Title: Complex Manifolds

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Abstract:

L_m

Eg Tangent bundle on $\mathbb{P}^1$ is L_2

Given a real manifold, is it a complex manifold?

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such that $J^2 = -1$, i.e. $J^\mu{}_\nu J^\nu{}_\xi = -\delta^\mu{}_\xi$

On a complex manifold

$$J\, dx^i = -dy^i$$

$$J\, dy^i = dx^i$$

$$J\, dz^i = i\, dz^i$$

$$J\, dz^{\bar{i}} = -i\, dz^{\bar{i}}$$

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Almost Complex Structure

J is a linear map on cotangent bundle

$$dx^\mu = J^\mu{}_\nu \, dx^\nu \qquad \mu, \nu = 1, \ldots, 2n$$

such that $J^2 = -1$, i.e. $J^\mu{}_\nu J^\nu{}_\xi = -\delta^\mu{}_\xi$

On a complex manifold

$$J\, dx^k = -dy^k \qquad k = 1, \ldots, n$$

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An almost complex structure defines a complex structure if it can be "integrable"

Nijenhuis tensor

$$N^{\mu}{}_{\nu\rho} = J^{\lambda}{}_{\nu}\left(\partial_{\lambda} J^{\mu}{}_{\rho} - \partial_{\rho} J^{\mu}{}_{\lambda}\right) - J^{\lambda}{}_{\rho}\left(\partial_{\lambda} J^{\mu}{}_{\nu} - \partial_{\nu} J^{\mu}{}_{\lambda}\right)$$

$N^{\mu}{}_{\nu\rho} = 0 \iff$ almost cx str is cx str

E.g. S^2 is complex

S^8 has no almost complex structure

S^6 has an obvious almost complex which is not a complex structure

Metrics $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$

in complex coords

$$ds^2 = g_{jk}\, dz^j dz^k + g_{j\bar{k}}\, dz^j dz^{\bar{k}} + g_{\bar{j}k}\, dz^{\bar{j}} dz^k + g_{\bar{j}\bar{k}}\, dz^{\bar{j}} dz^{\bar{k}}$$

Metric is "Hermitian" if $g_{jk} = g_{\bar{j}\bar{k}} = 0$

Metric is Hermitian $\Leftrightarrow$

$$\therefore \quad g_{\mu\nu} J^{\mu}{}_{\xi} J^{\nu}{}_{\zeta} = g_{\xi\zeta}$$

This implies $\quad \overline{J}_{\mu\nu} \overset{\text{def}}{=} g_{\mu\xi} J^{\xi}{}_{\nu} = -J_{\nu\mu}$

It is natural to define 2-form

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$$J = \frac{1}{2} J_{\mu\nu}\, dx^\mu \wedge dx^\nu \quad \text{is a real 2-form on } M$$

$$= \frac{i}{2} g_{j\bar{k}}\, dz^j \wedge dz^{\bar{k}} \quad \text{is a real } (1,1)\text{-form}$$

A complex manifold with Hermitian metric has a "Kähler" metric if $dJ = 0$, J is Kähler form

A complex manifold is Kähler if it admits a Kähler metric

M is compact

For a Kähler manifold one can show that

$$H^k_{dR}(M, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}(M)$$

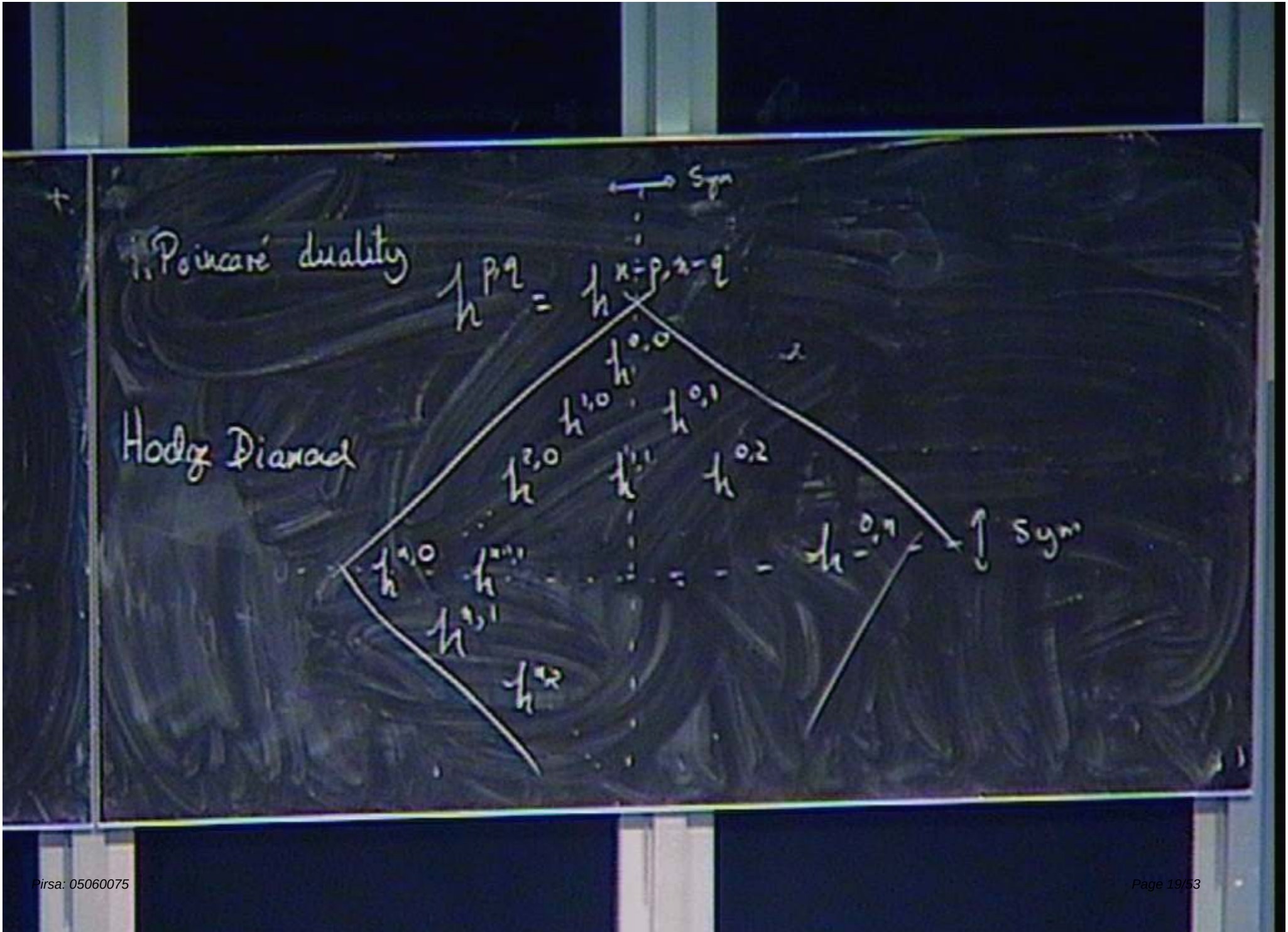
$$\Rightarrow \quad b_k = \sum_{p+q=k} h^{p,q}$$

$$H^{p,q}_{\bar{\partial}}(M) \cong H^{q,p}(M)$$

$$\Rightarrow \quad h^{p,q} = h^{q,p}$$

1. Poincaré duality $h^{p,q} = h^{n-p,n-q}$

Hodge Diamond


Sym
1. Poincaré duality
$h^{p,q} = h^{n-p,n-q}$
Hodge Diamond
$h^{0,0}$
$h^{1,0}$ $h^{0,1}$
$h^{2,0}$ $h^{1,1}$ $h^{0,2}$
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Sym

Algebraic Geometry.

An algebraic variety is the intersection of m equations

$$\left.\begin{aligned} f_1(Z_0, Z_1, \ldots, Z_N) &= 0 \\ f_2(Z_0, Z_1, \ldots, Z_N) &= 0 \\ &\vdots \\ f_m(Z_0, Z_1, \ldots, Z_N) &= 0 \end{aligned}\right\} \text{homogeneous polynomials}$$

in $\mathbb{P}^N$

Algebraic Geometry.

An algebraic variety X is the intersection of m equations

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If $m = 1$, X is a hypersurface

If $\dim X = N - m$, X is called a complete intersection

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In general $\dim X \geq N - m$

E.g. $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$

Let Z_i be hom coords of $\mathbb{P}^1$

W_j $\mathbb{P}^2$

Let $X_{ij} = Z_i W_j$ $X_{00} \dots X_{12}$ are hom coords of $\mathbb{P}^5$

$$X_{00}X_{11} = X_{01}X_{10}$$
$$X_{00}X_{12} = X_{02}X_{10}$$
$$X_{01}X_{12} = X_{02}X_{11}$$

$\mathbb{P}^N$ in Kähler (Fubini Study: $\omega_0 = \frac{1}{2\pi} \partial\bar{\partial} \log\left(1 + \sum_{i=1}^{N} \left|\frac{z_i}{z_0}\right|^2\right)$)

$\mathbb{P}^N$ is Kähler (Fubini-Study)

with $J = \frac{1}{2\pi} \partial\bar{\partial} \log\left(1 + \sum_{i=1}^{n} \left|\frac{z_i}{z_0}\right|^2\right)$

Algebraic Varieties are Kähler

Kodaira Embedding Theorem

A compact complex manifold M is an algebraic variety if and only if it has a closed, positive $(1,1)$-form, J, such that $[J] \in H^2(M, \mathbb{Q})$.

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(Positive $(1,1)$-form means that the induced Hermitian metric is positive definite)

Sheaves

Let X be a topological space

A "presheaf" $\mathcal{F}$ does the following:

a) Associates an abelian group $\mathcal{F}(U)$ to each open set $U \subset X$ such that $\mathcal{F}(\emptyset) = 0$

b) If $U \subset V$, we define a restriction map (homomorphism)

$$r_{VU} : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$$

c) If $U \subset V \subset W$, then $r_{WU} = r_{VU}\, r_{WV}$.

We use the notation $\sigma|_U$ for $r_{VU}(\sigma)$, $\sigma \in \mathcal{F}(V)$

$U \subset X$

A presheaf is a sheaf if

d) If $U, V \subset X$ and $\sigma \in \mathcal{F}(U)$, $\tau = \mathcal{F}(V)$ such that $\sigma|_{U \cap V} = \tau|_{U \cap V}$

e) If $\sigma \in \mathcal{F}(U \cup V)$ and $\sigma|_U = \sigma|_V = 0$, then $\sigma = 0$

E.g. Suppose $\mathcal{F}(U) = \mathbb{Z}$ for any $U \neq \emptyset$

and $r = \mathbb{1}$

This is a presheaf and not a sheaf.

$U = U_1 \amalg U_2$

Initial data sheaf $\mathcal{Z}$ so that

$$\mathcal{F}(U) = \mathcal{Z}^{\Theta(\#\text{ disconnected components of } U)}$$

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$$\mathcal{F}(U) = \mathbb{Z}^{\oplus (\#\text{ disconnected components of } U)}$$

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$C^\infty(U)$ (under +) is the group of differentiable functions over U

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$C^\infty(U)$ is the group (under $+$) of differentiable functions over U.

$r = 1$

$\mathcal{A}^p$ is sheaf of p-forms

If X is a complex manifold

$\mathcal{O}$ = holomorphic 'functions'

$\mathcal{O}^*$ = nowhere zero hol. functions (under mult)

$\mathcal{A}^{p,q}$ = (p,q)-forms

$Z^{p,q}$ = $\bar{\partial}$-closed (p,q)-forms

$\Omega^p = Z^{p,0}$ "holomorphic p-forms"

$f(z)\, dz^1 \wedge dz^2 \wedge \ldots dz^p$

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Eg. skyscraper sheaf $\mathcal{O}_x$ at a point $x \in X$

$$\mathcal{O}_x(U) = \mathbb{C} \text{ if } x \in U$$
$$= 0 \text{ if } x \notin U$$

$(r = id)$

Suppose $X \subset W$

We may define $\mathcal{O}_X$ as a sheaf on W by

$$\mathcal{O}_X(U) = \text{holomorphic functions on } X \text{ in } X \cap U$$

$U \subset W$

Eg If $x \in X$

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A sheaf which corresponds to sections of a hol. vector bundle is called "locally-free"

Defn. A "map of sheaves" $f: \mathcal{E} \to \mathcal{F}$ satisfies

$$\begin{array}{ccc} \mathcal{E}(U) & \xrightarrow{\rho_{UV}} & \mathcal{E}(V) \\ \downarrow f & & \downarrow f \\ \mathcal{F}(U) & \xrightarrow{f(\rho_{UV})} & \mathcal{F}(V) \end{array}$$

commutes

Subsheaf $\mathcal{E} \subset \mathcal{F} \Rightarrow \mathcal{E}(U) \subset \mathcal{F}(U)$ $\forall U \subset X$

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If $f: \mathcal{E} \to \mathcal{F}$ is a sheaf map,

then $\ker(f)$ and $\mathrm{im}(f)$ are subsheaves of $\mathcal{E}$ and $\mathcal{F}$ resp.