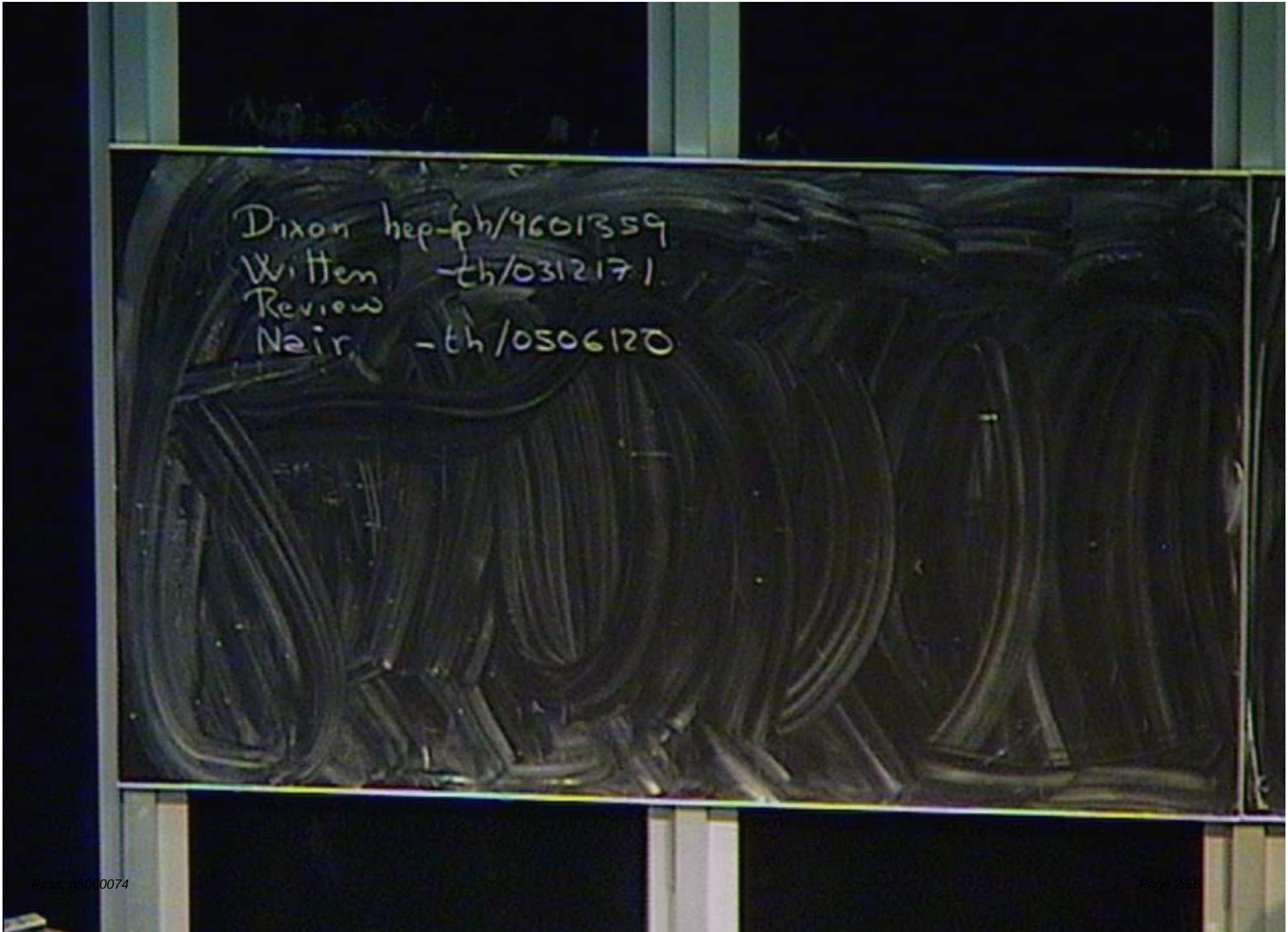
Title: Twistor String Theory and Perturbative Yang-Mills 2

Date: Jun 21, 2005 04:30 AM

URL: http://pirsa.org/05060074

Abstract:


Dixon hep-ph/9601359
Witten -th/0312171
Review
Nair -th/0506120

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Review

Topolog. String Theory
B-model on $CP^{3|4}$ $\longleftrightarrow$ PHSYM
+ N D5-branes
+ Dirac Fermions $\leftarrow$ Challenges

Conformal Group

$\lambda_\alpha, \bar{\lambda}^{\dot{\alpha}} \to (\lambda_\alpha, i\frac{\partial}{\partial \mu_{\dot{\alpha}}})$

$SL(4,\mathbb{C}) \to \mathbb{C}^4 \to \mathbb{C}^3 = (\lambda_1, \lambda_2, \mu_{\dot{1}}, \mu_{\dot{2}})$

Twistor Space — Twistor

$H_n(++--)$

Conformal Group

$\lambda_\alpha, \bar{\lambda}^{\dot{\alpha}} \to (\lambda_\alpha, i\frac{\partial}{\partial \mu^{\dot{\alpha}}})$

$SL(4,\mathbb{C}) \to \mathbb{C}^4 \to \mathbb{C}^3 = (\lambda_1, \lambda_2, \mu_1, \mu_2)$

Twistor Space — Twistor

$H_n\ (++--)$ $\quad \lambda, \bar{\lambda}$ Real

$SO(2,2) \xrightarrow[\text{Conf}]{} SO(3,3) \cong SL(4,\mathbb{R}) \to \mathbb{R}^4$

- What about a given Amplitude?

$$\{\lambda_j, \tilde{\lambda}_j, h_j\}) = \int \prod_{i=1}^{n} \frac{d^2\tilde{\lambda}_i}{(2\pi)^2} e^{i[p_i \tilde{\lambda}_i]} \tilde{A}(\{\lambda_j, \tilde{\lambda}_j, h_j\})$$

$$\left(\lambda_i \frac{\partial}{\partial \tilde{\lambda}_i} - \tilde{\lambda}_i \frac{\partial}{\partial \lambda_i}\right) A(\{\lambda \; \tilde{\lambda}\}) = -2h_i A(\{\lambda \; \tilde{\lambda}\})$$

$$(\lambda_i \quad \frac{\cdot}{-} \quad \frac{+}{-} \quad -$$

$$- \quad +$$

- What about a given Amplitude?

$$\{\lambda_j, \mu_j, h_j\}) = \int \prod_{i=1}^{n} \frac{d^2\zeta_i}{(2\pi)^2} e^{i[\mu_i \zeta_i]} \tilde{A}(\{\lambda_j, \zeta_j, h_j\})$$

$$\left(\lambda_i \frac{\partial}{\partial \lambda_i} - \lambda_i \frac{\partial}{\partial \lambda_q}\right) A(\{\quad\}) = -2h_i A(\{\quad\})$$

$$\left(\lambda_i \frac{\partial}{\partial \lambda_i} + \mu_i \frac{\partial}{\partial \mu_i}\right) A(\quad) = (-2h_i - 2) A(\mu\ \zeta)$$

- What about a given Amplitude?

$$\{\lambda_j, \mu_j, h_j\}) = \int \prod_{i=1}^{n} \frac{d^2 z_i}{(2\pi)^2} e^{i[p_i z_i]} \tilde{A}(\{\lambda_j, \tilde{\lambda}_j, h_j\})$$

$$\left(\lambda_i \frac{\partial}{\partial \lambda_i} - \tilde{\lambda}_i \frac{\partial}{\partial \tilde{\lambda}_i}\right) A(\{\lambda, \tilde{\lambda}\}) = -2h_i A(\{\lambda, \tilde{\lambda}\})$$

$$\left(\lambda_i \frac{\partial}{\partial \lambda_i} + \mu_i \frac{\partial}{\partial \mu_i}\right) A(\quad) = (-2h_i - 2) A(\quad)$$

$$z^I \to t z^I \quad t \in \mathbb{C}^*, \qquad -2h_i - 2, \quad \mathbb{C}^4$$

$Z^I \sim t Z^I$ $\mathbb{CP}^3$ or $\mathbb{RP}^3$

$A \rightarrow$ section line bundle $\mathcal{O}(-2h;-2)$

MHV amplitude $(A^+_{\cdots}\, A^{(-)^+_j}\, \bar{\imath}^-_{\cdots}\, \bar{\jmath}^-_{\cdots}\, A^{h^+})$

$$\int \prod_{i=1}^{n} \frac{d^2\tilde{\lambda}_i}{(2\pi)^2}\, e^{i[\mu_i \tilde{\lambda}_i]}\, \delta^{(4)}\Big(\sum_{a=1}^{n} \lambda_a \tilde{\lambda}_a\Big) \left(\frac{\langle ij \rangle^4}{\prod_{a=1}^{n} \langle a\, a+1 \rangle} \right)$$

$Z^I \sim t Z^I$ $\mathbb{CP}^3$ or $\mathbb{RP}^3$

$A \rightarrow$ section line bundle $\mathcal{O}(-2h_i - 2)$

MHV amplitudes $(A^+_{\cdots}(-)^+_{\cdots}\tilde{\lambda}_{\cdots}\tilde{\lambda}_{\cdots}\tilde{\lambda}_{\cdots}A^+_{\cdots})$

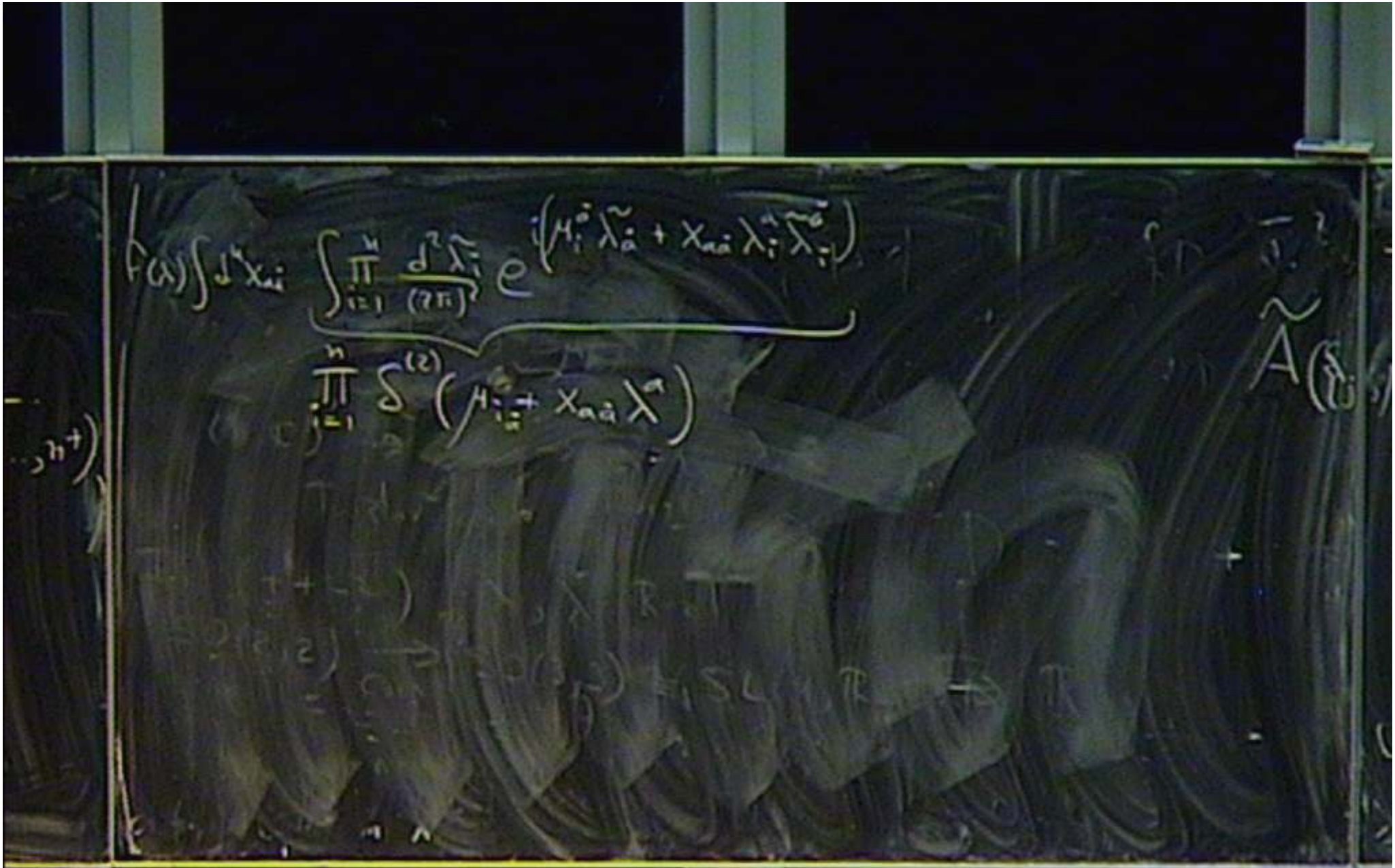
$$\int \prod_{i=1}^{n} \frac{d^2\tilde{\lambda}_i}{(2\pi)^2} e^{i[\mu_i \tilde{\lambda}_i]} \delta^{(4)}\left(\sum_{\alpha=1}^{n} \lambda_\alpha \tilde{\lambda}_\alpha\right) \left(\frac{\langle \cdots \rangle^4}{\prod_{\alpha=1}^{n} \langle \alpha\, \alpha+1 \rangle}\right) \int d^4 x_{\alpha\alpha} e^{i x_{\alpha\alpha} \sum_{i=1}^{n} \lambda_\alpha^i \tilde{\lambda}_\alpha^i} \frac{1}{\cdots} \quad P(\lambda)$$

$Z^I \sim t Z^I$ $\mathbb{CP}^3$ or $\mathbb{RP}^3$

A → section line bundle $\mathcal{O}(-2h_i-2)$

MHV amplitudes $(A^+_{\cdots}, A_{(-)^+_j \cdots}, \bar{A}^-_{\cdots}, A_{\cdots}, A^{n^+})$

$$\int \prod_{i=1}^{n} \frac{d^2\tilde{\lambda}_i}{(R\,m)^2} e^{i[\mu_i \lambda_i]} \underbrace{\delta^{(4)}\left(\sum_{\alpha=1}^{n} \lambda_\alpha \tilde{\lambda}_\alpha\right)}_{\int d^4 x_{aa} e^{i x_{aa} \sum_{i=1}^{n} \lambda_i \tilde{\lambda}_i}} \underbrace{\left(\frac{\langle ij\rangle^4}{\prod_{\alpha=1}^{n} \langle \alpha\, \alpha+1\rangle}\right)}_{P(\lambda)}$$


$f(\omega)\int d^4x_{ai}\ \underbrace{\prod_{i=1}^{n}\frac{d^2\tilde{\lambda}_i}{(2\pi)}\, e^{i(\mu_i^a\tilde{\lambda}_a + x_{ai}\lambda_i^a\tilde{\lambda}_a)}}$
$\prod_{i=1}^{n}\delta^{(2)}(\mu_i^a + x_{ai}\lambda^a)$
$\tilde{A}$

$$\int d^4x_{ai} \underbrace{\int \prod_{i=1}^{n} \frac{d^2\tilde{\lambda}_i}{(2\pi)^2} e^{i(\mu_i^a \tilde{\lambda}_a + x_{ai} \lambda_i^a \tilde{\lambda}_i^a)}}_{\prod_{i=1}^{n} \delta^{(2)}(\mu_{ia} + x_{ai}\lambda^a)}$$

$$\mu_{ia} + x_{ai}\lambda^a = 0$$

$(++--)$ $\quad \mathbb{R}^4 \to \mathbb{RP}^3 \quad (\lambda_1, \lambda_2, \mu_1, \mu_2)$

$\mathbb{R}^3$

$y_1 = \frac{\lambda_2}{\lambda_1}, \quad y_2 = \frac{\mu_1}{\lambda_1}, \quad y_3 = \frac{\mu_2}{y_2}$

$$f(\lambda)\int d^4x_{ai}\ \underbrace{\prod_{i=1}^{n}\frac{d^2\tilde{\lambda}_i}{(2\pi)^2}\, e^{i(\mu_i^a\tilde{\lambda}_a + x_{ai}\lambda_i^a\tilde{\lambda}_a)}}$$

$$\prod_{i=1}^{n}\delta^{(2)}(\mu_{ia} + x_{ai}\lambda^a)$$

$$\mu_{ia} + x_{ai}\lambda^a = 0$$

$(++--)$

$\mathbb{R}^4 \to \mathbb{RP}^3$

$\mathbb{R}^3$

$(\lambda_1, \lambda_2, \mu_1, \mu_2)$

$y_1 = \frac{\lambda_2}{\lambda_1}$, $y_2 = \frac{\mu_1}{\lambda_1}$, $y_3 = \frac{\mu_2}{y_2}$

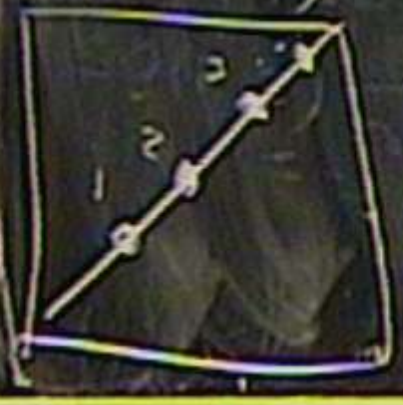
$$\int d^4x_{ai} \underbrace{\prod_{i=1}^{n} \frac{d^2\tilde{\lambda}_i}{(2\pi)^2} e^{i(\mu_i^a \tilde{\lambda}_a + x_{a\dot{a}} \lambda_i^a \tilde{\lambda}_i^{\dot{a}})}}$$

$$\prod_{j=1}^{n} \delta^{(2)}(\mu_{ia} + x_{a\dot{a}} \lambda^a)$$

$$\mu_{ia} + x_{a\dot{a}} \lambda^a = 0 \qquad d_1 = 1 \quad d_2 = 1$$

Holomorphic curve of $J=0$ $d=1$

$(++--)$ $\mathbb{R}^4 \to \mathbb{RP}^3$ $(\lambda_1, \lambda_2, \mu_1, \mu_2)$



$\mathbb{R}^3$

$\mathbb{CP}^3$

$y_1 = \frac{\lambda_2}{\lambda_1}$, $y_2 = \frac{\mu_1}{\lambda_1}$, $y_3 = \frac{\mu_2}{y_2}$

$f(z_1, \ldots, z_n) = 0$ $\quad g(z_1, \ldots, z_n) = 0$

d_1 $\qquad\qquad$ d_2

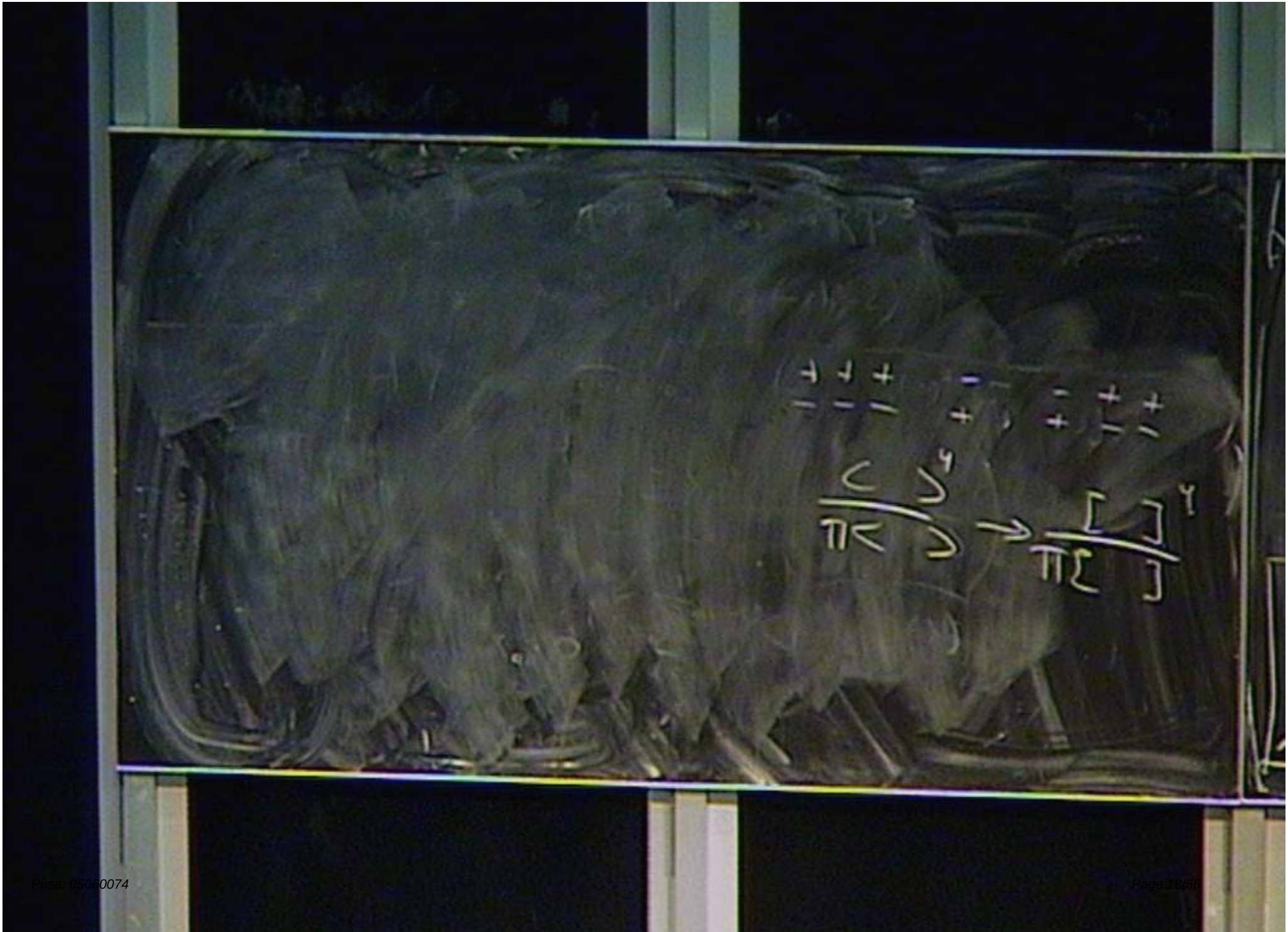
$d = d_1 d_2$

Witten. $A(1^+, 2^+ \ldots, i^-, j^-, k^-, \ldots n^+)$

q - helicity $\qquad$ p + helicity

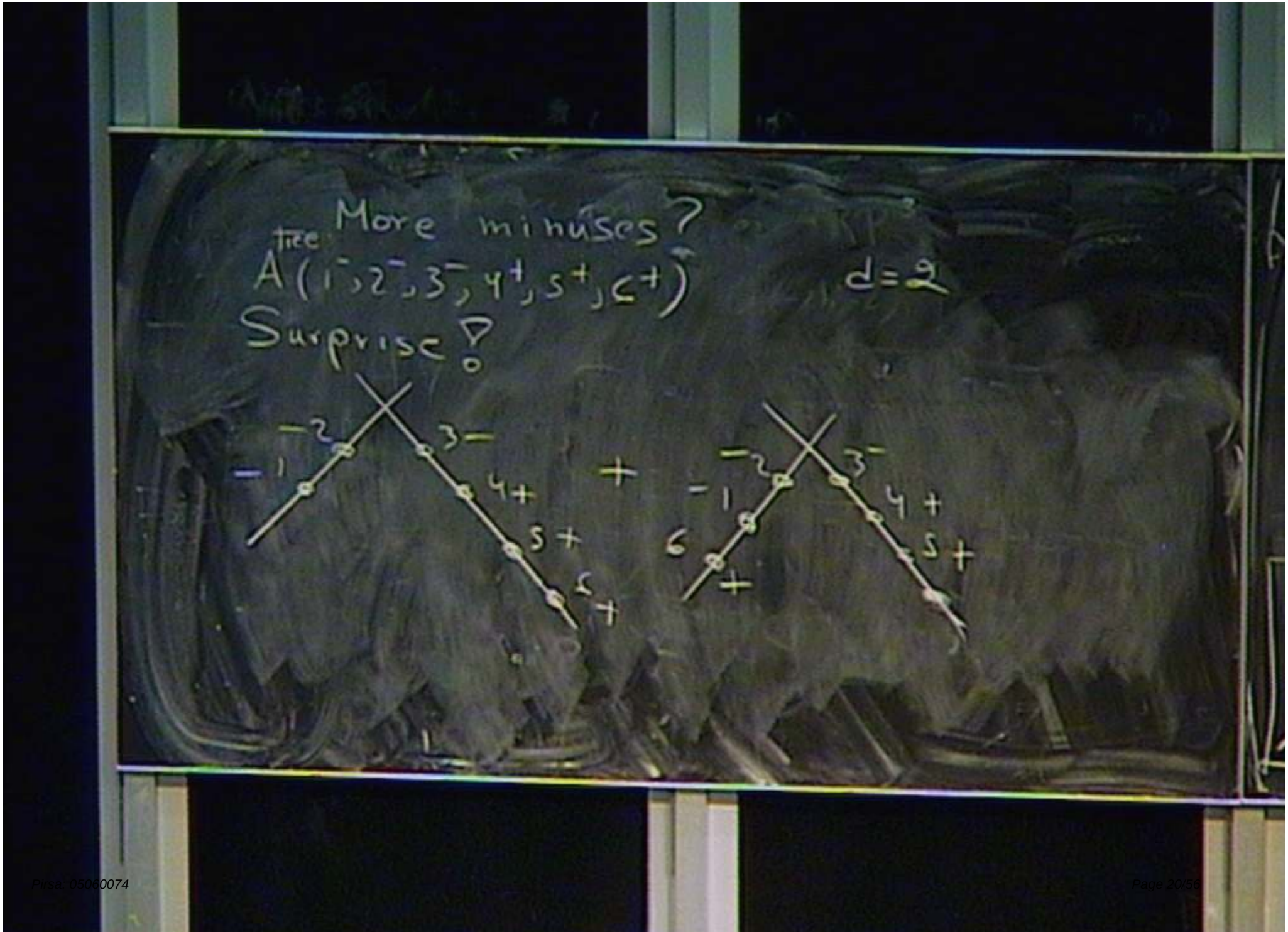
L - loop. $\qquad d = q - 1 + L$

$q \leq L$



More minuses?

$A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$

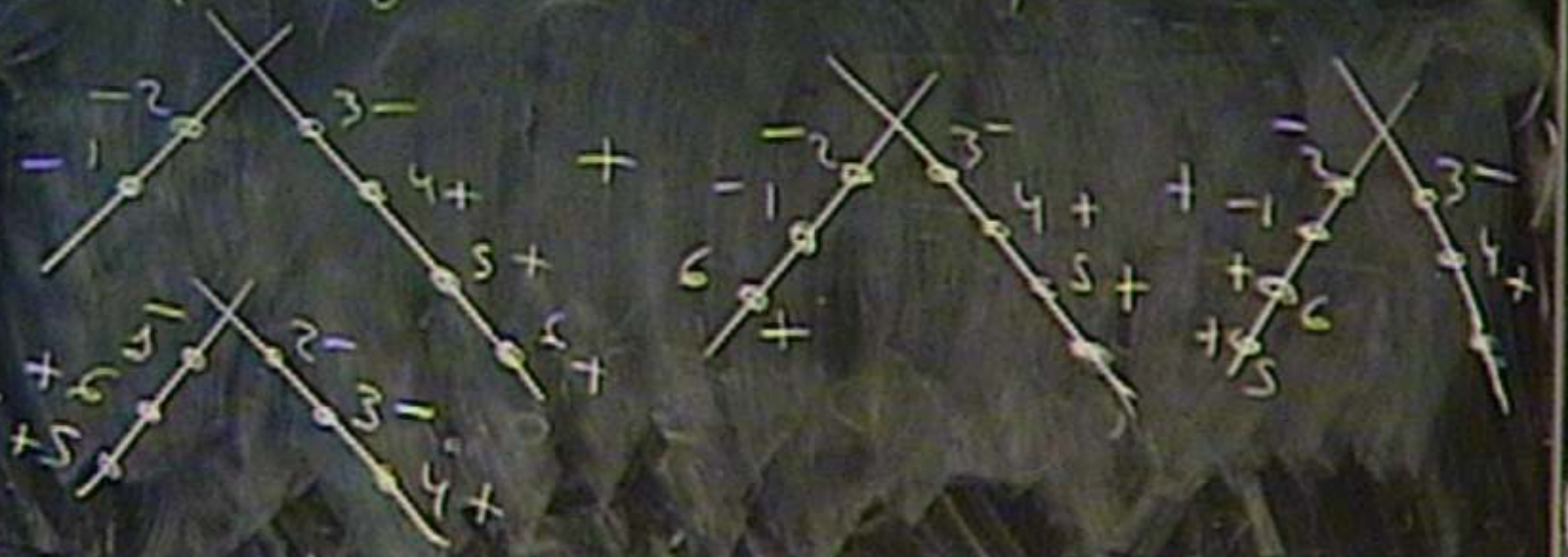

More minuses?
$A_{tree}(1^-,2^-,3^-,4^+,5^+,6^+)$
$d=2$
Surprise?
-2
3-
-1
4+
5+
6+
+
-2
3-
-1
4+
6
5+
+

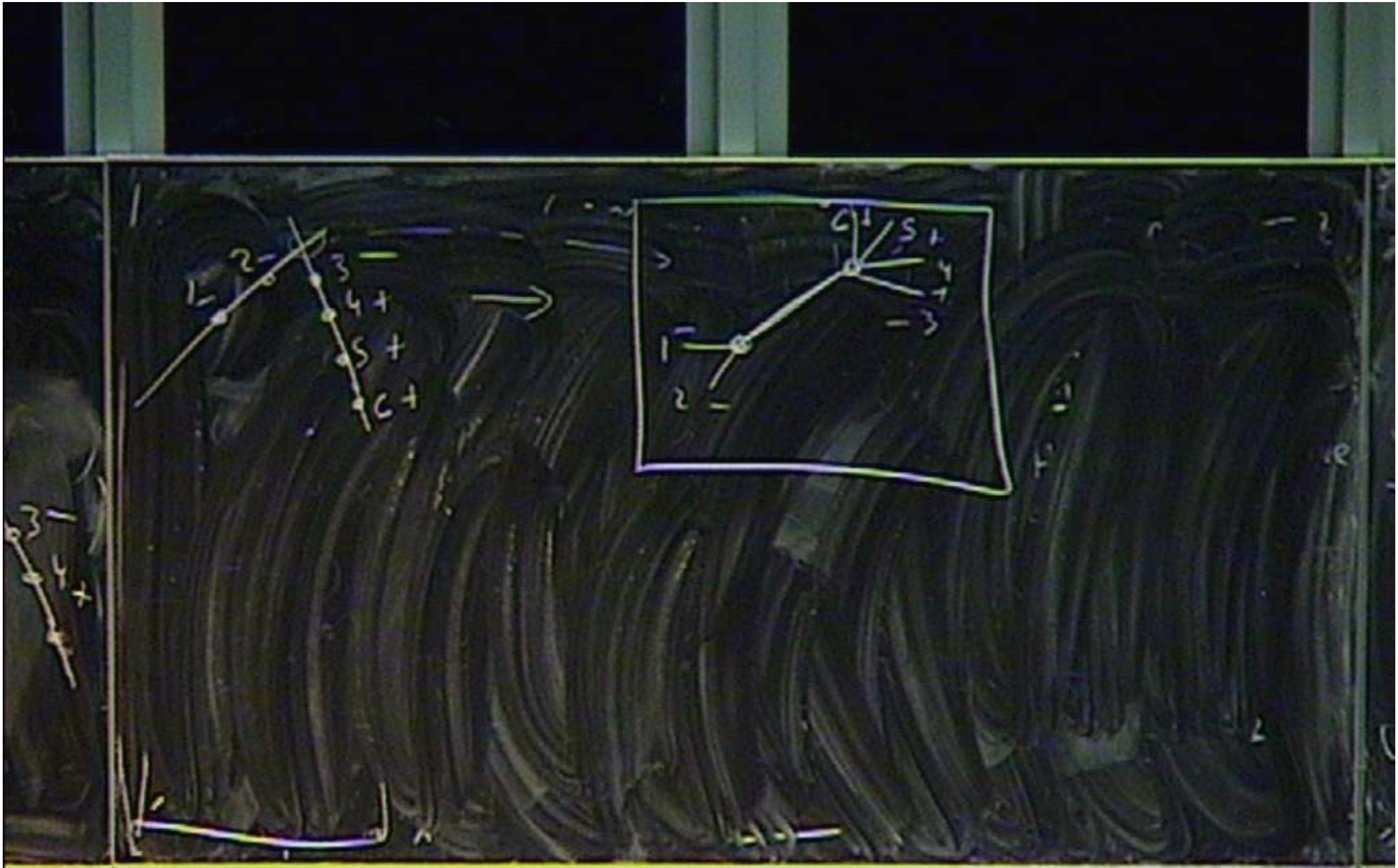
More minuses?

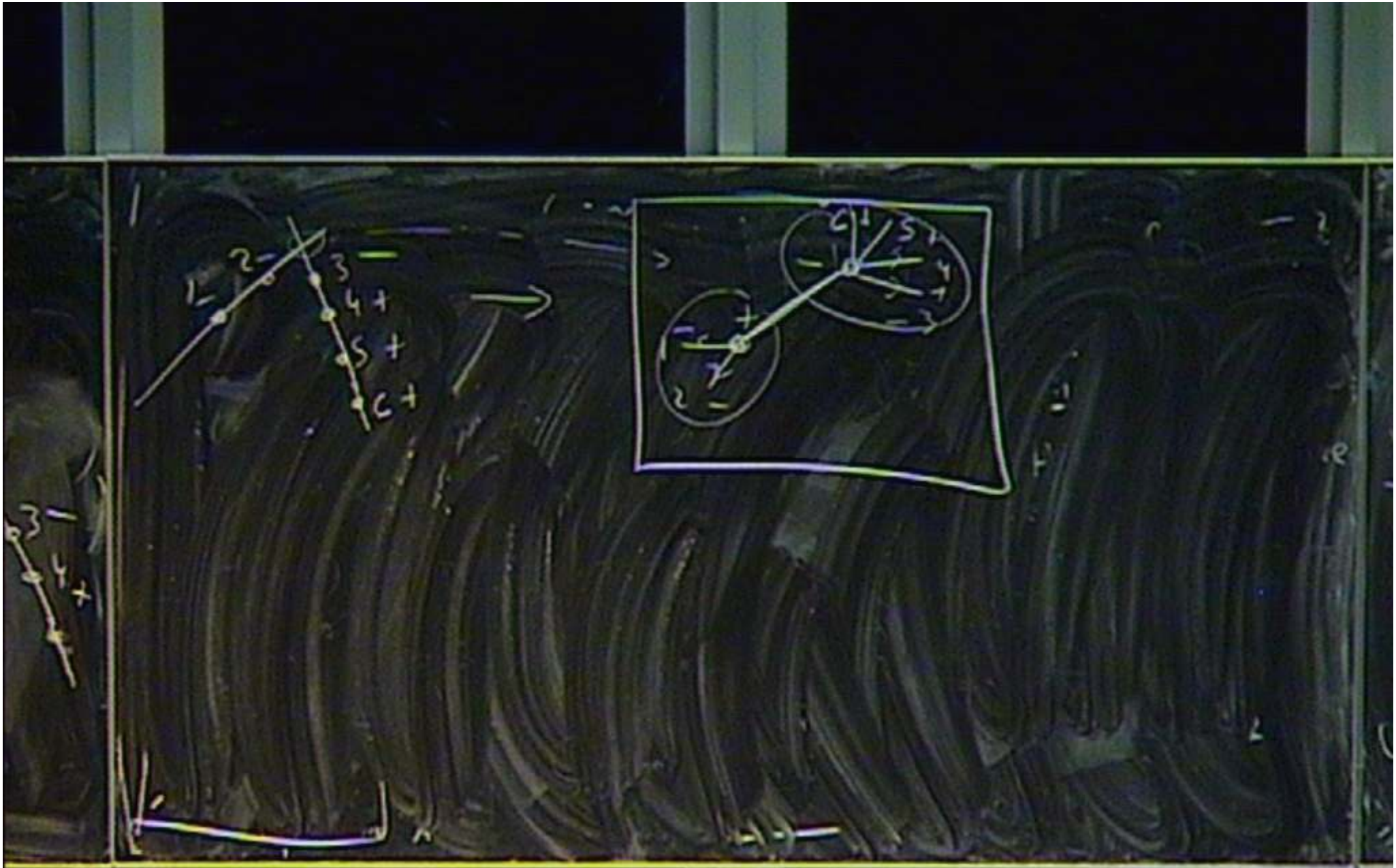
Tree

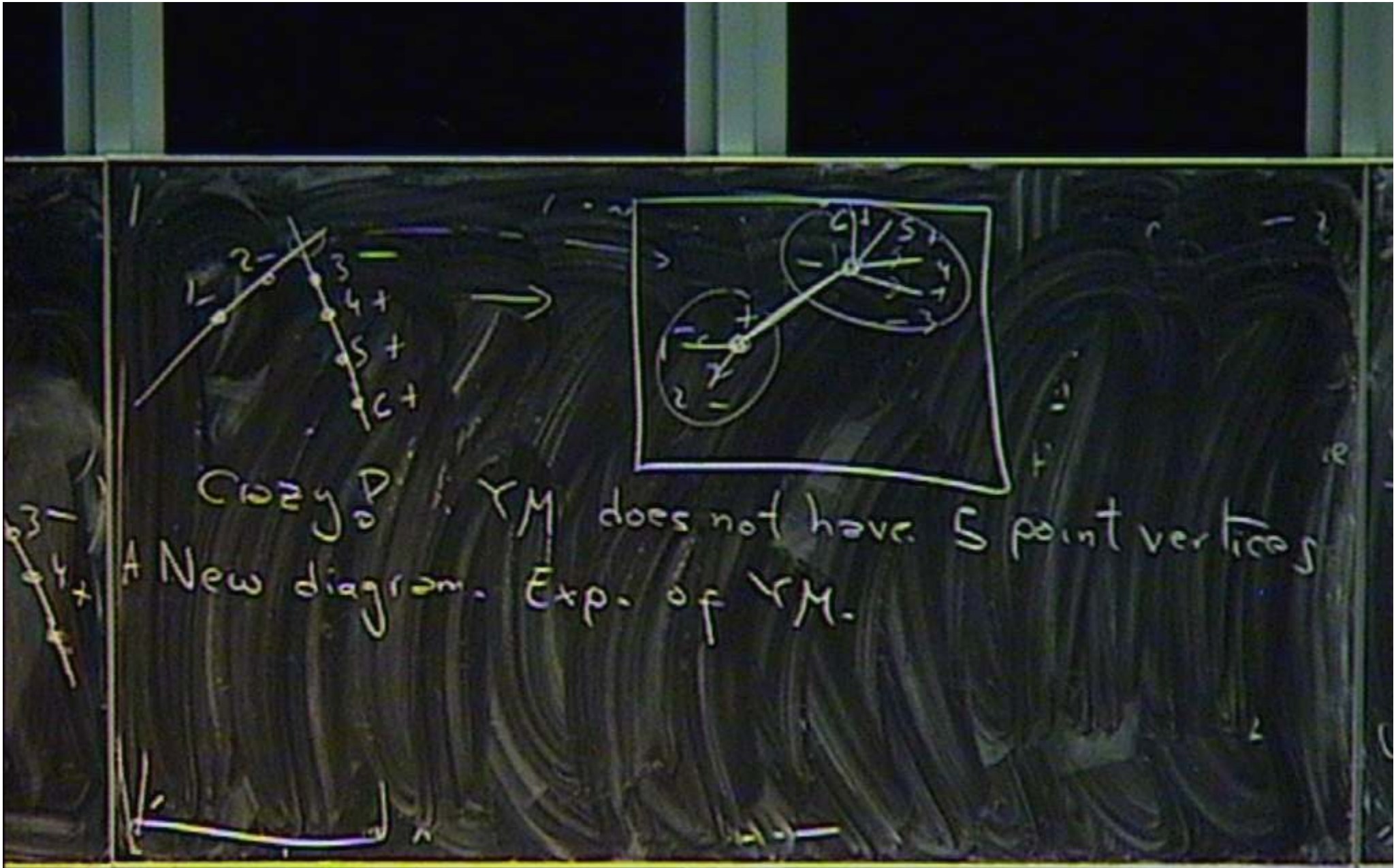
$A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ $d=2$

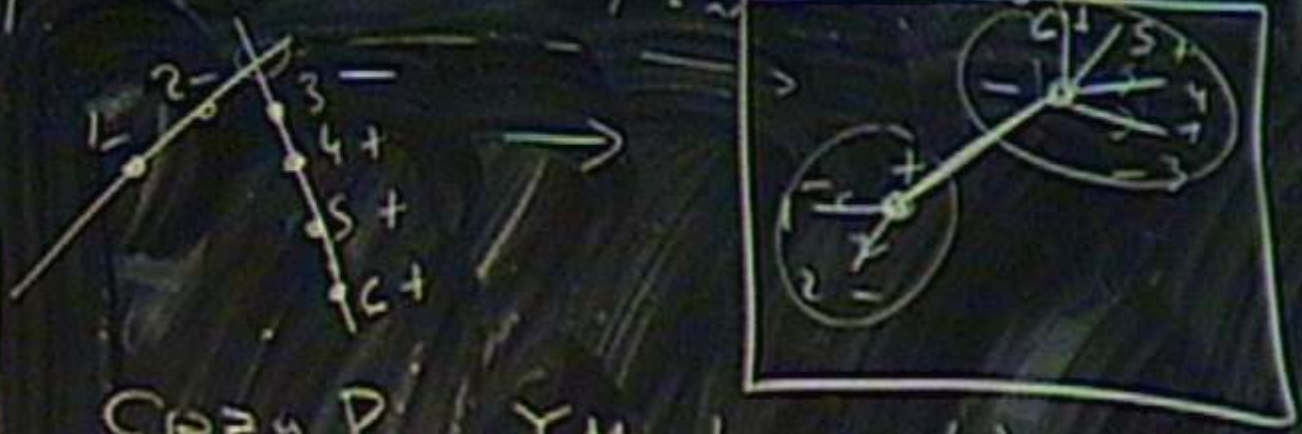
Surprise?




1
2
3
4
5
6
1
2
3
4
5
6
1
2
3
4


1
2
3
4 +
5 +
6 +
3
4 +
x


1−
2−
3−
4+
5+
6+
Crazy Pt. YM does not have 5 point vertices
A New diagram. Exp. of YM.



Crazy Do! YM does not have 5 point vertices

A New diagram. Exp. of YM.

- Vertices are MHV amplitudes
- Propagators

$A_L(1^-, 2^-, \hat{P}^+)$ $A_R(\hat{P}^-, 3^-, 4^+, 5^+, 6^+)$ $\rightarrow$ $d = 4 - n$

$4 - 3 = 1$ $4 - 5 = -1$ $= 4 - 6 = -2$

$A_L(1^-, 2^-, \hat{P}^+) \frac{1}{P^2} A_R(\hat{P}^-, 3^-, 4^+, 5^+, 6^+) \to$ dim $4-n$ $= 4-6 = -2$

$4-3=1$ $\qquad 4-5=-1$

$P = P_1 + P_2 \qquad P^2 \neq 0 \qquad \hat{P}^2 = 0$

$P_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} + \lambda'_a \tilde{\lambda}'_{\dot{a}}$

$\hat{P}_{a\dot{a}}$

$P_{a\dot{a}} \tilde{\lambda}^{\dot{a}} = \lambda_a \hat{P} [\tilde{\lambda}, \tilde{\lambda}'] + \lambda_a [\tilde{\lambda}', \tilde{\lambda}']$

0

$\lambda_{\hat{P}}$

$$\underbrace{A_L(1^-, 2^-, \hat{P}^+)}_{4-3=1} \frac{1}{P^2} A_R(\hat{P}^-, 3^-, 4^+, 5^+, 6^+) \longrightarrow \text{dim } 4-n = 4-6 = -2$$

$$4-5 = -1$$

$$P = P_1 + P_2 \qquad P^2 \neq 0 \qquad \hat{P}^2 = 0$$

$$P_{a\dot{a}} = \underbrace{\lambda_a \tilde{\lambda}_{\dot{a}}}_{\hat{P}_{a\dot{a}}} + \lambda'_a \tilde{\lambda}'_{\dot{a}}$$

$$\lambda_{\hat{P}} = \frac{P_{a\dot{a}} \tilde{\lambda}^{\dot{a}}}{[\tilde{\lambda}, \tilde{\lambda}']} \qquad P_{a\dot{a}} \tilde{\lambda}^{\dot{a}} = \lambda^{\hat{P}}_a [\tilde{\lambda}, \tilde{\lambda}'] + \lambda'_a \underbrace{[\tilde{\lambda}', \tilde{\lambda}']}_{0}$$

$$\frac{\langle 12\rangle^3}{\langle 2\hat{P}\rangle\langle\hat{P}1\rangle}\,\frac{1}{P^2}\,\frac{\langle\hat{P}3\rangle^3}{\langle 3|x_{45}|5\rangle\langle 5|x_{6\hat{P}}|\hat{P}]}$$

$$\lambda_{\hat{P}a} \equiv P_{a\dot{a}}\eta^{\dot{a}} \qquad \lambda^{\mu} = \eta$$

Fixed but arbitrary η.

$$\frac{\langle 12\rangle^3}{\langle 2\hat{P}\rangle\langle\hat{P}1\rangle}\,\frac{1}{P^2}\,\frac{\langle \hat{P}3\rangle^3}{\langle 3|x_{4\hat{P}}\rangle\langle\hat{P}|x_{\hat{P}4}|\hat{P}\rangle}$$

$$\lambda_{\hat{P}\alpha} \equiv P_{\hat{P}\dot{a}}\eta^{\dot{a}} \qquad \lambda^{\alpha} = \eta$$

Fixed but arbitrary η.

$$A(1^-, 2^-, 3^-, 4^+, 5^+, \ldots, n^+) = \sum_i$$

Is this correct?

Key; Rational functions in $\lambda, \tilde{\lambda}$ $P \in \mathbb{C}^4$

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- One can check that MHV diag. produce correct singul.

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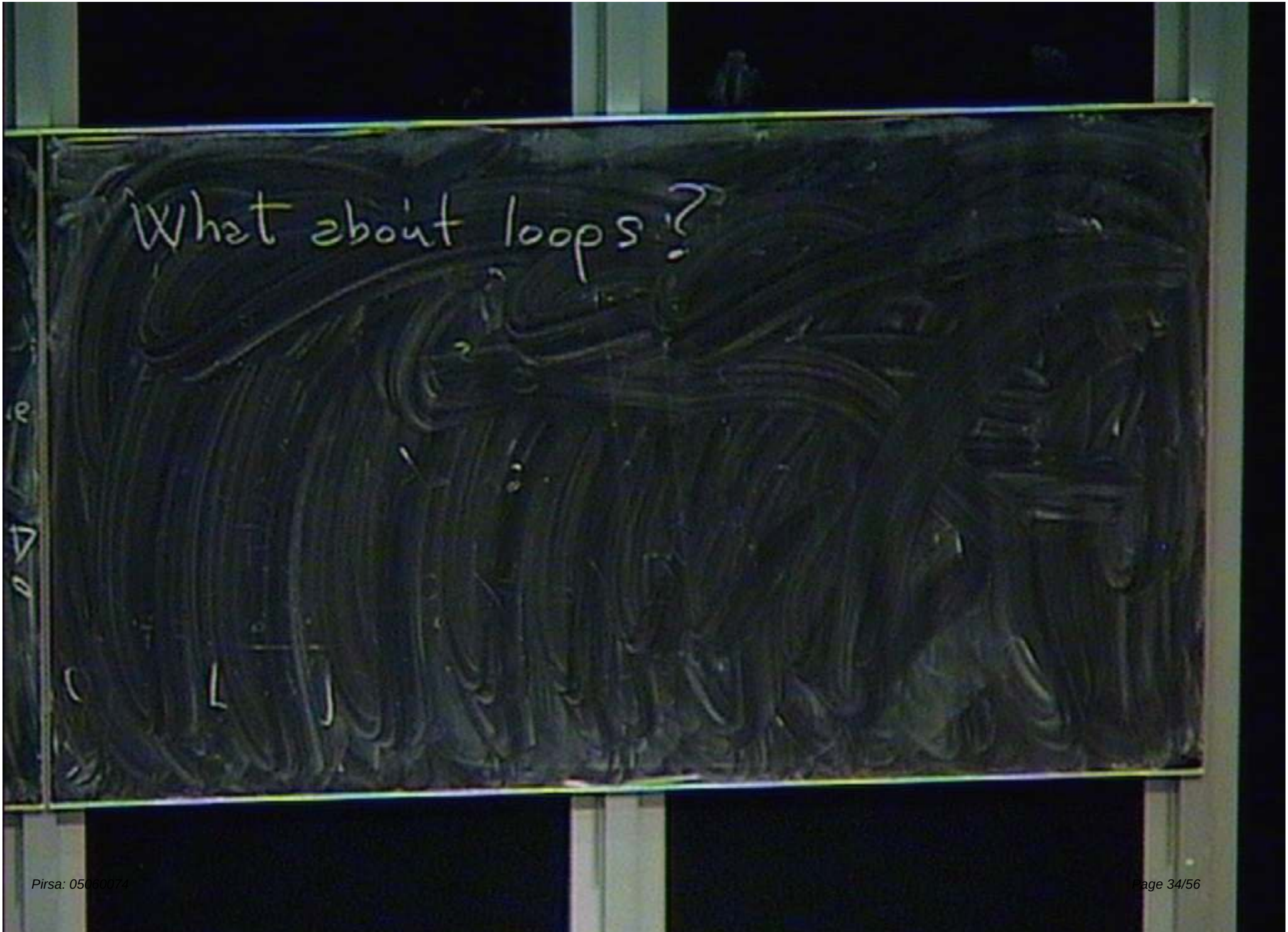
- One can check that MHV diag. produce correct singul.
- $\frac{1}{\langle P, \eta \rangle}$ η is fixed
- Sum of all MHV diag. is η-indep
- Restore LI.

$\frac{1}{\langle P, \eta \rangle}$ are spurious!


What about loops?

What about loops?

$\mathcal{N}=4$ SYM $\quad (W_\alpha, 3\Phi)$

$\mathcal{N}=1 \quad W_\alpha = (A_\mu, \gamma) \quad \Phi = (\phi, \gamma)$

$g \neq 4f + 3s$!

What about loops?

$\mathcal{N}=4$ SYM $\quad \mathcal{N}=1$ $\quad W_\alpha = (A_\mu, \gamma)$

$(W_\alpha, 3\Phi)$ $\quad SU_R(4)$ $\quad \Phi = (\phi, \gamma)$

$g \neq 4f + 3s$!

What about loops?

$\mathcal{N}=4$ SYM $\quad \mathcal{N}=1$ $\quad W_\alpha = (A_\mu, \gamma)$

$(W_\alpha, 3\Phi)$ $\quad SU_R(4)$ $\quad \Phi = (\phi, \gamma)$

$g \to 4f + 3S$

$S \to 6$

$f \to 4, \bar{4}$

$g \to 1$

(L)

MHV amplitudes at 1-loop

$d = 2g - 1 + L \qquad g \leq L$

$= 2 - 1 + 1 = 2 \qquad g \leq 1$

Surprise

MHV amplitudes at 1-loop

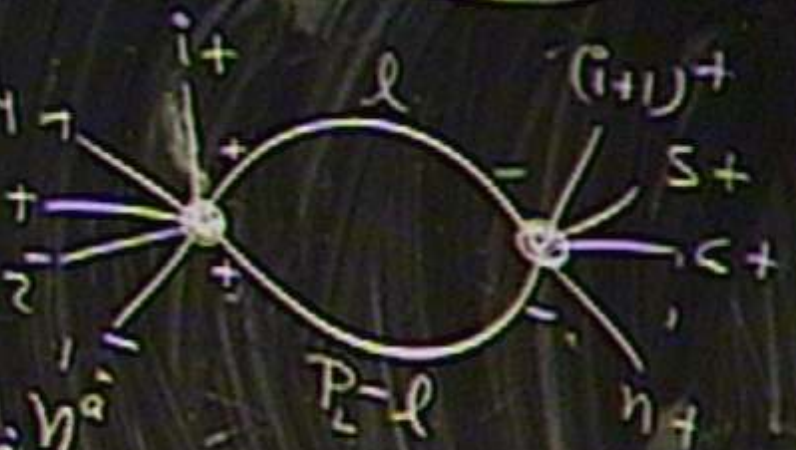
$d = 2g - 1 + L \qquad g \leq L$

$= 2 - 1 + 1 = 2 \qquad g \leq 1$

Surprise

$=$

$\sum_{\text{loops}} \int d^4\ell$



Open Problem

$\lambda_\ell = \ell_\alpha \eta^{\dot\alpha}$

$(\lambda_{P-\ell})_{\dot\alpha} = (P-\ell)_{\alpha\dot\alpha}\eta^{\dot\alpha}$

MHV amplitudes at 1-loop

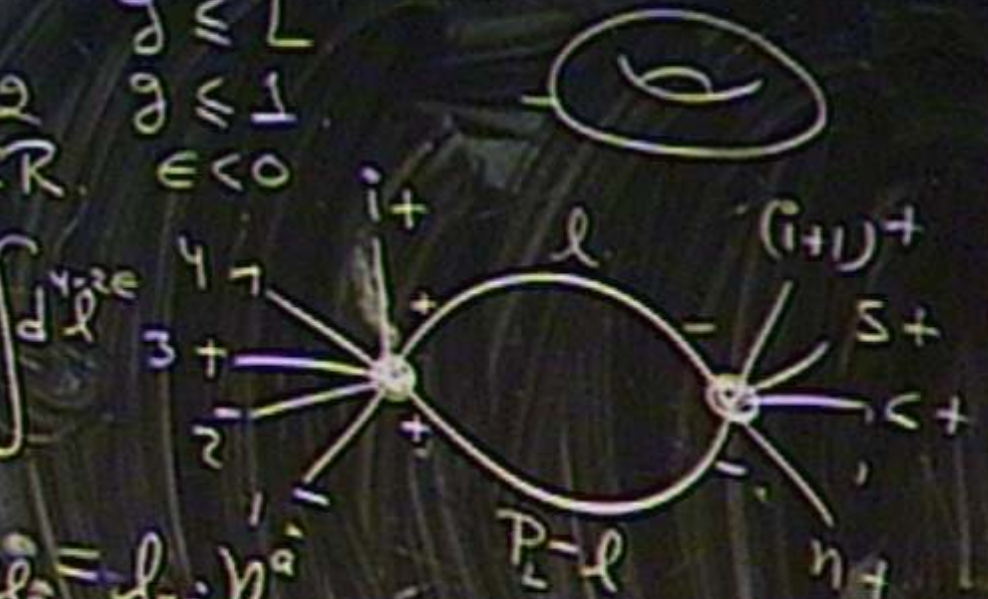
$d = 2g - 1 + L \qquad g \leq L$

$= 2 - 1 + 1 = 2 \qquad g \leq 1$

Surprise IR $\epsilon < 0$

$=$ Open Problem

$\sum_{\text{1h}} \int d^{4-2\epsilon}\ell$



$\lambda_\ell = \ell_{\alpha\dot{a}} \eta^{\dot{a}}$

$(\lambda_{P-\ell})_a = (P-\ell)_{\alpha\dot{a}} \eta^{\dot{a}}$

Q: Does this work for non-MHV 1-loop amplitudes?

Ex Open problem.

Q: Does this work for non-MHV 1-loop amplitudes?

Ex. Open problem.

$$= \text{(one-loop diagram)} + \text{Permut.}$$

$$\text{1-loop}\left(\quad\right) = \sum \underbrace{\mathrm{Tr}(T^{a_1}\cdots T^{a_n})\, A(\quad)}_{\text{Leading color}} + \text{partial ampl.}$$

$$+ \sum_c \mathrm{Tr}(T^{a_1}\cdots T^{a_k})\,\mathrm{Tr}(T^{a_{k+1}}\cdots T^{a_n})\, A(\quad)$$

What is known about Loop amplitudes?

90's

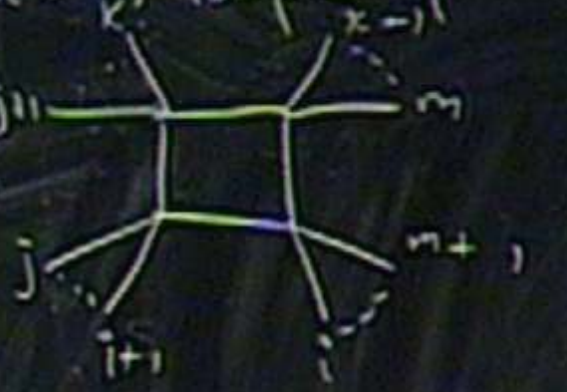
1-loop

$A(1, \cdots, n) =$

What is known about 1-loop ampl.tudes ?

90's
1-loop

$$A(1,\ldots,n) = \sum_{1\le i<j<k<m\le n} B_{ijkm}$$

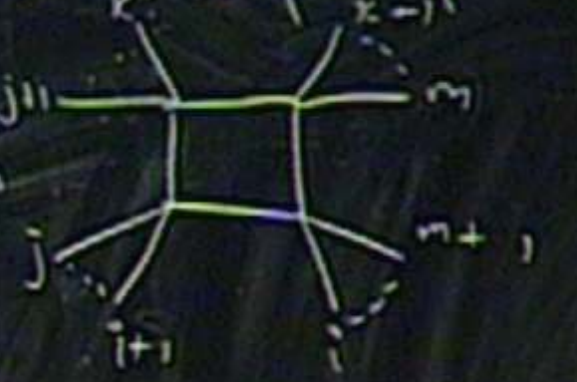


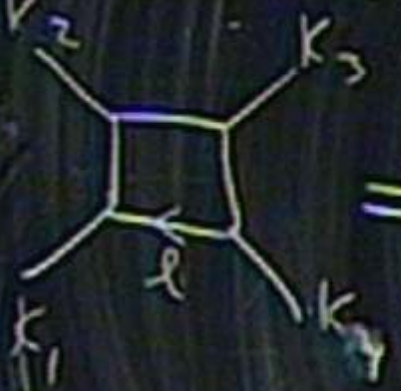
What is known about 1-loop amplitudes?

90's

1-loop

$$A(1,\ldots,n) = \sum_{1\le i<j<k<m} B_{ijkm}$$





$$= \int d^4\ell^{2\epsilon} \frac{1}{\ell^2(\ell-k_1)^2(\ell-k_1-k_2)^2(\ell+k_4)^2}$$

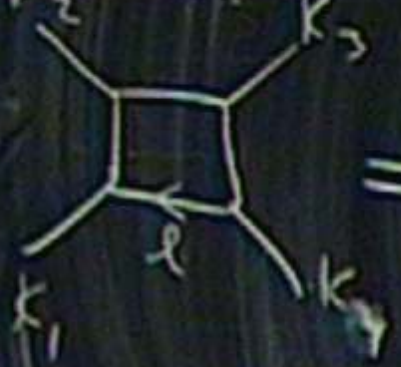
Scalar Box integral

What is known about 1-loop amplitudes?

90's

$$A^{1\text{-loop}}(1,\ldots,n) = \sum_{1<i<j<k<l<m} B_{ijkm}$$

Rational funct. $\langle \rangle, L]$



$$= \int d^4\ell^{2\epsilon} \frac{1}{\ell^2(\ell-k_1)^2(\ell-k_1-k_2)^2(\ell+k_4)^2}$$

$Li_2(x)$, $\ln(x)$

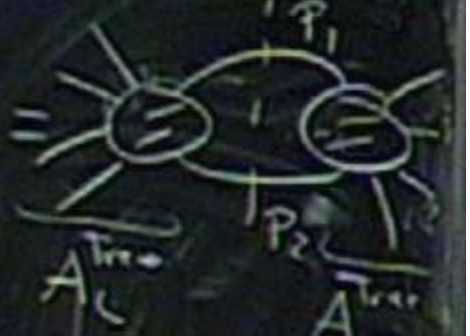
Scalar Box integral

Natural singularity → Unique.

A Branch cuts

Discont. across the B.C

$$\frac{1}{P^2+i\varepsilon} = PV\left(\frac{1}{P^2}\right) + \delta(P^2)$$



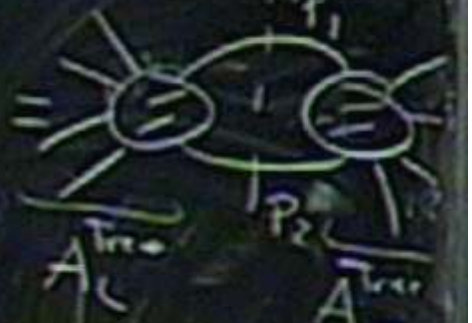
Natural singularity → Unique.

A Branch cuts

Discont. across the B.C

$$\frac{1}{P^2+i\varepsilon} = PV\left(\frac{1}{P^2}\right) + \delta(P^2)$$

Quadruple cut

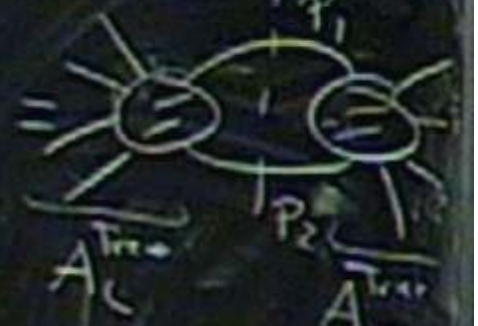


Natural singularity → Unique.

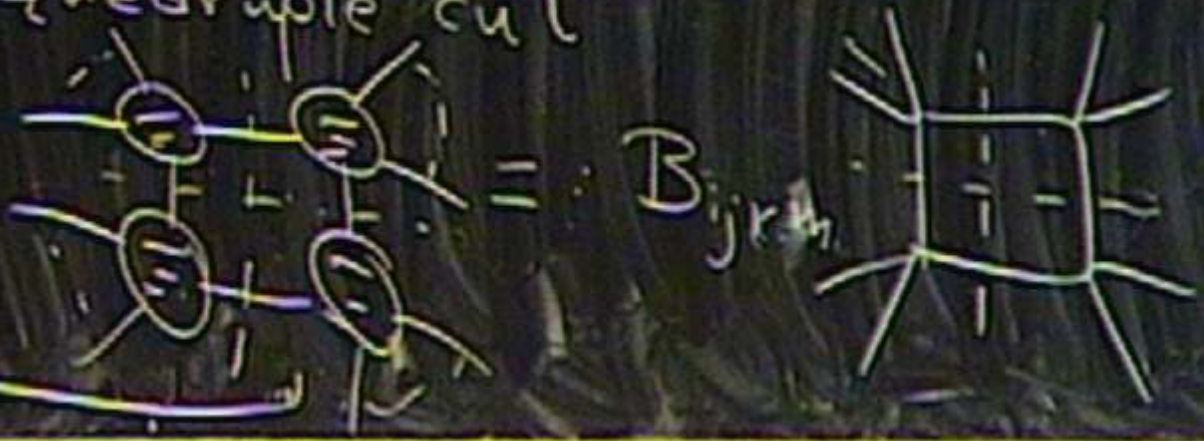
A Branch cuts

Discont. across the B.C

$$\frac{1}{P^2+i\varepsilon} = PV\left(\frac{1}{P^2}\right) + \delta(P^2)$$



Quadruple cut



$$S = \left\{ \ell^2 = 0 \,,\ (\ell - k_1)^2 = 0 \,,\ (\ell - k_1 - k_2)^2 \neq 0 \quad (\ell + k_1)^2 \neq 0 \right\}$$

$$S = \{ l^2 = 0,\ (l-k_1)^2 = 0,\ (l-k_1-k_2)^2 \neq 0\ \ (l+k_4)^2 \neq 0 \}$$

Trick is: cplx. l $\Rightarrow$ $(++--)$

$|S| = 2$

$$\int d^4l\, \delta(\cdot)\, \delta(\cdot)\, A_1 A_2 A_3 A_4 = B_{ijkm} \int d^4l\, \delta(\cdot)\delta(\cdot)\delta(\cdot)\delta(\cdot)$$

$$= \frac{1}{2} \sum_{S,h} A_1 A_2 A_3 A_4 = B_{ijkm}$$

$$S = \{ \ell^2 = 0,\ (\ell - k_1)^2 = 0,\ (\ell - k_1 - k_2)^2 \neq 0 \quad (\ell + k_4)^2 \neq 0 \}$$

Trick is: cplx. ℓ $\Rightarrow$ $(++--)$

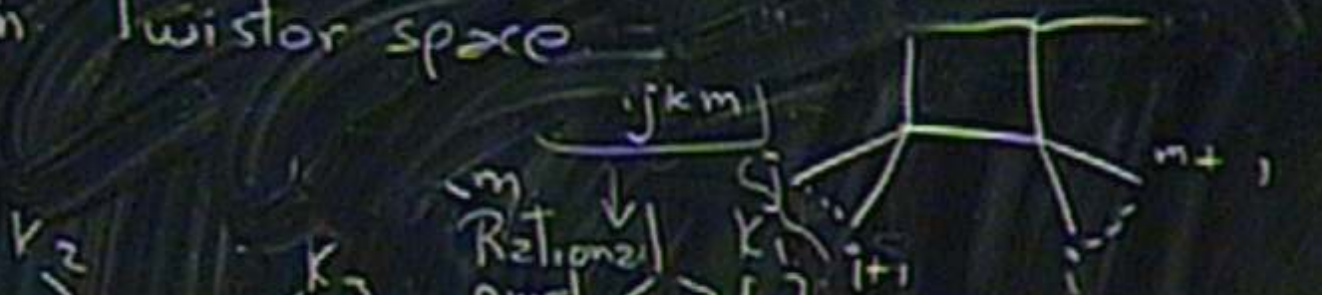
$|S| = 2$ $\quad \ell_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$ $\quad \tilde{\lambda}_{\dot{\alpha}} \neq \pm \tilde{\lambda}_{\dot{\alpha}}$

$$\int d^4\ell\, \delta(\ell)\, \delta(\ell)\, A_1 A_2 A_3 A_4 = B_{ijkm} \int d^4\ell\, \delta(\ell)\, \delta(\ell)\, \delta(\ell)\, \delta(\ell)$$

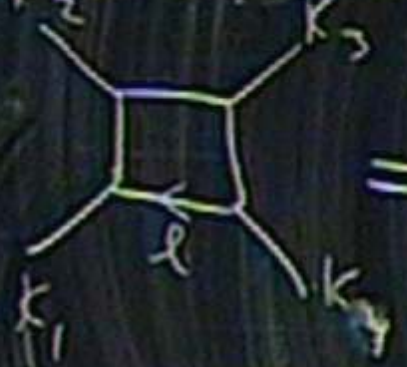
$$= \frac{1}{2} \sum_{S,h} A_1 A_2 A_3 A_4 = B_{ijkm}$$

To find a picture or an interpretation of $\mathcal{A}^{MHV}_{AAAA}$?

in Twistor space

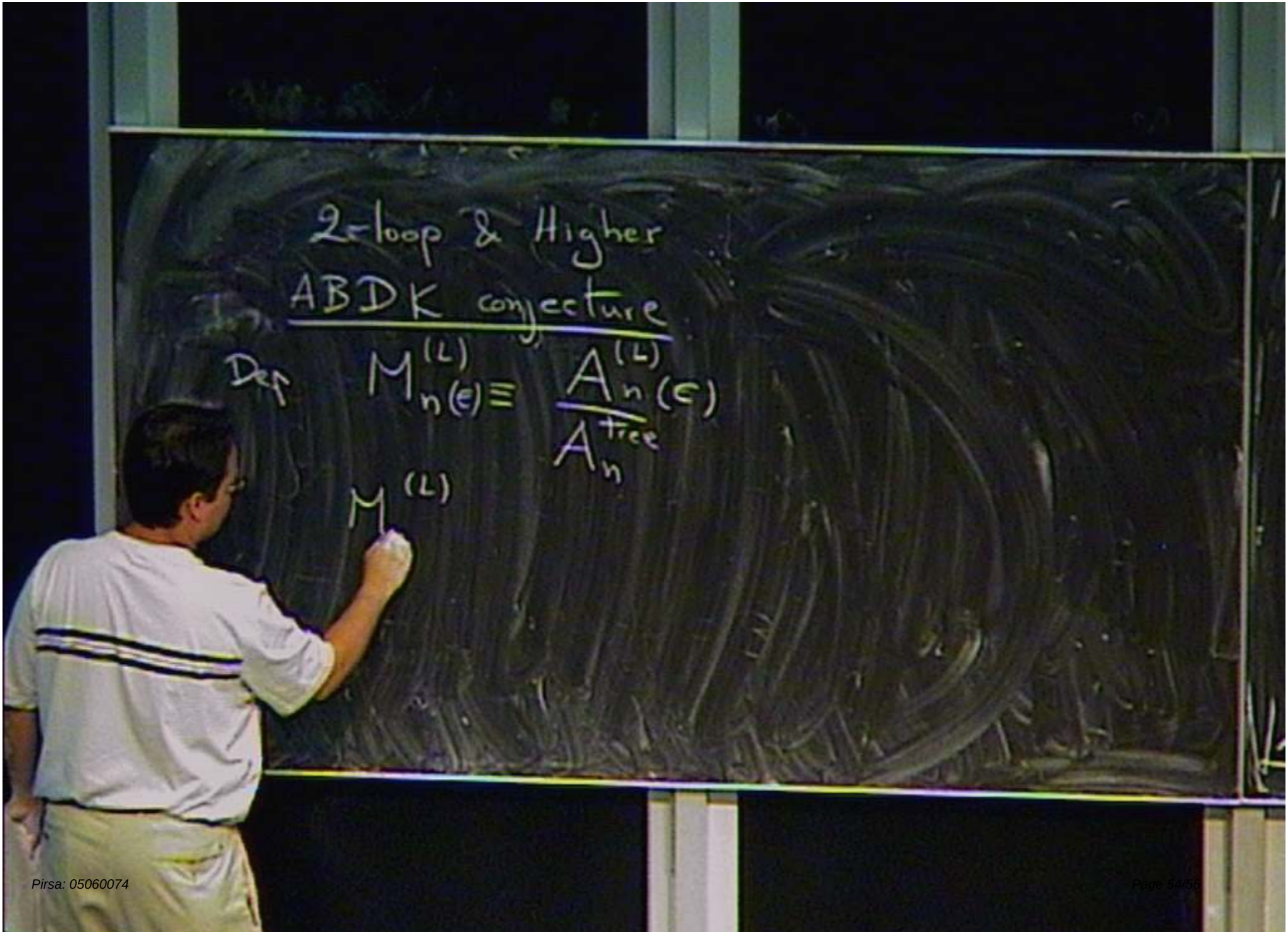


$Li_2(x)$, $\ln(x)$



$$= \int d^4\ell^{2\epsilon} \frac{1}{\ell^2(\ell-k_1)^2(\ell-k_1-k_2)^2(\ell+k_4)^2}$$

Scalar Box integral


2-loop & Higher
ABDK conjecture
Def
$M_{n(\epsilon)}^{(L)} \equiv \frac{A_{n(\epsilon)}^{(L)}}{A_n^{tree}}$
$M^{(L)}$

2-loop & Higher

ABDK conjecture

$$f(\epsilon) = \frac{\psi(1-\epsilon) - \psi(1)}{\epsilon}$$

Def $$M_n^{(L)}(\epsilon) \equiv \frac{A_n^{(L)}(\epsilon)}{A_n^{tree}}$$

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x)$$

$$M_n^{(L)}(\epsilon) = \text{Poly}\left(M_n^{(1)}, \ldots, M_n^{(L-1)} \right)$$

$$M_n^{(2)}(\epsilon) = \frac{1}{2}\left(M_n^{(1)}(\epsilon)\right)^2 + f(\epsilon) M_n^{(1)}(2\epsilon) + \frac{5}{4}\zeta_4 + \mathcal{O}(\epsilon)$$

2-loop & Higher

ABDK conjecture

$$f(\epsilon) = \frac{\psi(1-\epsilon) - \psi(1)}{\epsilon}$$

Def $\quad M_n^{(L)}(\epsilon) \equiv \dfrac{A_n^{(L)}(\epsilon)}{A_n^{\text{tree}}}$

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x)$$

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$$\boxed{M_n^{(2)}(\epsilon) = \frac{1}{2}\left(M_n^{(1)}(\epsilon)\right)^2 + f(\epsilon) M_n^{(1)}(2\epsilon) + \frac{5}{4}\zeta_4 + \mathcal{O}(\epsilon)}$$