

Title: Complex Manifolds

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Abstract:

Complex Manifolds

Differential Geometry

Algebraic Geometry

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Differential Geometry

Algebraic Geometry

- computations in cohomology

(- Orbifolds
- Spectral Sequences)

Differential Geometry

- Familiar from C.R
- Any manifold

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- Put a metric on manifold using local coords

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$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

- compute connection, curvatures, etc.

Vector Bundle.

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E, M are manifolds, $\pi^{-1}(x) \simeq \mathbb{R}^k$ for any $x \in M$.

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 π is a C^∞ -map

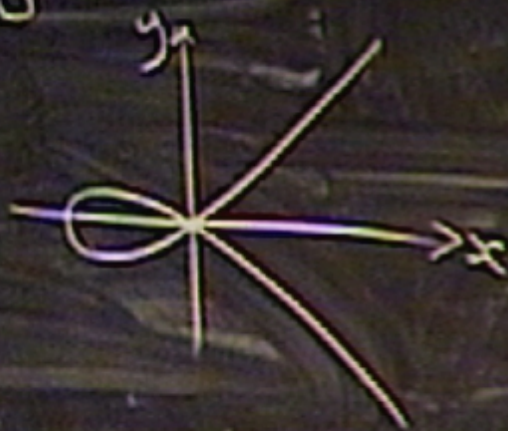
For every $x_0 \in M$, $\exists$ an open nbd of x_0
such that

$$\varphi_U: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$$

is a diffeomorphism.

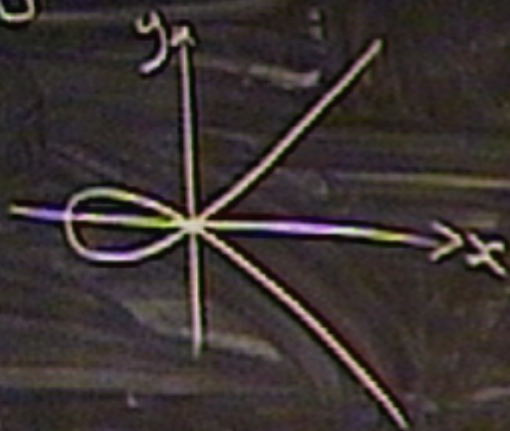
Algebraic Geometry

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Connection of alg. geom. to physics?

1) Alg. geom. is often equiv. to diff. geom. but easier to analyze

2)

Conjecture

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- 2) It might, in some cases better represent "true physics!"
- mirror symmetry

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"algebraically closed field"

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Alg geom is only easy if we work over an
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$\mathbb{C}$ is good - supersymmetry.

Diff Geom. & Complex Manifolds.

A complex manifold of dim n is a manifold with open cover $\{U_\alpha\}$

Diff Geom. of M Complex Manifolds.

A complex manifold of dim n is a manifold with open cover $\{U_\alpha\}$, and coordinate maps

$$\varphi_\alpha: U_\alpha \rightarrow \mathbb{C}^n$$

such that $\varphi_\alpha \cdot \varphi_\beta^{-1}$ is holomorphic on $U_\alpha \cap U_\beta$

function of z_j^2 and not z_j^1

$\rightarrow U \times \mathbb{R}^n$
is a diffeomorphism

such that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is holomorphic on $\varphi_\beta(U_\alpha \cap U_\beta)$

function of z_j^2 and not z_j^1

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Clearly, putting $z_j = x_j + iy_j$,
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E.g. $\mathbb{C}^n$

$\mathbb{P}^n$ (aka $\mathbb{CP}^n$)

$$[Z_0, Z_1, \dots, Z_n] \sim [\lambda Z_0, \lambda Z_1, \dots, \lambda Z_n] \text{ in } \mathbb{C}^{n+1} - \{0\}$$

where $\lambda \in \mathbb{C} - \{0\}$

To see this is a complex manifold,

put $U_k = \mathbb{P}^n - \{ \text{all points with } Z_k = 0 \}$

$$\varphi_j(u_j) \text{ is given by } \varphi_j([Z_0, Z_1, Z_n]) \\ = \left(\frac{Z_0}{Z_1}, \frac{Z_1}{Z_2}, \dots, \frac{Z_{n-1}}{Z_n} \right)$$

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$$\varphi_j(u_j) \text{ is given by } \varphi_j([Z_0, Z_1, \dots, Z_n]) = \left(\frac{Z_0}{Z_j}, \frac{Z_1}{Z_j}, \dots, \frac{Z_n}{Z_j} \right)$$

↑ coord of point in $\mathbb{C}^n$

$\varphi_j: U_j \rightarrow U_j \times \mathbb{R}^n$
is a diffeomorphism

or

$$\varphi_j(u_j) \text{ is given by } \varphi_j([Z_0, Z_1, \dots, Z_n]) = \left(\frac{Z_0}{Z_k}, \frac{Z_1}{Z_k}, \dots, \frac{Z_k}{Z_k}, \frac{Z_n}{Z_k} \right)$$

↑ coord of point in $\mathbb{C}^n$

$$\varphi_j \circ \varphi_k^{-1}(z^1, z^2, \dots, z^n) = \left(\frac{z^1}{z^n}, \frac{z^2}{z^n}, \dots, \frac{z^n}{z^n}, \frac{1}{z^n}, \dots, \frac{z^n}{z^n} \right)$$

is a d.h.f. coordinate position

Complex Vector Bundle

$$\pi: E \rightarrow M$$

E, M are manifolds (not nec. complex)

π is a C^∞ -map

- such that
- $\pi^{-1}(x) \cong \mathbb{C}^k$ for all $x \in M$ or
 - There exist an open nbd U of any $x \in M$,
with a map
$$\varphi_U: \pi^{-1}(U) \rightarrow U \times \mathbb{C}^k$$

• $\mathbb{C}^\infty$ -diffeomorphism

$$\pi \varphi_u^{-1}(x, v) = x \quad \text{for all } v \in \mathbb{C}^k$$

• $v \mapsto \varphi_u^{-1}(x, v)$ for a fixed $x \in M$ yields
isomorphism $\mathbb{C}^n \cong \pi^{-1}(x)$

k is the "rank" of bundle E .

$k=1$ is called a (complex) line bundle

If U and V are open sets with trivializations φ_U and φ_V as above, then

$$g_{UV}(x) = \varphi_U \circ \varphi_V^{-1}$$

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defines transition functions

$$g_{uv} : U \cap V \xrightarrow[\text{map}]{\text{diff}} GL(k, \mathbb{C})$$

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$$g_{uv} : \varphi(U \cap V) \xrightarrow[\text{map}]{\text{diff}} GL(k, \mathbb{C})$$

Clearly

$$\left. \begin{aligned} g_{uv} g_{vu} &= 1 \\ g_{uv} g_{vw} g_{wu} &= 1 \end{aligned} \right\} *$$

Conversely, given open-cover $\{U_\alpha\}$ of M
and C^∞ -maps $g_{\alpha\beta}$ satisfying $*$
then there exists a unique (upto change of coords)
complex vector bundle with these transition functions

Given $E \rightarrow M$

• ${}^t g_{uv}^{-1}$ defines a "dual bundle" E^* or $E^\vee$

• $E \oplus F$ defined by $\left(\begin{array}{c|c} g_{uv} & 0 \\ \hline 0 & h_{uv} \end{array} \right)$ — transform for F

• $E \otimes F$ defined by given h_{uv} .

• $\wedge^m E$ is antisym of $\underbrace{E \otimes E \otimes \dots \otimes E}_m$

A 'global section' σ is a map.

$$E \xrightarrow[\sigma]{\pi} M, \quad \pi\sigma = 1$$

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Similarly local sections can be defined on $U \subset M$.

Holomorphic Vector Bundles

If M is a complex manifold and the transition functions

$$g_{\alpha\beta}: U_{\alpha} \cap U_{\beta} \rightarrow GL(k, \mathbb{C})$$

are holomorphic, then E is holomorphic.

Sim. hol. sections

Tangent Bundles

• Real space of directional derivatives

Dir derivative is $V^\mu \frac{\partial}{\partial x^\mu}$ for some tangent vector V^μ

- basis of $\mathbb{R}^n$ for each $x \in M$ is $\left\{ \frac{\partial}{\partial x^\mu} \right\}$

If we have a complex manifold of complex dimension n
then $\left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \right\}$ form a basis for the fibre $\mathbb{R}^{2n}$
for a vector bundle.

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Define $T_{\mathbb{C}}$ to be the complex vector bundle with
basis $\left\{ \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j} \right\}$ over each pt
- fibre is $\mathbb{C}^{2n}$

If $\left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j} \right\}$ are basis vectors for each $\mathbb{C}^{2n}$ -fibre

then $\left\{ \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j} \right\}$ also forms a basis

If manifold has holomorphic transition function, then $\frac{\partial}{\partial \bar{z}_j}$ in one patch is only func of $\frac{\partial}{\partial z_j}$

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$$T_{\mathbb{C}} = T \oplus \bar{T}$$

$\uparrow$
 holomorphic
 tangent
 bundle

so

$$T_{\mathbb{C}} = T \oplus \bar{T}$$

↑
isomorphic
tangent
bundle

We may also define dual of tangent bundles by using
 dx^i etc. as basis.

Cotangent bundle T^*

Differential Form

$$T_e^* = T^* \oplus \overline{T}^*$$

$$\wedge^k T_e^* = \bigoplus_{p+q=k} (\wedge^p T^* \otimes \wedge^q \overline{T}^*)$$

Let $A^{p,q}(M)$ be the space of C^∞ -sections of $\Lambda^p T^* \otimes \Lambda^q \overline{T}^*$

Define $\partial: A^{p,q} \rightarrow A^{p+1,q}$

$\overline{\partial}: A^{p,q} \rightarrow A^{p,q+1}$

$$\partial \left(\sum_{j,k} a_{jk} dz^j dz^k \cdot dz^{\overline{l}} \right) = \sum_{s,j,k} \frac{\partial a_{jk}}{\partial z^s} dz^s dz^j dz^k \cdot dz^{\overline{l}}$$

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$$d = \partial + \bar{\partial}$$

De Rham cohomology $H_{dR}^k(M) = \frac{\text{Ker } d}{\text{Im } d}$

$$b_k = \dim H_{dR}^k$$

Dolbeault Cohomology

$$H_{\bar{\partial}}^{p,q}(M) = \frac{\text{Ker } \bar{\partial} \text{ (on } A^{p,q})}{\text{Im } \bar{\partial} \text{ (on } A^{p,q-1})}$$

$$h^{p,q} = \dim H_{\bar{\partial}}^{p,q}(M)$$

Hodge Numbers

$$A^k \xrightarrow{d} A^{k+1} \xrightarrow{d} A^{k+2}$$

$$d = \partial + \bar{\partial}$$

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Betti Numbers

E.g. Line bundle on $\mathbb{P}^n$ L_n

Recall U_j is patch with $z_j \neq 0$

Suppose $g_{U_j U_k}$ is given by $\left(\frac{z_j}{z_k} \right)^n \times \dots$
 $GL(1, \mathbb{C})$

E.g. L_0 is $\mathbb{P}^1 \times \mathbb{C}$ - trivial line bundle
 L_1 on $\mathbb{P}^1$: Consider a global section

Ex L_0 is $\mathbb{P}^n \times \mathbb{C}$ - trivial line bundle

L_1 on $\mathbb{P}^1$: Consider a global section
which looks like "1" in U_0

So this global section is
never infinite but it has one zero at $z_0 = 0$

Global section has one zero

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Global section has one zero

A global section of $\mathcal{L}_m$ has m zeroes.