

Title: Complex Manifolds

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Abstract:

L_n

E.g. Tangent bundle on P^1 is L_2 .

Given a real manifold, is it a complex manifold?

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such that $J^2 = -1$, i.e. $J^\mu_\nu J^\nu_\xi = -\delta^\mu_\xi$

On a complex manifold

$$J \cdot dx^1 = -dy^1$$

$$J \cdot dy^1 = dx^1$$

$$J \cdot dz^1 = i dz^1$$

$$J \cdot dz^{\bar{1}} = -i dz^{\bar{1}}$$

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Almost Complex Structure

J is a linear map on cotangent bundle

$$dx^\mu = J^\mu_\nu dx^\nu \quad \mu, \nu = 1, \dots, 2n$$

$$\text{such that } J^2 = -1, \text{ i.e. } J^\mu_\nu J^\nu_\xi = -\delta^\mu_\xi$$

On a complex manifold

$$J \, dx^k = -dy^k \quad k=1, \dots, n$$

$$J \, dy^k = dx^k$$

$$J \, dz^k = i \, dz^k$$

$$J \, dz^{\bar{k}} = -i \, dz^{\bar{k}}$$

An almost complex structure defines a complex structure if it can be "integrated"

Nijenhuis tensor

$$N^\mu_{\nu\lambda} = J^\lambda_\nu (\partial_\lambda J^\mu_\lambda - \partial_\lambda J^\mu_\nu) - J^\lambda_\nu (\partial_\lambda J^\mu_\nu - \partial_\nu J^\mu_\nu)$$

$$N^\mu_{\nu\lambda} = 0 \Leftrightarrow \text{almost cplx str is cplx str}$$

E.g. S^2 is complex

S^8 has no almost complex structure

S^6 has an obvious almost complex which
is not a complex structure

Metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

in complex coords

$$ds^2 = g_{jk} dz^j dz^k + g_{j\bar{k}} dz^j d\bar{z}^k + g_{\bar{j}k} d\bar{z}^j dz^k$$

Metric is "Hermitian"

$$g_{jk} = \overline{g_{\bar{j}\bar{k}}} \quad \text{if} \quad g_{j\bar{k}} = \overline{g_{\bar{j}k}} = 0$$

Metric is Hermitian $\Leftrightarrow$

$$g_{\mu\nu} J^\mu_\xi J^\nu_\zeta = g_{\xi\zeta}$$

This implies

$$\bar{J}_{\mu\nu} \stackrel{\text{def}}{=} g_{\mu\xi} J^\xi_\nu = -J_{\nu\mu}$$

It is natural to define 2-form

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$$\begin{aligned} J &= \frac{1}{2} J_{\mu\nu} dx^\mu \wedge dx^\nu \text{ is a real 2-form on } M \\ &= \frac{i}{2} g_{j\bar{k}} dz^j \wedge d\bar{z}^{\bar{k}} \text{ is a real } (1,1)\text{-form} \end{aligned}$$

has a
A complex manifold with Hermitian metric is "Kähler"
metric if $dJ = 0$, J is Kähler form

A complex manifold is Kähler if it admits a Kähler metric

M is compact

For a Kähler manifold one can show that

$$H_{dR}^k(M, \mathbb{C}) = \bigoplus_{p+q=k} H_{\bar{\partial}}^{p,q}(M)$$

$$\Rightarrow b_k = \sum_{p+q=k} h^{p,q}$$

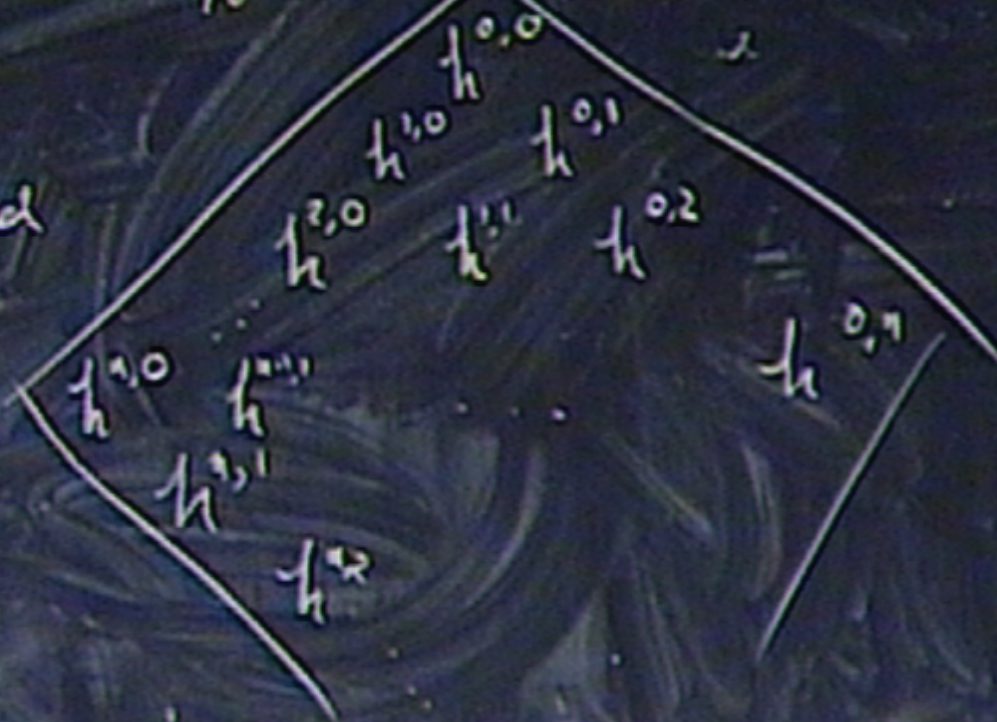
$$H_{\bar{\partial}}^{p,q}(M) = H_{\bar{\partial}}^{q,p}(M)$$

$$\Rightarrow h^{p,q} = h^{q,p}$$

Poincaré duality

$$h^{p,q} = h^{n-p, n-q}$$

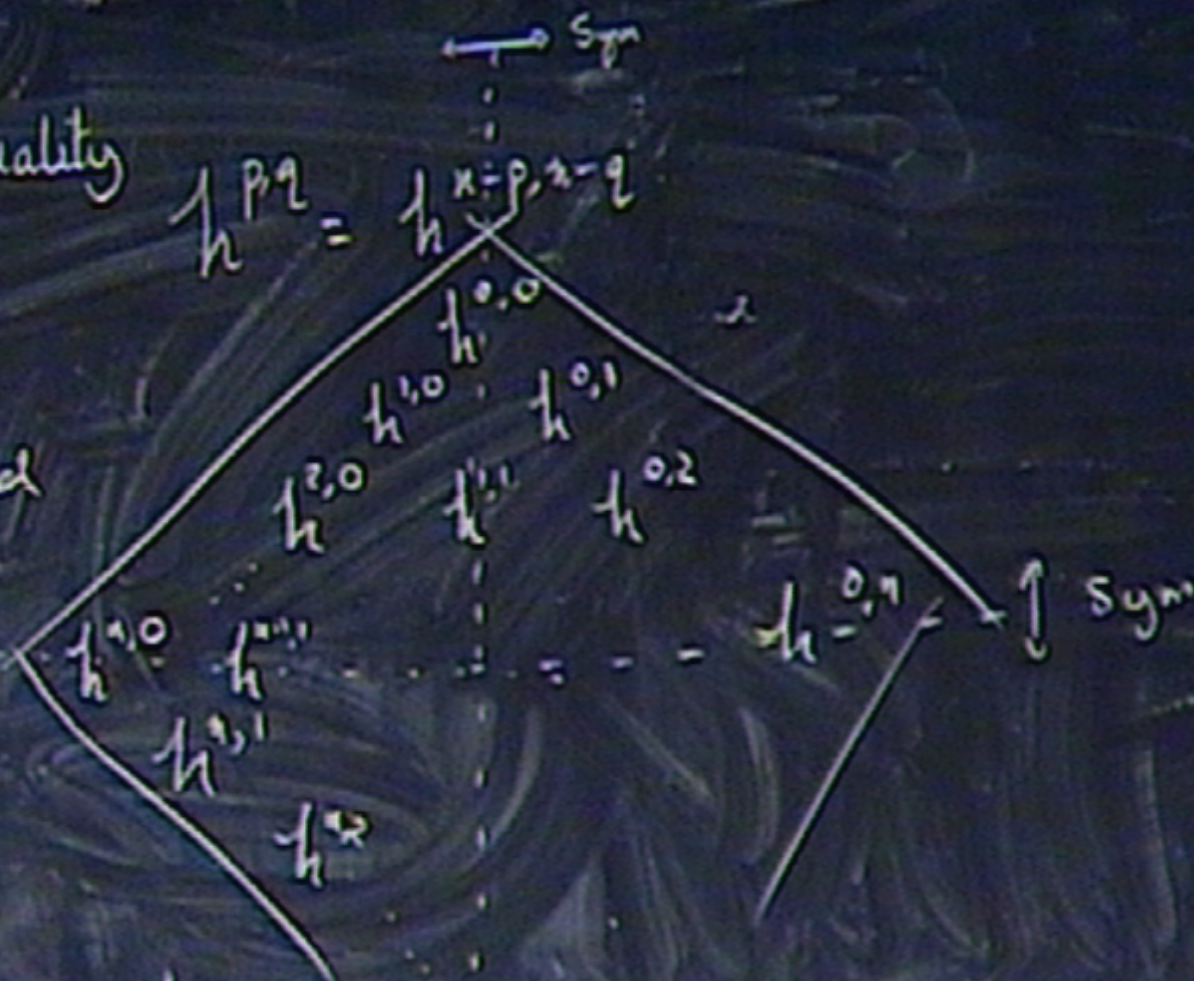
Hodge Diamond



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Algebraic Geometry.

An algebraic variety is the intersection of m equations

$$\left. \begin{aligned} f_1(Z_0, Z_1, \dots, Z_N) &= 0 \\ f_2(Z_0, Z_1, \dots, Z_N) &= 0 \\ &\vdots \\ f_m(Z_0, Z_1, \dots, Z_N) &= 0 \end{aligned} \right\} \begin{array}{l} \text{homogeneous} \\ \text{polynomials} \end{array}$$

in $\mathbb{P}^N$

Algebraic Geometry.

An algebraic variety X is the intersection of m equations

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$$\text{in } \mathbb{P}^N$$

If $m=1$, X is a hypersurface

If $\dim X = N - m$, X is called a complete intersection

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In general $\dim X \geq N - m$

$$\text{E.g. } \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$$

Let Z_i be hom. coords of $\mathbb{P}^1$
 W_j $\mathbb{P}^2$

Let $X_{ij} = Z_i W_j$ $X_{00} \dots X_{12}$ are hom. coords of $\mathbb{P}^5$

$$X_{00} X_{11} = X_{01} X_{10}$$

$$X_{00} X_{12} = X_{02} X_{10}$$

$$X_{01} X_{12} = X_{02} X_{11}$$

$\mathbb{P}^n$ is Kähler

(Fubini Study)

$$\text{ex llo } J = \frac{1}{2\pi} \partial \bar{\partial} \left(\log \left(1 + \sum_{i=1}^n \left| \frac{z_i}{z_0} \right|^2 \right) \right)$$

P^n is Kähler (Fubini Study)

$$\text{or llo } J = \frac{1}{2\pi} \partial \bar{\partial} \left(\log \left(1 + \sum_{i=1}^n \left| \frac{z_i}{z_0} \right|^2 \right) \right)$$

Algebraic Varieties are Kähler

Kodaira Embedding Theorem

A compact complex manifold M is an algebraic variety if and only if it has a closed, positive $(1,1)$ -form ω , such that $[\omega] \in H^2(M, \mathbb{Q})$.

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(Positive $(1,1)$ -form means that the induced Hermitian metric is positive definite)

Sheaves

Let X be a topological space

A "presheaf" $\mathcal{F}$ does the following:

- a) Associates an abelian group $\mathcal{F}(U)$ to each open set $U \subset X$ such that $\mathcal{F}(\emptyset) = 0$

b) If $U \subset V$, we define a restriction map (homomorphism)

$$\uparrow_{vu}: \mathcal{F}(V) \rightarrow \mathcal{F}(U)$$

c) If $U \subset V \subset W$, then $\uparrow_{wu} = \uparrow_{vu} \uparrow_{wv}$.

We use the notation $\sigma|_u$ for $\uparrow_{vu}(\sigma)$, $\sigma \in \mathcal{F}(V)$

$U \subset X$

A presheaf is a sheaf if

d) If $U, V \subset X$ and $\sigma \in \mathcal{F}(U)$, $\tau \in \mathcal{F}(V)$
such that $\sigma|_{U \cap V} = \tau|_{U \cap V}$.

c) If $\sigma \in \mathcal{F}(U \cup V)$ and $\sigma|_U = \sigma|_V = 0$,
then $\sigma = 0$

E.g. Suppose $\mathcal{F}(U) = \mathbb{Z}$ for any $U \neq \emptyset$
and $|r| = 1$

This is a preheaf and not a sheaf.

$$U = U_1 \amalg U_2$$



U

Instead define sheaf $\mathbb{Z}$ so that

$$\mathcal{F}(U) = \mathbb{Z}^{\oplus (\# \text{ disconnected components of } U)}$$

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$$\Gamma = \mathbb{1}$$

$\mathcal{A}^p$ is the set of p -forms

If X is a complex manifold

$\mathcal{O} =$ holomorphic 'functions'

$\mathcal{O}^* =$ nowhere zero hol. functions (under mult.)

$\mathcal{A}^{p,q} = (\phi, \eta)$ -forms

$$\sum^{p,q}$$

$\bar{\partial}$ -closed (p,q) -forms

$$\Omega^p = \sum^{p,0}$$

"holomorphic p-forms"

$$\int \omega dz^1 \wedge dz^2 \wedge \dots \wedge dz^n$$

To any holomorphic vector bundle we may associate
the "sheaf of $\mathcal{A}$ -holomorphic sections."

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On $\mathbb{P}^n$ we denote the sheaf of sections of L_m by $\mathcal{O}(m)$

Eg. skyscraper sheaf $\mathcal{O}_x$ at a point $x \in X$

$$\begin{aligned}\mathcal{O}_x(U) &= \mathbb{C} \text{ if } x \in U \\ &= 0 \text{ if } x \notin U \\ (r = \text{id})\end{aligned}$$

Suppose $X \subset W$

We may define $\mathcal{O}_X$ as a sheaf on W by

$$\mathcal{O}_X(U) = \begin{array}{l} \text{holomorphic functions on } X \\ \text{in } X \cap U \end{array}$$

$U \subset W$

Ex. If $x \in X$

define $\mathcal{J}_x$ as the ideal of functions which vanish at x

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A sheaf which corresponds to sections of a hol. vector bundle
is called "locally-free".

Defn. A "map of sheaves" $f: \mathcal{E} \rightarrow \mathcal{F}$ satisfies

$$\mathcal{E}(U) \xrightarrow{f_U} \mathcal{E}(V)$$

$$\downarrow f$$

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$$\mathcal{F}(U) \xrightarrow{f_U} \mathcal{F}(V)$$

commutes

Subsheaf

$\mathcal{E} \subset \mathcal{F} \rightarrow \mathcal{E}(U) \subset \mathcal{F}(U) \forall U \subset X$
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If $f: \mathcal{E} \rightarrow \mathcal{F}$ is a sheaf map,

then $\ker(f)$ and $\operatorname{im}(f)$ are subsheaves of $\mathcal{E}$ and $\mathcal{F}$ resp.