

Title: Twistor String Theory and Perturbative Yang-Mills 2

Date: Jun 21, 2005 04:30 AM

URL: <http://pirsa.org/05060074>

Abstract:

Dixon hep-ph/9601359

Witten -th/0312171

Review

Nair -th/0506120

Dixon hep-ph/9601359

Witten -th/0312171

Review

Nair -th/0506120

Review

Dixon hep-ph/9601359

Witten -th/0312171

Review

Nair -th/0506120

Review

Topolog. String Theory

B-model on CP^3 ↔ PMSYM

+ N D5-branes

+ D-instantons

← Challenges

Conformal Group.

$$\lambda_a, \lambda_i \rightarrow (\lambda_a, i \frac{\partial}{\partial \mu_i})$$

$$SL(4, \mathbb{C}) \rightarrow \mathbb{CP}^3 \xrightarrow{\lambda_i} \mathcal{Z}^I = (\lambda_1, \lambda_2, \mu_1, \mu_2)$$

Twistor Space Twistor

$$I_n (+ + - -)$$

Conformal Group.

$$\lambda_a, \tilde{\lambda}_a \rightarrow (\lambda_a, \frac{\tilde{\lambda}_a}{\lambda_a})$$

$$SL(4, \mathbb{C}) \rightarrow \mathbb{C}^4 \xrightarrow{\lambda_a, \tilde{\lambda}_a} \mathcal{Z}^I = (\lambda_1, \lambda_2, \mu_1, \mu_2)$$

Twistor Space Twistor

$$I_n (+ + - -) \quad \lambda, \tilde{\lambda} \text{ Real}$$

$$SO(2, 2) \xrightarrow{\text{Comp}} SO(3, 3) \cong SL(4, \mathbb{R}) \rightarrow \mathbb{R}^4$$

- What about a given Amplitude?

$$\langle \lambda_j, \mu_j, h_j \rangle = \int \prod_{i=1}^n \frac{d^2 \lambda_i}{(2\pi)^2} e^{i[\mu_j \lambda_i]} A(\{\lambda_j, \mu_j, h_j\})$$

$$\left(\lambda_i \frac{\partial}{\partial \lambda_i} - \lambda_i^2 \frac{\partial}{\partial \lambda_i^2} \right) A(\{\lambda\}) = -2h_i A(\{\lambda\})$$

$$(\lambda_i \frac{\partial}{\partial \lambda_i} - \lambda_i^2 \frac{\partial}{\partial \lambda_i^2}) A(\{\lambda\}) = -2h_i A(\{\lambda\})$$

- What about a given Amplitude?

$$\langle \{ \lambda_j, \mu_j, h_j \} \rangle = \int \prod_{i=1}^n \frac{d^2 \lambda_i}{(2\pi)^2} e^{i[\sum \lambda_i]} A(\{ \lambda_j, \mu_j, h_j \})$$

$$\left(\lambda_i^a \frac{\partial}{\partial \lambda_i^a} - \lambda_i^{\bar{a}} \frac{\partial}{\partial \lambda_i^{\bar{a}}} \right) A(\{ \lambda \}) = -2h_i A(\{ \lambda \})$$

$$\left(\lambda_i^a \frac{\partial}{\partial \lambda_i^a} + \mu_i^{\bar{a}} \frac{\partial}{\partial \mu_i^{\bar{a}}} \right) A(\{ \lambda, \mu \}) = (-2h_i - 2) A(\{ \lambda, \mu \})$$

- What about a given Amplitude?

$$\langle \{ \lambda_j, \mu_j, h_j \} \rangle = \int \prod_{i=1}^n \frac{d^2 \lambda_i}{(2\pi)^2} e^{i[\sum \lambda_i]} A(\{ \lambda_j, \mu_j, h_j \})$$

$$\left(\lambda_i \frac{\partial}{\partial \lambda_i} - \lambda_i^2 \frac{\partial}{\partial \lambda_i^2} \right) A(\{ \lambda \}) = -2h_i A(\{ \lambda \})$$

$$\left(\lambda_i \frac{\partial}{\partial \lambda_i} + \mu_i^2 \frac{\partial}{\partial \mu_i^2} \right) A(\{ \lambda, \mu \}) = (-2h_i - 2) A(\{ \lambda, \mu \})$$

$$Z^I \rightarrow t Z^I + t e^{\frac{1}{2} \sum \lambda_i^2} e^{-2h_i - 2} \mathbb{C}^4$$

$z^+ \sim t z^+$ $\mathbb{CP}^3$ or $\mathbb{RP}^3$
 $A \rightarrow$ sections line bundle $\mathcal{O}(-2h; -2)$

MHV amplitudes $(1^+, \dots, (-1)^+, i^-, \dots, j^-, \dots, n^+)$

$$\int \prod_{i=1}^n \frac{d\tilde{\lambda}_i}{(2\pi)^2} e^{i\langle \gamma_i, \tilde{\lambda}_i \rangle} \delta\left(\sum_{i=1}^n \lambda_i \tilde{\lambda}_i\right) \left(\frac{\langle ij \rangle^4}{\prod_{i=1}^n \langle i, i+1 \rangle} \right)$$

$z^\pm \sim t z^\pm$ $\mathbb{CP}^3$ or $\mathbb{RP}^3$
 $A \rightarrow$ sections line bundle $\mathcal{O}(-zh; -z)$

MHV amplitude $(1^+, \dots, (-)^+, i^-, \dots, j^-, \dots, n^+)$

$$\int \prod_{i=1}^n \frac{d\tilde{\lambda}_i}{(2\pi)^2} e^{i[L(\lambda; \tilde{\lambda})]} \underbrace{\delta\left(\sum_{i=1}^n \lambda_i \tilde{\lambda}_i\right)}_{\int d^4x_{aa'} e^{i x_{aa'} \sum_{i=1}^n \lambda_i \tilde{\lambda}_i}} \underbrace{\left(\frac{\langle ij \rangle^4}{\prod_{i=1}^n \langle i, i+1 \rangle} \right)}_{f(\lambda)}$$

$$Z^T \sim t Z^T \quad \mathbb{CP}^3 \text{ or } \mathbb{RP}^3$$

$$A \rightarrow \text{sections line bundle } \mathcal{O}(-2h; -2)$$

MHV amplitude $(1^+, \dots, (-)^+, i^-, \dots, j^-, \dots, n^+)$

$$\int \prod_{i=1}^n \frac{d\tilde{\lambda}_i}{(2\pi)^2} e^{i\langle i, \tilde{\lambda}_i \rangle} \underbrace{\delta\left(\sum_{i=1}^n \lambda_i \tilde{\lambda}_i\right)}_{\int d^4 x_{aa'} e^{i x_{aa'} \sum_{i=1}^n \lambda_i \tilde{\lambda}_i}} \underbrace{\left(\frac{\langle ij \rangle^4}{\prod_{i=1}^n \langle i, i+1 \rangle} \right)}_{f(\lambda)}$$

$$f(\alpha) \int d^4 x_{ai} \underbrace{\int \prod_{i=1}^n \frac{d^2 \tilde{\lambda}_i}{(2\pi)} e^{i(\mu_i^a \tilde{\lambda}_a + x_{ai} \lambda_i^a \tilde{\lambda}_i)}}_{\prod_{i=1}^n \delta^{(2)}(\mu_i^a + x_{ai} \lambda^a)}$$

$$\tilde{A}(\tilde{\lambda}_i)$$

$$f(\alpha) \int d^4 x_{ai} \underbrace{\int \prod_{i=1}^n \frac{d^2 \tilde{\lambda}_i}{(2\pi)^2} e^{i(\mu_i^{\dot{a}} \tilde{\lambda}_a + x_{ai} \lambda_i^{\dot{a}} \tilde{\lambda}_{\dot{a}})}}_{\prod_{i=1}^n \delta^{(2)}(\mu_i^{\dot{a}} + x_{ai} \lambda_i^{\dot{a}})}$$

$$\mu_i^{\dot{a}} + x_{ai} \lambda_i^{\dot{a}} = 0$$

$$\Pi(+ + - -) \quad \mathbb{R}^4 \rightarrow \mathbb{RP}^3$$

$$(\lambda_1, \lambda_2, \mu_1, \mu_2)$$

$$SO(2,2)$$

$$\mathbb{R}^3$$

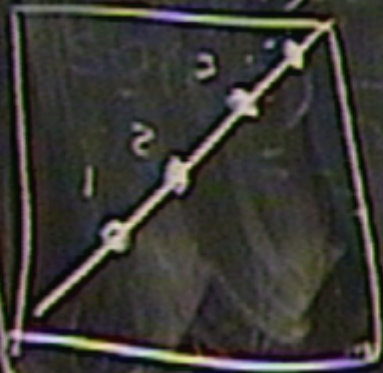
$$y_1 = \frac{\lambda_2}{\lambda_1}, \quad y_2 = \frac{\mu_1}{y_1}, \quad y_3 = \frac{\mu_2}{y_2}$$

$$f(\alpha) \int d^4 x_{ai} \int \prod_{i=1}^n \frac{d^2 \tilde{\lambda}_i}{(2\pi)^2} e^{i(\mu_i^{\dot{a}} \tilde{\lambda}_{\dot{a}} + x_{a\dot{a}} \lambda_i^a \tilde{\lambda}_{\dot{a}})}$$

$$\prod_{i=1}^n \delta^{(2)}(\mu_i^{\dot{a}} + x_{a\dot{a}} \lambda_i^a)$$

$$\mu_i^{\dot{a}} + x_{a\dot{a}} \lambda_i^a = 0$$

$\Pi(+ + - -)$



$$\mathbb{R}^4 \rightarrow \mathbb{RP}^3$$

$$\mathbb{R}^3$$

$$(\lambda_1, \lambda_2, \mu_1, \mu_2)$$

$$y_1 = \frac{\lambda_2}{\lambda_1}, \quad y_2 = \frac{\mu_1}{y_1}, \quad y_3 = \frac{\mu_2}{y_2}$$

$$F(\omega) \int d^4 x_{ai} \underbrace{\int \prod_{i=1}^n \frac{d^2 \tilde{\lambda}_i}{(2\pi)^2} e^{i(\mu_i^a \tilde{\lambda}_a + x_{ai} \lambda_i^a \tilde{\lambda}_{i-1}^a)}}_{\prod_{i=1}^n \delta^{(2)}(\mu_i^a + x_{ai} \lambda_i^a)}$$

$$\prod_{i=1}^n \delta^{(2)}(\mu_i^a + x_{ai} \lambda_i^a)$$

$$\mu_i^a + x_{ai} \lambda_i^a = 0$$

$$d_1=1 \quad d_2=1$$

Holomorphic curve
of $g=0$ $d=1$

(++--)

$$\mathbb{R}^4 \rightarrow \mathbb{RP}^3$$

$$(\lambda_1, \lambda_2, \mu_1, \mu_2)$$



$\mathbb{R}^3$

$\mathbb{CP}^3$



$$y_1 = \frac{\lambda_2}{\lambda_1}, \quad y_2 = \frac{\mu_1}{y_1}, \quad y_3 = \frac{\mu_2}{y_2}$$

$$f(z_1, z_2) = 0 \quad g(z_1, z_2) = 0$$

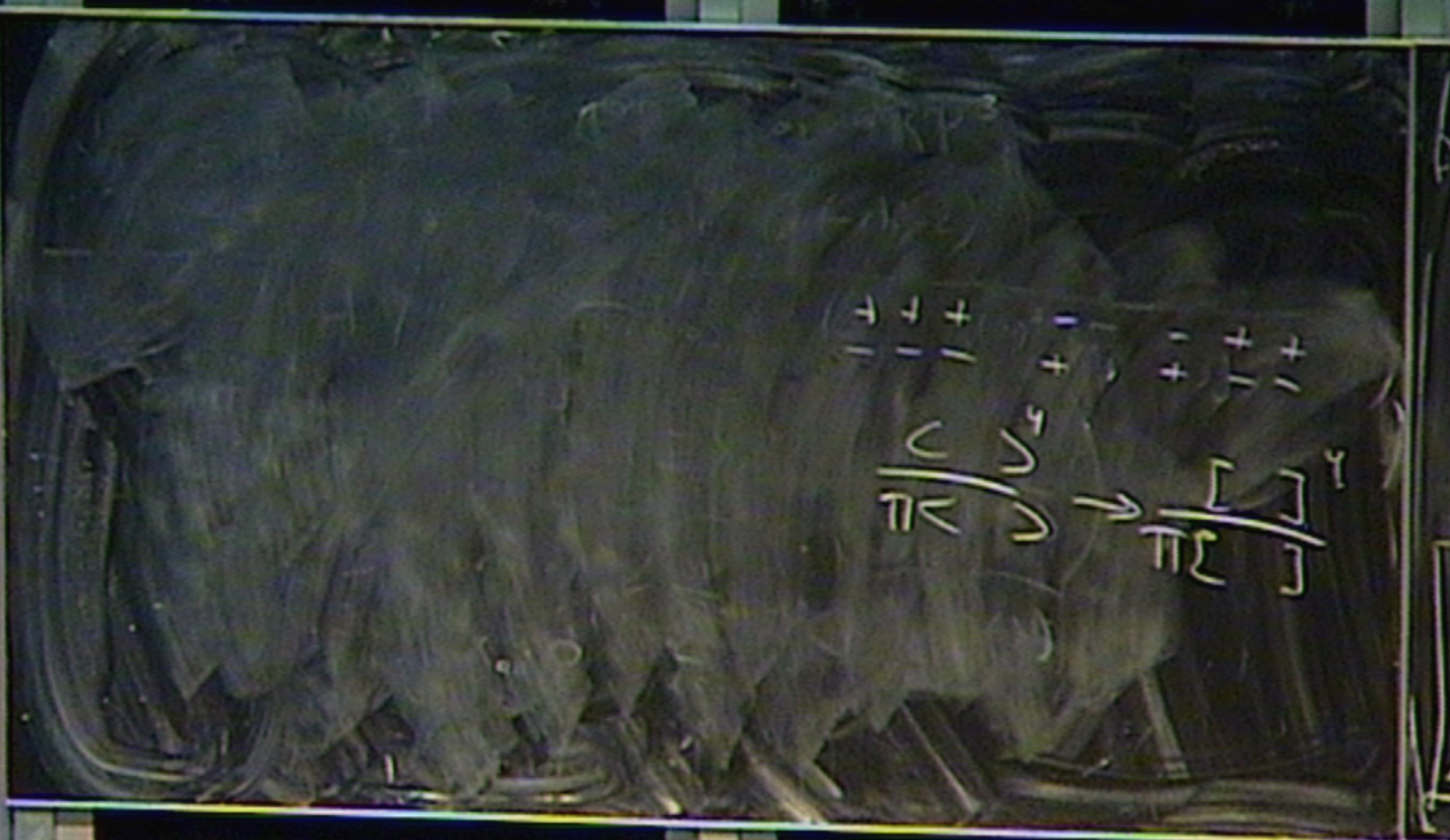
$$d = d_1 d_2$$

Witten. $A(1^+, 2^+, \dots, i^-, \dots, j^-, \dots, k^-, \dots, n^+)$

q - helicity p + helicity

$$L = \text{loop.} \quad d = q - 1 + L$$

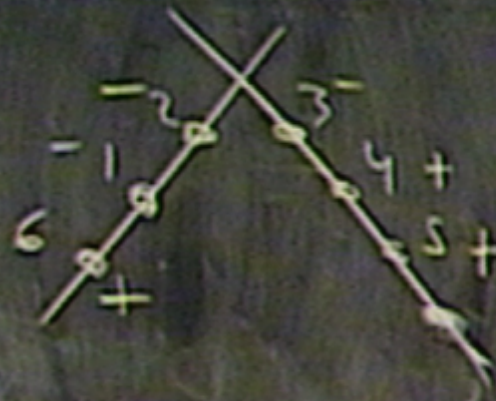
$$q \geq L$$



More minuses?
 $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$

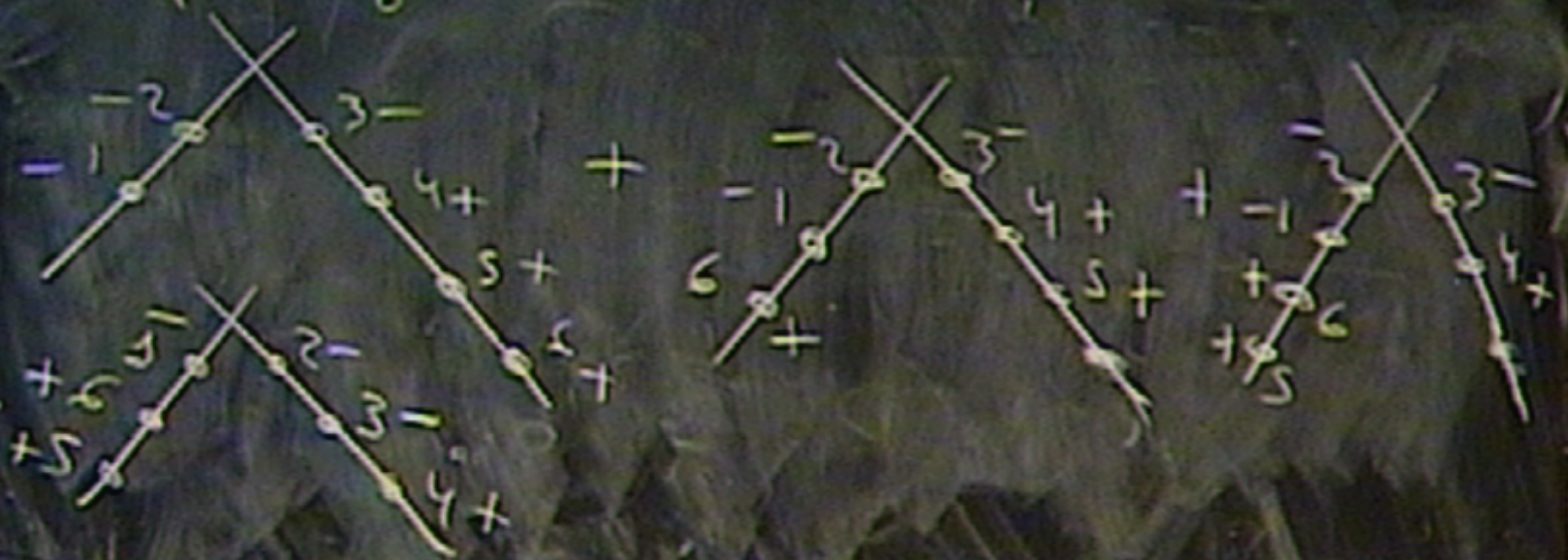
tree More minus?
 $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$
 Surprise?

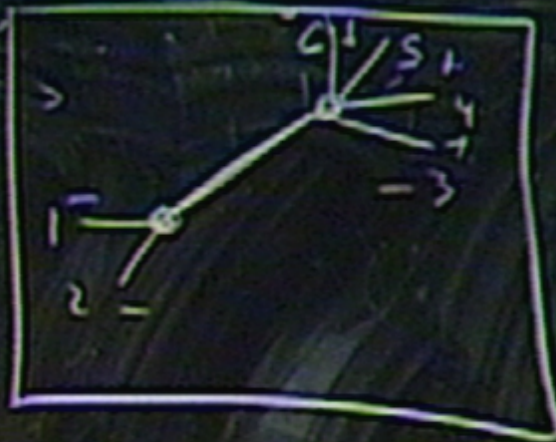
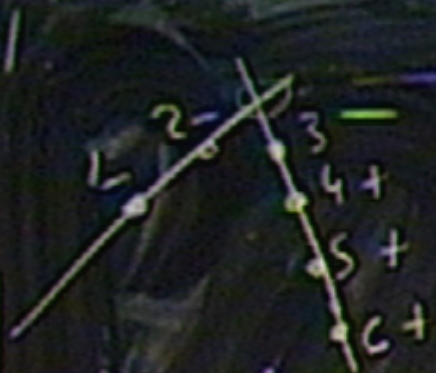
$d=2$

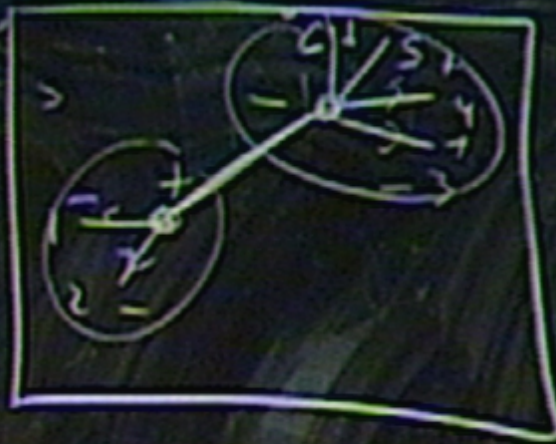
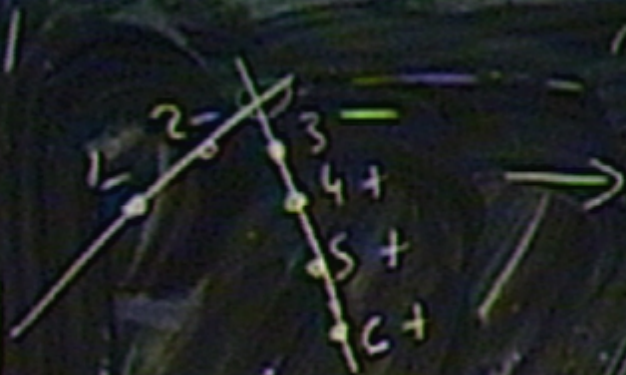


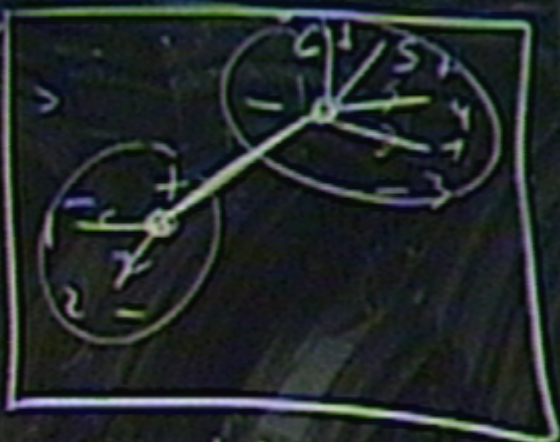
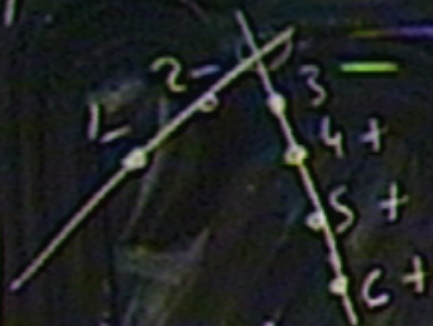
More minuses?
 $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$
 Surprise?

$d=2$

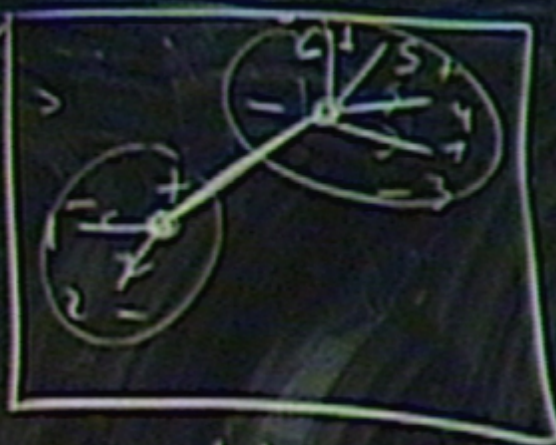
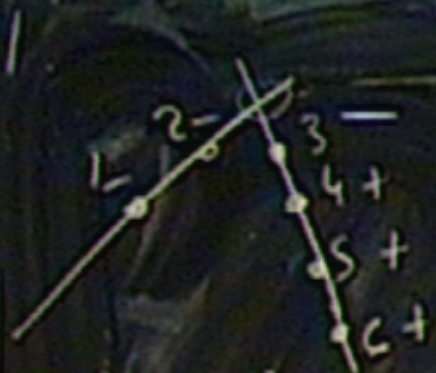








Crazy D: YM does not have 5 point vertices
 A New diagram. Exp. of YM.



Crazy D

YM does not have 5 point vertices

A New diagram. Exp. of YM.

- Vertices are MHV amplitudes
- Propagators

$$\hat{A}_L(\bar{1}, \bar{2}, \hat{p}^+)$$

$$\underbrace{\quad}_{4-3=1}$$

$$\hat{A}_R(\hat{p}_1^-, \bar{3}, 4^+, 5^+, \bar{6}^+)$$

$$4-5=-1$$

$$\rightarrow d = 4-n$$

$$= 4-6 = -2$$

$$\underbrace{A_L(\bar{1}, \bar{2}, \dot{p}^+)}_{4-3=1} \frac{1}{p^2} A_R(\dot{p}^-, 3^-, 4^+, 5^+, 6^+) \quad \begin{matrix} 4-5=-1 \\ \hat{p}^2=0 \end{matrix}$$

$$\rightarrow \dim 4-n = 4-6 = -2$$

$$P = P_1 + P_2 \quad P^2 \neq 0 \quad \hat{P}^2 = 0$$

$$P_{aa} = \underbrace{\lambda_a \tilde{\lambda}_a}_{\hat{P}_{aa}} + \lambda'_a \tilde{\lambda}'_a$$

$$P_{aa} \tilde{\lambda}^{aa} = \lambda_a^{\hat{a}} [\tilde{\lambda}, \tilde{\lambda}'] + \lambda'_a \underbrace{[\tilde{\lambda}', \tilde{\lambda}']}_0$$

$$\underbrace{A_L(\bar{1}, 2^-, \hat{p}^+)}_{4-3=1} \frac{1}{p^2} \underbrace{A_R(\hat{p}^-, 3^-, 4^+, 5^+, 6^+)}_{4-5=-1}$$

$$\rightarrow \dim 4-n = 4-6 = -2$$

$$P = p_1 + p_2 \quad P^2 \neq 0 \quad \hat{P}^2 = 0$$

$$P_{a\dot{a}} = \underbrace{\lambda_a \tilde{\lambda}_{\dot{a}}}_{\hat{P}_{a\dot{a}}} + \lambda'_a \tilde{\lambda}'_{\dot{a}}$$

$$\lambda_{\hat{p}} = \frac{P_{a\dot{a}} \tilde{\lambda}'^{\dot{a}}}{[\tilde{\lambda}, \tilde{\lambda}']} \quad P_{a\dot{a}} \tilde{\lambda}'^{\dot{a}} = \lambda_a [\tilde{\lambda}, \tilde{\lambda}'] + \lambda'_a \underbrace{[\tilde{\lambda}, \tilde{\lambda}']}_0$$

$$\frac{\langle 12 \rangle^3}{\langle 2\hat{p} \rangle \langle \hat{p} 1 \rangle} \quad \frac{1}{p^2} \frac{\langle \hat{p} 3 \rangle^3}{\langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 6\hat{p} \rangle}$$

$$\lambda_{\hat{p}a} \equiv P_{a\hat{p}} \eta^a$$

$$\tilde{\lambda}' = \eta$$

Fixed but arbitrary η .

$$\frac{\langle 12 \rangle^3}{\langle 2\hat{p} \rangle \langle \hat{p} 1 \rangle} = \frac{1}{p^2} \frac{\langle \hat{p} 3 \rangle^3}{\langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 6\hat{p} \rangle}$$

$$\lambda_{\hat{p}a} \equiv p_{a\hat{p}} \eta^a$$

$$\tilde{\lambda} = \eta$$

Fixed but arbitrary η .

$$A(1^-, 2^-, 3^-, 4^+, 5^+, \dots, n^+)$$

$$= \sum_i$$



$$+ \sum_i$$



Is this correct?

Key: Rational functions α, λ^2 $p \in \mathbb{C}^4$

Is this correct?

Key: Rational functions in $\lambda, \bar{\lambda}$ $P \in \mathbb{C}^4$

- One can check that MHV diag. produce correct singul.

Is this correct?

Key: Rational functions in $\lambda, \bar{\lambda}$ $p \in \mathbb{C}^4$

- One can check that MHV diag. produce correct singul.

$$\langle p, \epsilon \rangle$$

η is fixed

- Location is not Lorentz invariant
- "Sum" of all MHV diag. is η -independent

Restore L.I.

$$\langle p, \epsilon \rangle$$

are spurious

What about loops?

What about loops?

$$N=4 \text{ SYM}$$
$$(W_\alpha, 3\bar{\Phi})$$

$$g + 4f + 3S$$

$$N=1$$

$$W_\alpha = (A_\mu, \psi)$$

$$\bar{\Phi} = (\phi, \chi)$$

What about loops?

$$\begin{array}{ll}
 N=4 \text{ SYM} & N=1 \\
 (W_\alpha, 3\bar{\Phi}) & SU(4) \quad W_\alpha = (A_\mu, \psi) \\
 g + 4f + 3S & \quad \quad \quad \bar{\Phi} = (\phi, \chi)
 \end{array}$$

What about loops?

$$N=4 \text{ SYM} \quad N=1 \quad W_\alpha = (A_\mu, \psi)$$

$$(W_\alpha, 3\bar{\Phi}) \quad SU(4) \quad \bar{\Phi} = (\phi, \bar{\psi})$$

$$g + 4f + 3s$$

$$s \rightarrow 6$$

$$f \rightarrow 4, \bar{4}$$

$$g \rightarrow 1$$

MHV amplitudes at 1-loop.

$$d = g - 1 + L \quad g \leq L$$

$$= 2 - 1 + 1 = 2 \quad g \leq 1.$$

Surprise



MHV amplitudes at 1-loop

$$d = g - 1 + L \quad g \leq L$$

$$= 2 - 1 + 1 = 2 \quad g \leq 1$$

Surprise

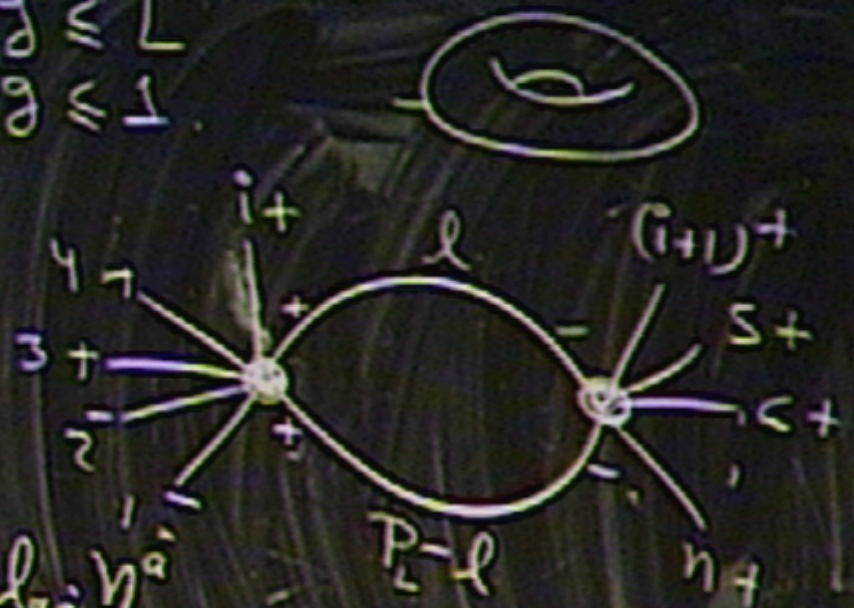


Open
problem

$$\sum_{i,j} \int d^4 \ell$$

$$\lambda_{\ell} = \ell_{\alpha} \gamma^{\alpha}$$

$$(\lambda_{P-\ell})_{\alpha} = (P-\ell)_{\alpha} \gamma^{\alpha}$$



MHV amplitudes at 1-loop

$$d = g - 1 + L \quad g \leq L$$

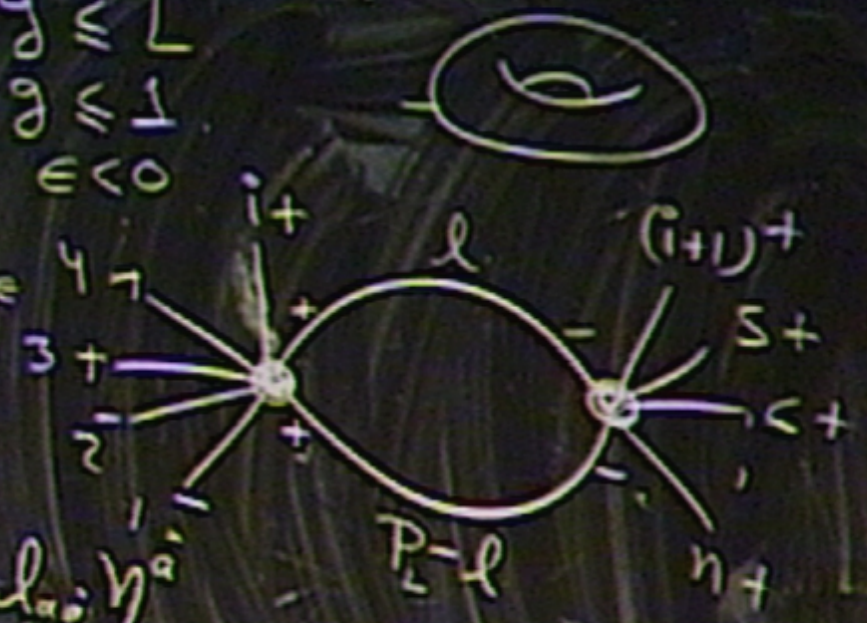
$$= 2 - 1 + 1 = 2 \quad g \leq 1$$

Surprise

IR $\epsilon < 0$



$$\sum_{i,j,h} \int d^4 \ell$$



$$\lambda_{\ell} = \ell_{\alpha} \eta^{\alpha}$$

$$(\lambda_{P-L})_{\alpha} = (P-L)_{\alpha} \eta^{\alpha}$$

Q: Does this work for non-MHV 1-loop amplitudes?
Ex. Open problem.

Q: Does this work for non-MHV 1-loop amplitudes?

Ex Open problem.


 $=$

 $+ \text{Permut.}$



$$eA^{(1\text{-loop})}(\dots) = \sum_{\text{Leading color}} \text{tr}(T^{a_1} \dots T^{a_n}) A(\dots) +$$

Color ordered partial amplitudes

$$+ \sum_{\text{Color ordered}} \text{tr}(T^{a_1} \dots T^{a_n}) \text{tr}(T^{a_{n+1}} \dots T^{a_m}) A(\dots)$$

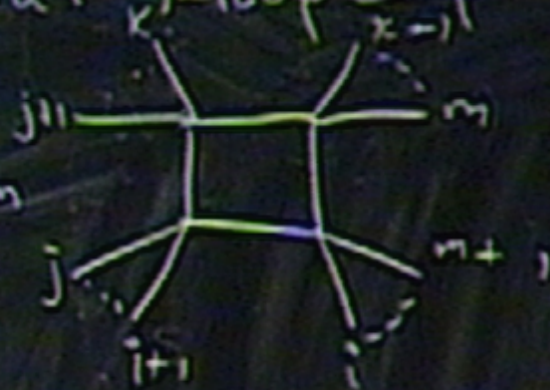
What is known about 1-loop amplitudes?

90's
1-loop
 $A(1, n) =$

What is known about 1 -loop amplitudes?

90's
1-loop

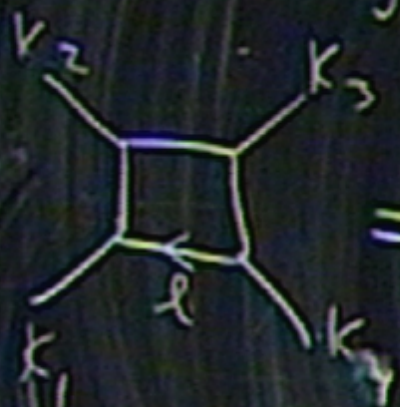
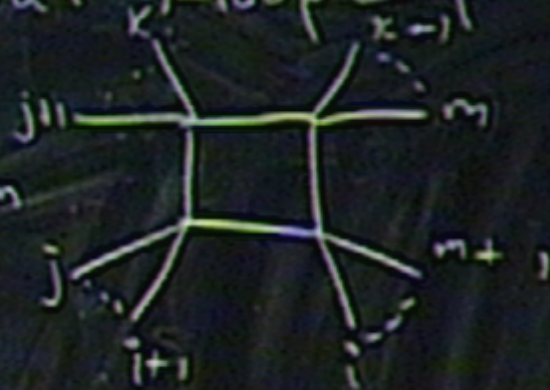
$$A(1, n) = \sum_{1 \leq i < j < k < m} B_{ijklm}$$



What is known about 1-loop amplitudes?

90's
1-loop

$$A(l, n) = \sum_{1 \leq i < j < k < m} B_{ijklm}$$



$$= \int d^4 l \frac{1}{l^2 (l-k_1)^2 (l-k_1-k_2)^2 (l+k_4)^2}$$

Scalar Box integral

What is known about Loop amplitudes?

90's
1-loop

$A(l, n)$

$= \sum$

$i < j < k < m$

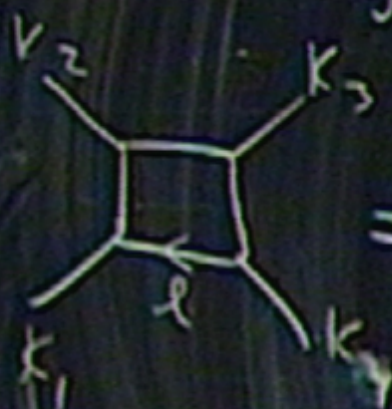
B_{ijklm}

Rational
funct. $\langle \rangle, []$

$\int d^4x$

1

$l^2(l-k_1)^2(l-k_1-k_2)^2(l+k_4)^2$



Scalar Box integral



$Li_2(z),$
 $\ln(z)$

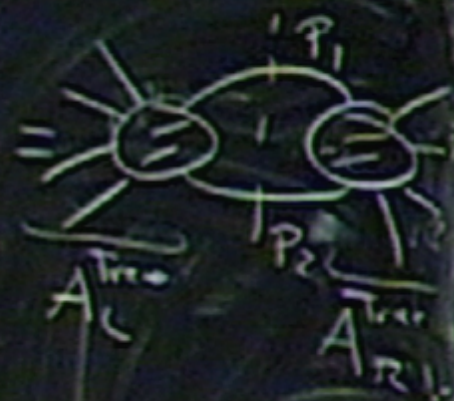
$\odot$
 $A_{\text{tree}}^{\text{tree}}$
 A_{tree}

$1) +$
 $S +$
 $-C +$
 1
 $1 +$

Natural singularity $\rightarrow$ Unique.
 Branch cuts

$$\frac{1}{P^2 + i\epsilon} = \text{PV} \left(\frac{1}{P^2} \right) + \delta(P^2)$$

Discont.
 across the B.C



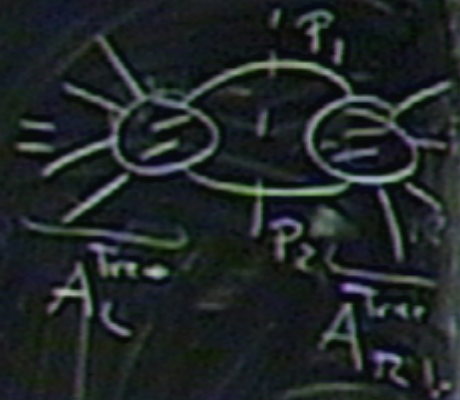
$\circ$
 $A_{\text{tree}}^{\text{tree}}$
 $1) +$
 $S +$
 $-C +$
 1
 14

Natural singularity $\rightarrow$ Unique.
 A Branch cuts.

$$\frac{1}{P^2 + i\epsilon} = \text{PV} \left(\frac{1}{P^2} \right) + \delta(P^2)$$

Quadruple cut

Discont. across the B.C



$\bigcirc$
 $A_{\text{tree}}^{\text{tree}}$
 A_{tree}

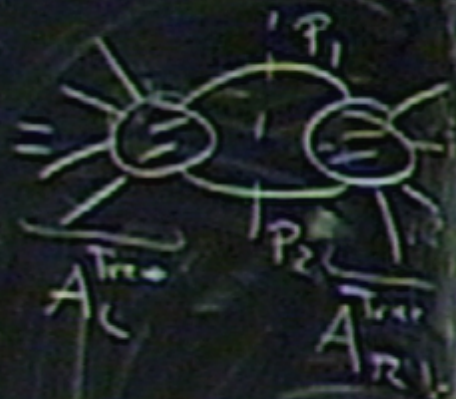
$1) +$
 $S +$
 $-C +$
 $n +$

Natural singularity $\rightarrow$ Unique.

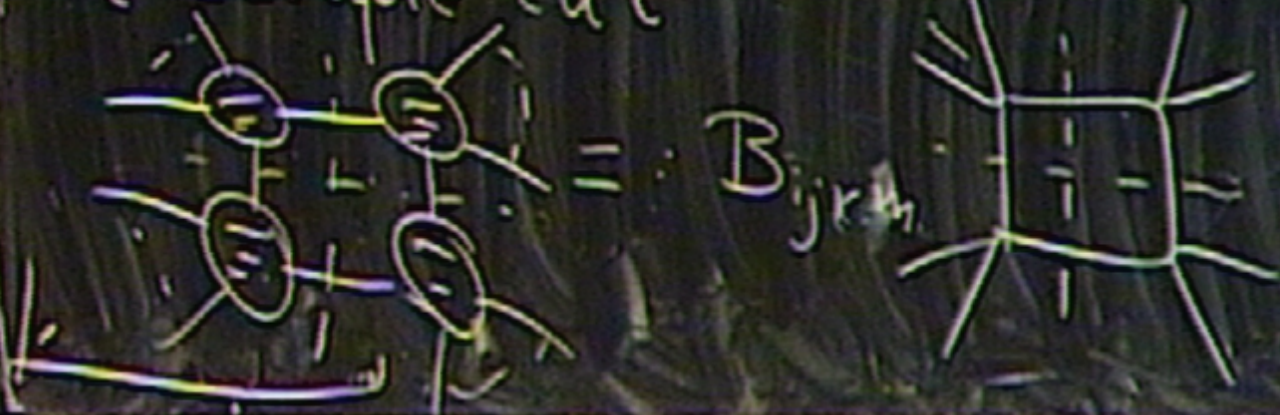
A Branch cuts

$$\frac{1}{P^2 + i\epsilon} = PV \left(\frac{1}{P^2} \right) + \delta(P^2)$$

Discont.
across the B.C



Quadruple cut



$$S = \{ \ell^2 = 0, (\ell - k_1)^2 = 0, (\ell - k_1 - k_2)^2 = 0, (\ell + k_1)^2 = 0 \}$$

$$S = \{l^2=0, (l-k_1)^2=0, (l-k_1-k_2)^2=0, (l+k_1)^2=0\}$$

Trick is: cplx. $l \rightarrow (+ + - -)$

$$|S|=2$$

$$\int d^4l \delta(l) \delta(l-k_1) \delta(l-k_1-k_2) \delta(l+k_1) A_1 A_2 A_3 A_4 = B_{ijklm} \int d^4l \delta(l) \delta(l-k_1) \delta(l-k_1-k_2) \delta(l+k_1)$$

$$= \frac{1}{2} \sum_{S,h} A_1 A_2 A_3 A_4 = B_{ijklm}$$

$$S = \{ l^2=0, (l-k_1)^2=0, (l-k_1-k_2)^2=0, (l+k_4)^2=0 \}$$

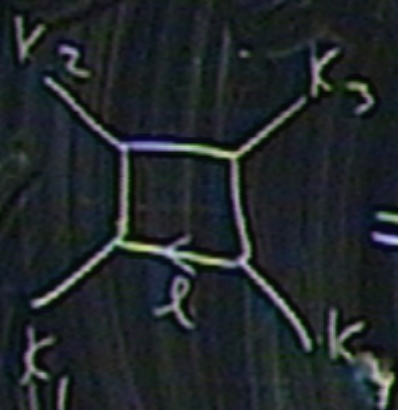
Trick is: cplx. $l \rightarrow (+ + - -)$

$$|S|=2 \quad l_{a\bar{a}} = \lambda_a \tilde{\lambda}_{\bar{a}} \quad \tilde{\lambda}_{\bar{a}} \neq \tilde{\lambda}_a$$

$$\int d^4l \delta(l) \delta(l-k_1) \delta(l-k_1-k_2) \delta(l+k_4) A_1 A_2 A_3 A_4 = B_{ijklm} \int d^4l \delta(l) \delta(l-k_1) \delta(l-k_1-k_2) \delta(l+k_4)$$

$$= \frac{1}{2} \sum_{S,h} A_1 A_2 A_3 A_4 = B_{ijklm} \quad 2 \times$$

To find a picture or an interpretation of $F^{AAAA} = B$ in Twistor space



Scalar Box integral

$$= \int d^4 l \frac{1}{l^2 (l-k_1)^2 (l-k_1-k_2)^2 (l+k_4)^2}$$

$\downarrow$
 Rational
 func. $\langle \rangle, []$

(jkm)
 $\downarrow$
 k_1
 $i+1$
 $m+$

$Li_2(z),$
 $\ln(z)$

2-loop & Higher ABDK conjecture

Def $M_n^{(L)}(\epsilon) \equiv \frac{A_n^{(L)}(\epsilon)}{A_n^{\text{tree}}}$

$M^{(L)}$

2-loop & Higher ABDK conjecture

Def $M_n^{(L)}(\epsilon) \equiv \frac{A_n^{(L)}(\epsilon)}{A_n^{\text{tree}}}$

$$f(\epsilon) = \frac{\psi(1-\epsilon) - \psi(1)}{\epsilon}$$

$$\psi(x) = \frac{d}{dx} \ln T(x)$$

$$M_n^{(L)}(\epsilon) = \text{Poly}(M_n^{(1)}, \dots, M_n^{(L-1)})$$

$$M_n^{(2)}(\epsilon) = \frac{1}{2} (M_n^{(1)}(\epsilon))^2 + f(\epsilon) M_n^{(1)}(\epsilon) = \frac{5}{4} \zeta_1 + \mathcal{O}(\epsilon)$$

2-loop & Higher
ABDK conjecture

Def $M_n^{(L)}(\epsilon) \equiv \frac{A_n^{(L)}(\epsilon)}{A_n^{\text{tree}}}$

$$f(\epsilon) = \frac{\psi(1-\epsilon) - \psi(1)}{\epsilon}$$

$$\psi(x) = \frac{d}{dx} \ln T(x)$$

$$M_n^{(L)}(\epsilon) = \text{Poly}(M_n^{(1)}, \dots, M_n^{(L-1)})$$

$$M_n^{(2)}(\epsilon) = \frac{1}{2} (M_n^{(1)}(\epsilon))^2 + f(\epsilon) M_n^{(1)}(\epsilon) = \frac{5}{4} \zeta_1 + \mathcal{O}(\epsilon)$$