check{H}^0(\mathcal{E}) \rightarrow \check{H}^0(\mathcal{F}) \rightarrow \check{H}^0(\mathcal{G}) \rightarrow \check{H}^1(\mathcal{E}) \rightarrow \check{H}^1(\mathcal{F}) \\ \rightarrow \check{H}^1(\mathcal{G}) \rightarrow \check{H}^2(\mathcal{E}) \rightarrow \dots \end{aligned}$$

is exact.

E.g. Holomorphic Tangent Sheaf of $\mathbb{P}^n$ $\mathcal{T}_{\mathbb{P}^n}$

E.g. Holomorphic Tangent Sheaf of $\mathbb{P}^n$ $\mathcal{T}_{\mathbb{P}^n}$

Hol. tangent bundle on $\mathbb{C}^{n+1}$ has a basis $\frac{\partial}{\partial z^j}$ for $j=0, \dots, n$
i.e. a vector field is $f^j(z^*) \frac{\partial}{\partial z^j}$

These sections of the hol bundle of $\mathcal{C}^{n+1}$ are invariant under
 $[Z^0, Z', \dots] \rightarrow [\lambda Z^0, \lambda Z', \dots]$

These sections of the hol bundle over $\mathbb{C}^{n+1}$ are invariant under
 $[Z^0, Z^1, \dots] \rightarrow [\lambda Z^0, \lambda Z^1, \dots]$
 if f_j 's are linear in Z^k .

These sections of the hol bundle over $\mathbb{C}^{n+1}$ are invariant under

$$[Z^0, Z^1, \dots] \rightarrow [\lambda Z^0, \lambda Z^1, \dots]$$

if f_j 's are linear in Z^k 's

A linear function in Z^k 's is a section of $\mathcal{O}(1)$

$\Rightarrow$

$$\mathcal{O}(1)^{\Theta(n+1)} \rightarrow \mathcal{I}_{P^*} \rightarrow 0$$

$$\Rightarrow \mathcal{O}(1)^{\oplus(n+1)} \longrightarrow \mathcal{T}_{P^n} \longrightarrow 0$$

If $x^j = \frac{z^j}{z^0}$ are affine coords in U_0

$$\left. \begin{aligned} \text{then } \frac{\partial}{\partial z^0} &= -\frac{z^j}{(z^0)^2} \frac{\partial}{\partial x^j} \\ \frac{\partial}{\partial z^j} &= \frac{1}{z^0} \frac{\partial}{\partial x^j} \end{aligned} \right\} \sum_j z^j \frac{\partial}{\partial z^j} = 0$$

$$\begin{array}{c}
 \text{So} \\
 0 \rightarrow 0 \xrightarrow{(z, z^2, \dots, z^n)} 0(1)^{\Theta(n+1)} \rightarrow \mathcal{I}_P \rightarrow 0
 \end{array}$$

So

$$0 \rightarrow \mathcal{O} \xrightarrow{(z, z', \dots, z^n)} \mathcal{O}(1)^{\oplus (n+1)} \rightarrow \mathcal{I}_P \rightarrow 0$$

is a short exact sequence of sheaves

Define $T_{P^*}(k) = T_{P^*} \otimes \mathcal{O}(k)$

$$0 \rightarrow \mathcal{O}(k) \rightarrow \mathcal{O}(k+1)^{\oplus(n+1)} \rightarrow T_{P^*}(k) \rightarrow 0$$

Define $T_{P^n}(k) = T_{P^n} \otimes \mathcal{O}(k)$

$$0 \rightarrow \mathcal{O}(k) \rightarrow \mathcal{O}(k+1)^{\oplus(n+1)} \rightarrow T_{P^n}(k) \rightarrow 0$$

Also

$$0 \rightarrow \Omega_{P^n}^1 \rightarrow \mathcal{O}(-1)^{\oplus(n+1)} \rightarrow \mathcal{O} \rightarrow 0 \quad (\text{Euler exact sequence})$$

$$H^0(\mathcal{I}_{P^n}) = \mathbb{C}^{n^2+2n}$$

$$H^r(\mathcal{I}_{P^n}) = 0 \quad \text{for } r \neq 0$$