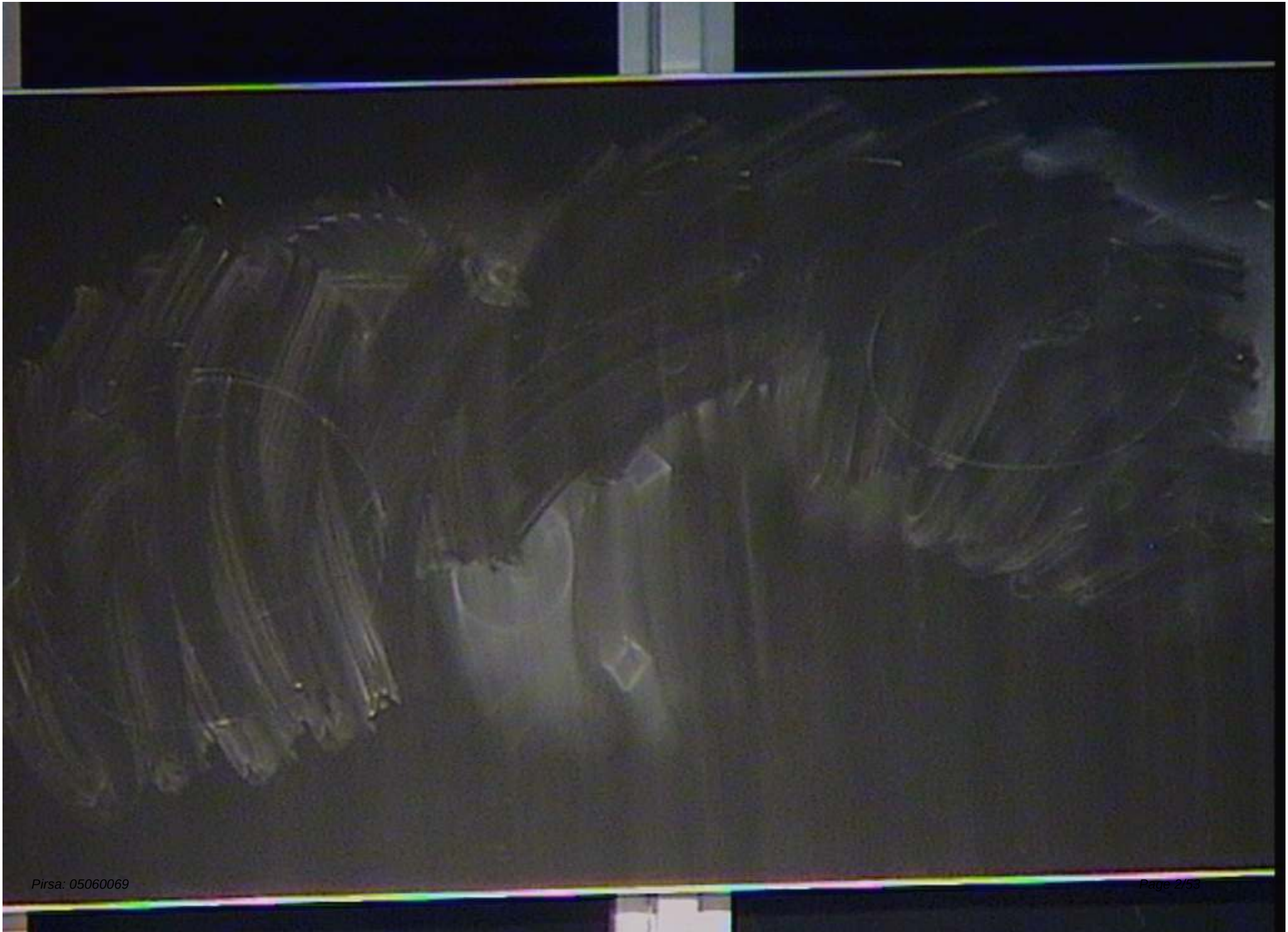
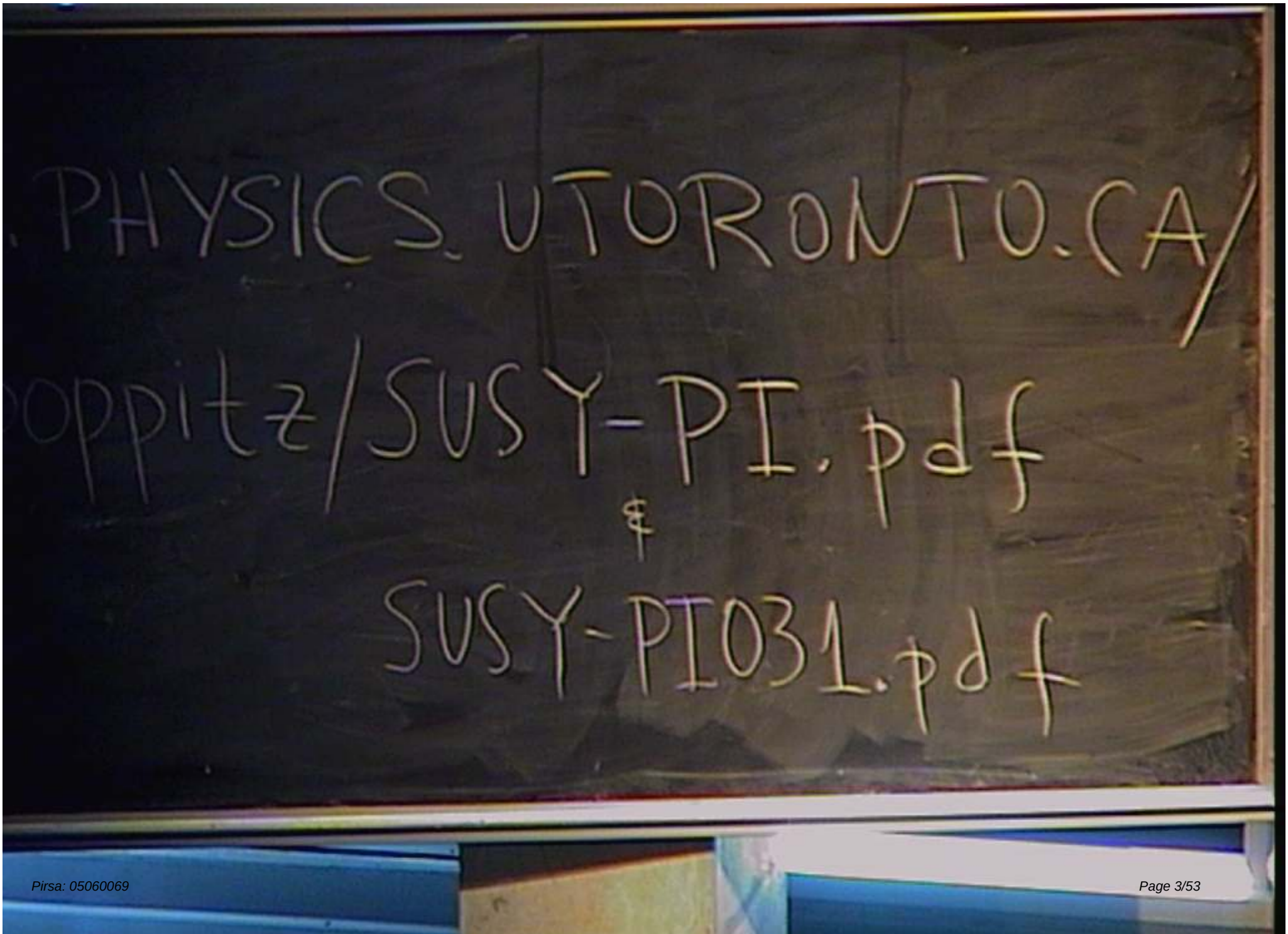
Title: Supersymmetry Basics

Date: Jun 20, 2005 09:30 AM

URL: http://pirsa.org/05060069

Abstract:

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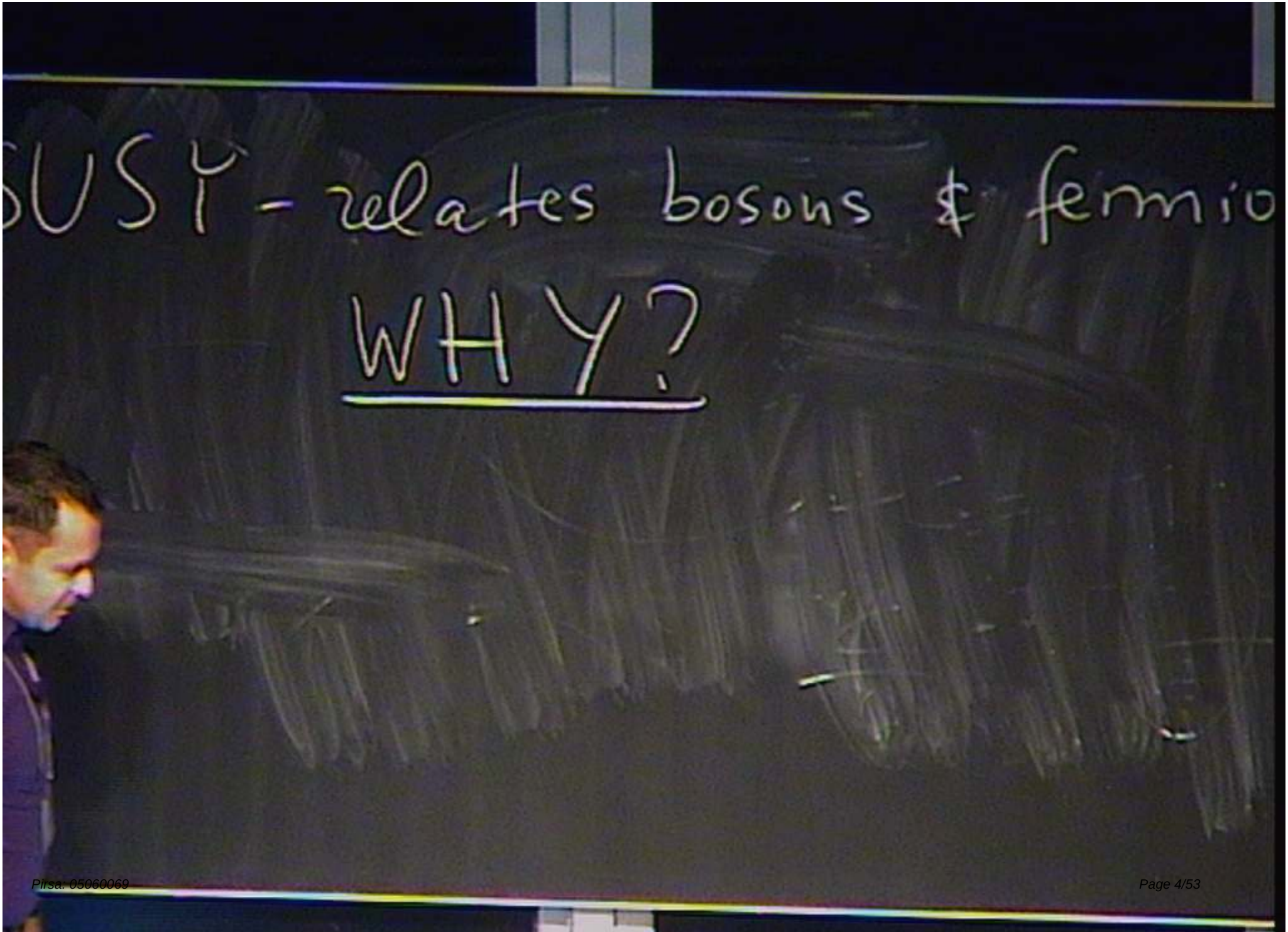
SUSY - relates bosons & fermio
WHY?

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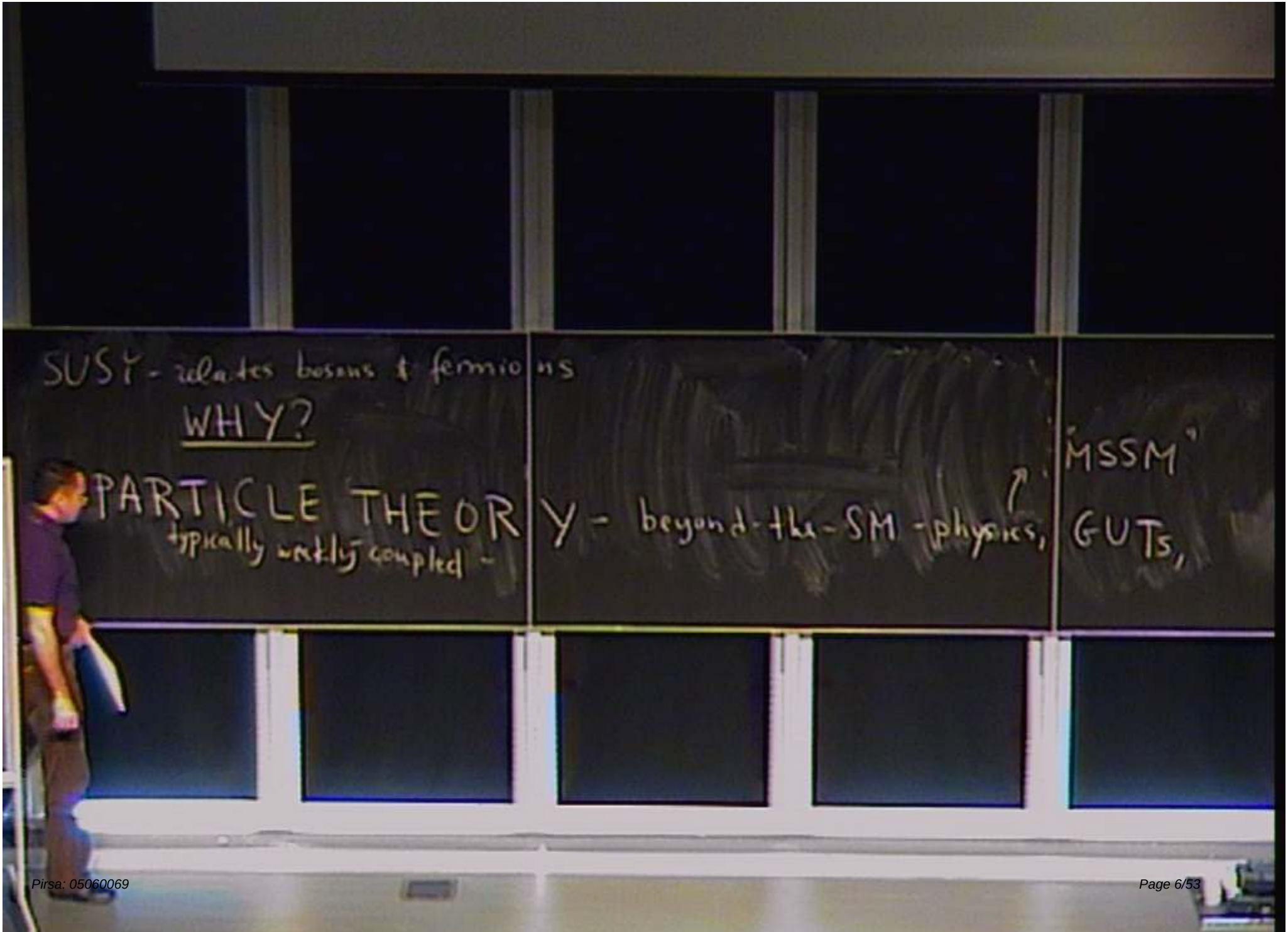
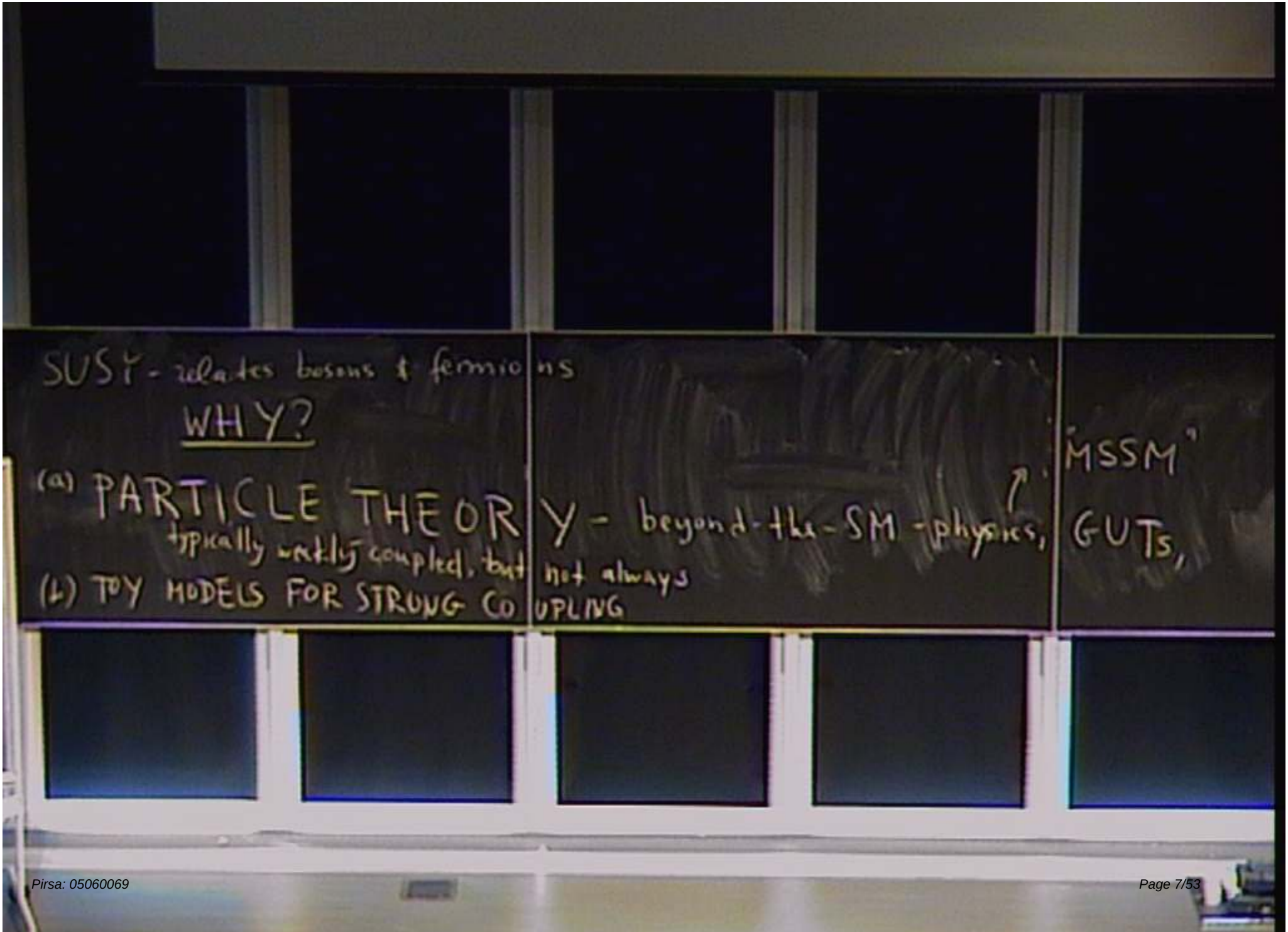

SUSY - relates bosons & fermions
WHY?
PARTICLE THEORY - beyond-the-SM-physics,
typically weakly coupled -
'MSSM'
GUTs,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(a) PARTICLE THEORY - beyond-the-SM - physics,
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(b) TOY MODELS FOR STRONG COUPLING
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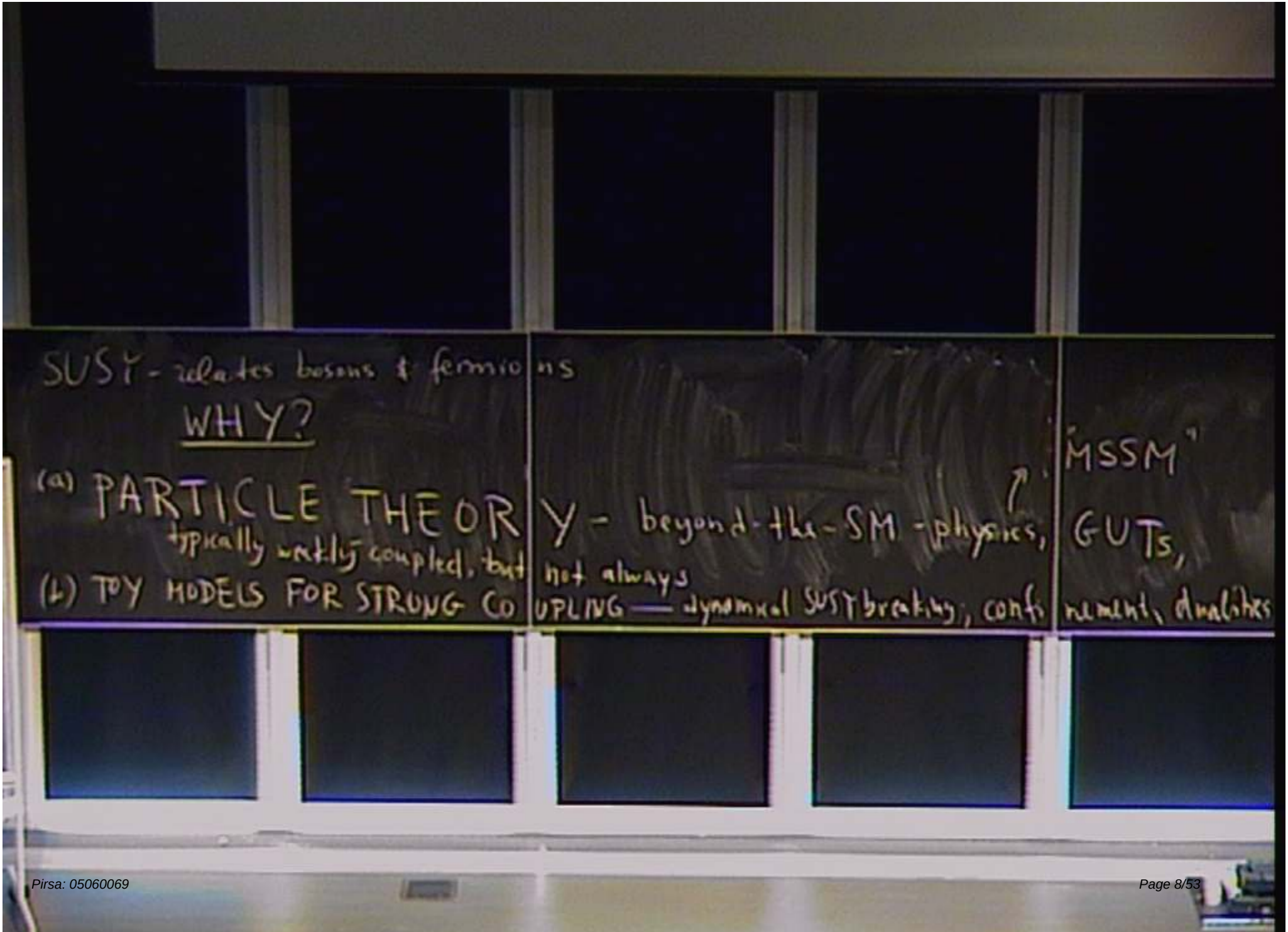

SUSY - relates bosons & fermions
WHY?
(a) PARTICLE THEORY - beyond-the-SM-physics,
typically weakly coupled, but not always
(b) TOY MODELS FOR STRONG COUPLING — dynamical SUSY breaking, confinement, dualities
"MSSM"
GUTs,

"MSSM"

GUTs,

nement, dualities ...

both in 4d as well as
5d & 6d

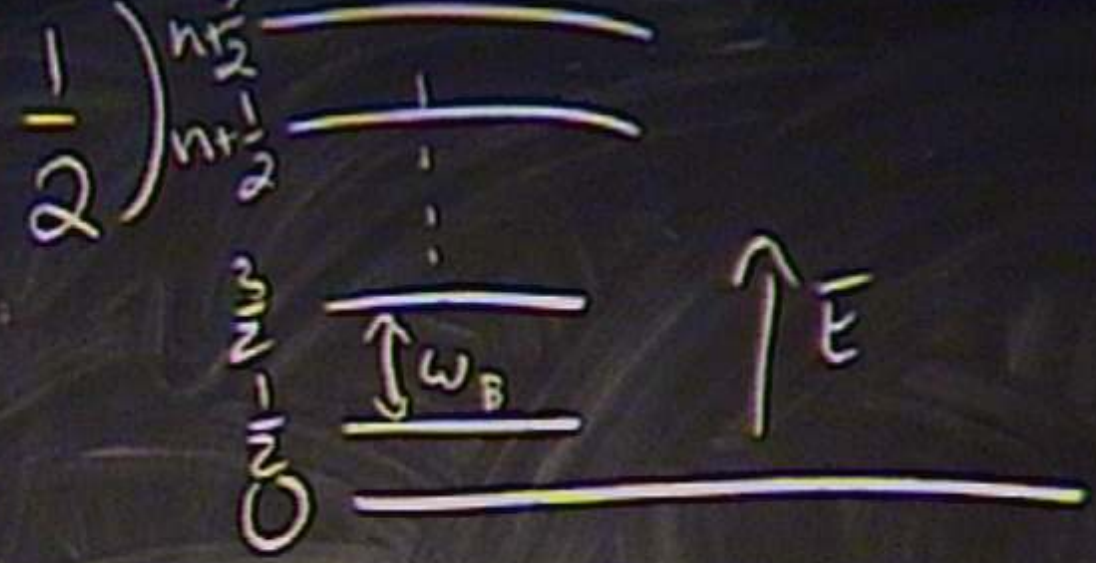
& lower d = 2, 3

(c) string theory

(d) mathematical results - index theorems ...

(e) getting jobs

$$E_n = \omega_B \left(n + \frac{1}{2} \right)$$

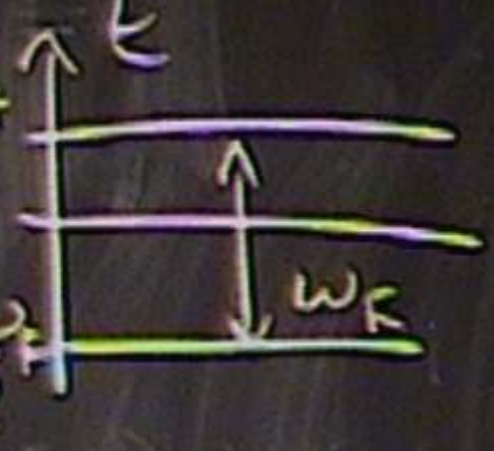


) fermionic oscillator

f, f^+ $\quad f^2 = f^{+2} = 0 \quad \{f, f^+\} =$

$$= \omega_F \left(f^+ f^- - \frac{1}{2} \right) = \frac{\omega}{2} \left[f^+, f^- \right]$$

$$= \omega_F \left(k - \frac{1}{2} \right), \; k = 0, 1$$



oscillator = a + b, with $\omega_B = \omega_F$

$$H_{SUSY} = H_B + H_F \Big|_{\omega_B = \omega_F} = \omega \left(b^\dagger b + f^\dagger f \right)$$

Energy levels (n,k), $n = 0, \ldots \infty$

$k = 0, 1$

$(n,0)$	$(n-1,1)$
$(3,0)$	$(2,1)$
$(2,0)$	$(1,1)$
$(1,0)$	$(0,1)$
$(0,0)$	

* all $E \geq 0$ levels are doubly degenerate
* $E_0 = 0$

$$H_{SUSY} = H_B + H_F\Big|_{\omega_B = \omega_F} = \omega\left(b^\dagger b + f^\dagger f\right)$$

Supersymmetry is responsible for these (*) properties (take $b^{(\dagger)}, f^{(\dagger)}$ to commute)

$$Q = b^\dagger f$$

$$Q^\dagger = b f^\dagger$$

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Supersymmetry is responsible for these (*) properties (take $b^{(\dagger)}, f^{(\dagger)}$ to commute)

$Q = b^\dagger f$ nilpotent $Q^2 = Q^{\dagger 2} = 0$

$Q^\dagger = b f^\dagger$ $\Big\}$ $H_{SUSY} = \omega\{Q, Q^\dagger\}$ ← SHOW!

$$[Q, H] = [Q^+, H] =$$

$$Q^2 = Q^{+2} = 0$$

$$\{Q, Q^+\} = H$$

Energy levels (n,k)

2d electrons in B along $\hat{z}$ axis w/ $g=2$

$(n,0)$ $\xrightarrow{a^\dagger}$ $(n+1,1)$

$(3,0)$ $(2,1)$

$(2,0)$ $(1,1)$

$(1,0)$ $(0,1)$

$(0,0)$

a

$a^\dagger$

$\vec{B}$

$k=1$

$n+1$

n

$k=0$

$[Q^+, H] = 0$

$Q^{+2} = 0$

$Q^+ = H$

for a system def'd like this

(1) all energy levels are ≥ 0

(2) all energy levels w/ $E > 0$ are doubly degenerate

$Q^+, H] = 0$ } for a system def'd like this

(1) all energy levels are ≥ 0

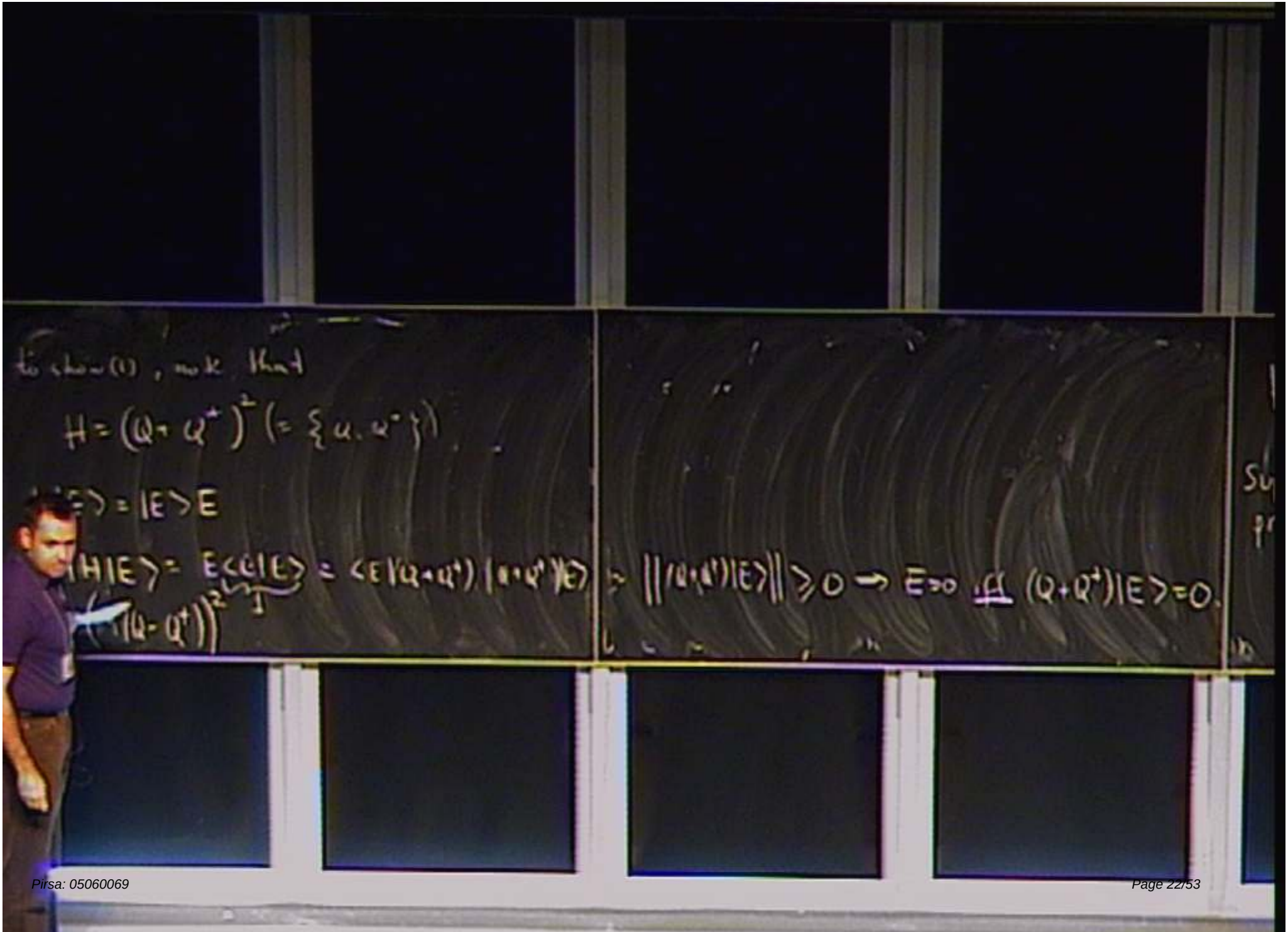
(2) all energy levels w/ $E > 0$ are doubly degenerate

to show (1), note that

$$H = (Q + Q^+)^2 \quad (= \{Q, Q^+\})$$

$$H|E\rangle = |E\rangle E$$

$$\langle E|H|E\rangle = E\langle E|E\rangle = \langle E|(Q+Q^+)(Q+Q^+)$$


$H = (Q + Q^\dagger)^2 \; (= \{Q, Q^\dagger\})$
$|E\rangle = |E\rangle E$
$\langle E|H|E\rangle = E\langle E|E\rangle = \langle E|(Q+Q^\dagger)(Q+Q^\dagger)|E\rangle = \|(Q+Q^\dagger)|E\rangle\|^2 \geq 0 \rightarrow E \geq 0$
$\|(Q+Q^\dagger)|E\rangle\|$
$(Q+Q^\dagger)|E\rangle = 0.$

to show (1), need that

$H = (Q + Q^\dagger)^2 \; (= \{Q, Q^\dagger\})$

$H|E\rangle = |E\rangle E$

$\langle E|H|E\rangle = E\langle E|E\rangle = \langle E|(Q+Q^\dagger)(Q+Q^\dagger)|E\rangle$

$H = \left((Q+Q^\dagger)\right)^2 \rightarrow$ same argument implies $E \geq 0$ iff $(Q+Q^\dagger)|E\rangle = 0$

(2) $E = 0$ iff $Q|E\rangle = 0$, $Q^\dagger|E\rangle = 0$

$\therefore \; \| (Q+Q^\dagger)|E\rangle \| \geq 0 \rightarrow E \geq 0$ iff $(Q+Q^\dagger)|E\rangle = 0$.

(1') $E=0$ iff $Q|E\rangle = 0$
$Q^\dagger|E\rangle = 0$

Supersymmetric ground states

→ in a system w/ unbroken Susy, $E_0 = 0$

$\| (Q+Q^\dagger)|E\rangle \| \geq 0 \Rightarrow E=0$ iff $(Q+Q^\dagger)|E\rangle = 0$

$(Q-Q^\dagger)|E\rangle = 0$,

our system

$N=2$ SUSY QM

$Q_1 = Q + Q^+$

$Q_2 = i(Q - Q^+)$

$H = Q_1^2 = Q_2^2$

$\{Q_1, Q_2\} = 0$

$SO(2)$ automorphism (acts on $(1,2)$)

$\{Q_i, Q_j\} = \delta_{ij} H$, $i,j = 1,2$

"$N=$"

More complicated SUSY QM w/ "inte

$[\hat{p}, x] = -i$; $\Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix}$

$$H = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\left(W'(x)\right)^2 + \frac{1}{2}\sigma_3 W''(x$$

$W(x)$ — superpotential

More complicated SUSY QM w/ "interactions"

$[\hat{p}, x] = -i$; $\Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix}$

states w/ $\Psi_2 = 0$ "Fermionic"

states w/ $\Psi_1 = 0$ "Bosonic"

$$H = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\left(W'(x)\right)^2 + \frac{1}{2}\sigma_3 W''(x)$$

$W(x)$ — superpotential

$$Q_1 = \frac{1}{2}\sigma_1 p + \frac{1}{2}\sigma_2 W'(x)$$

$$Q_2 = \frac{1}{2}\sigma_2 p - \frac{1}{2}\sigma_1 W'(x)$$

↓ check!

$$\{Q_i, Q_j\} = \delta_{ij} H$$

"fermion #"

$$F = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(\mathbb{1} + \sigma_3)$$

also $[F, H] = 0$

$$[F, Q] = Q$$

$$[F, Q^+] = -Q^+$$

↑ CHECK! (this sign!!)

define operator $(-1)^F$ = +1 on bosonic states
-1 on fermionic states

$(-1)^F H' = H(-1)^F$ ⇒ all states are either B or F.

$(-1)^F Q_1 = -Q_1 (-1)^F$ - when Q acts on a B(F) state it makes it into F(B) state

$(-1)^F |B\rangle = |B\rangle$

$(-1)^F |F\rangle = -|F\rangle$

So ready to show that all $E>0$ eigenstates are doubly degenerate ($\forall |B\rangle$ - eigenstate is an $|F\rangle$ eigenstate w/ same energy).

Let $|E\rangle$ be an eigenstate $H|E\rangle = |E\rangle E$

$\langle E|E\rangle = 1$

then $|E,1\rangle = \frac{Q_1|E\rangle}{\sqrt{E}}$ is also an eigenstate of same E

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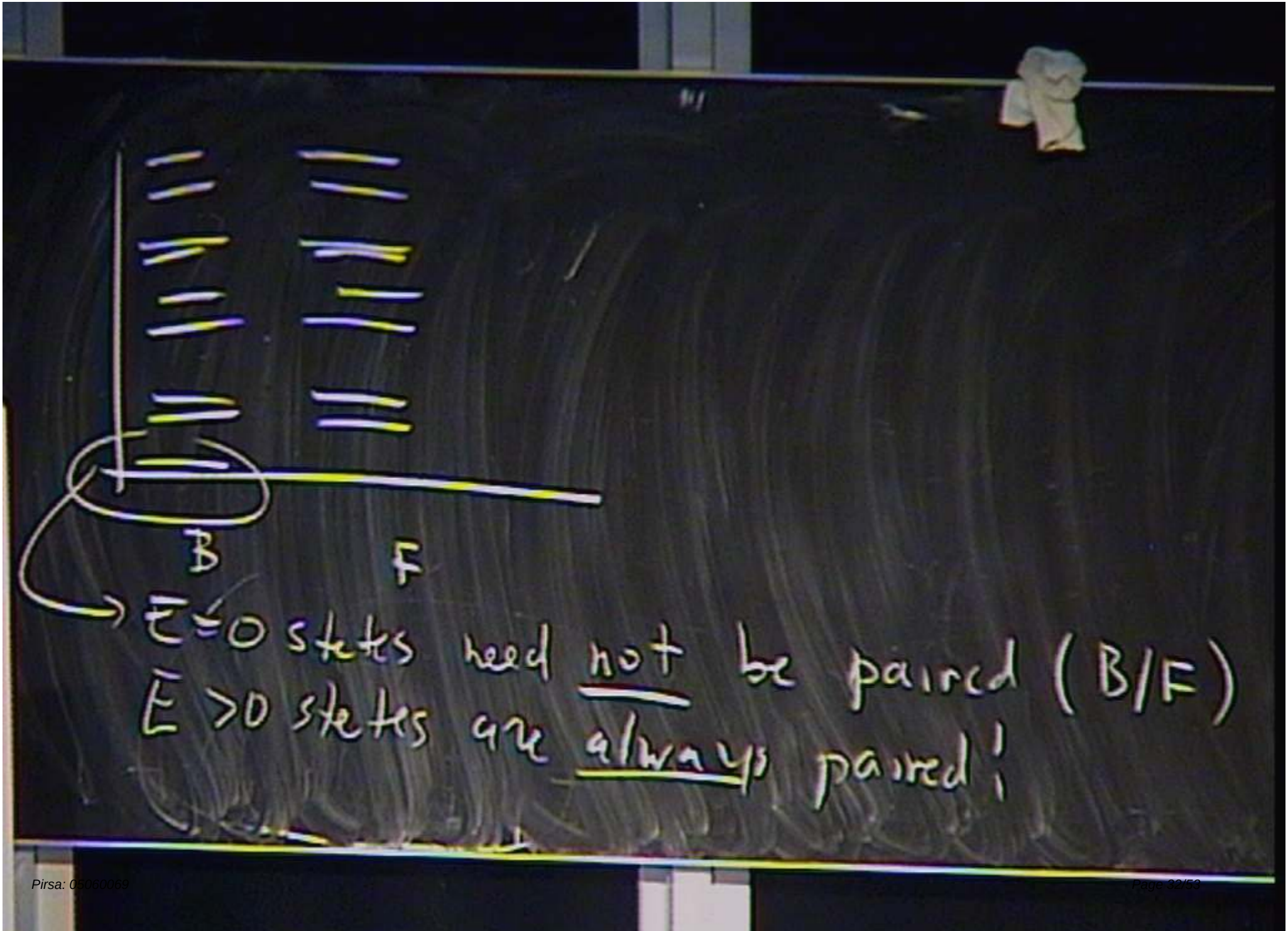
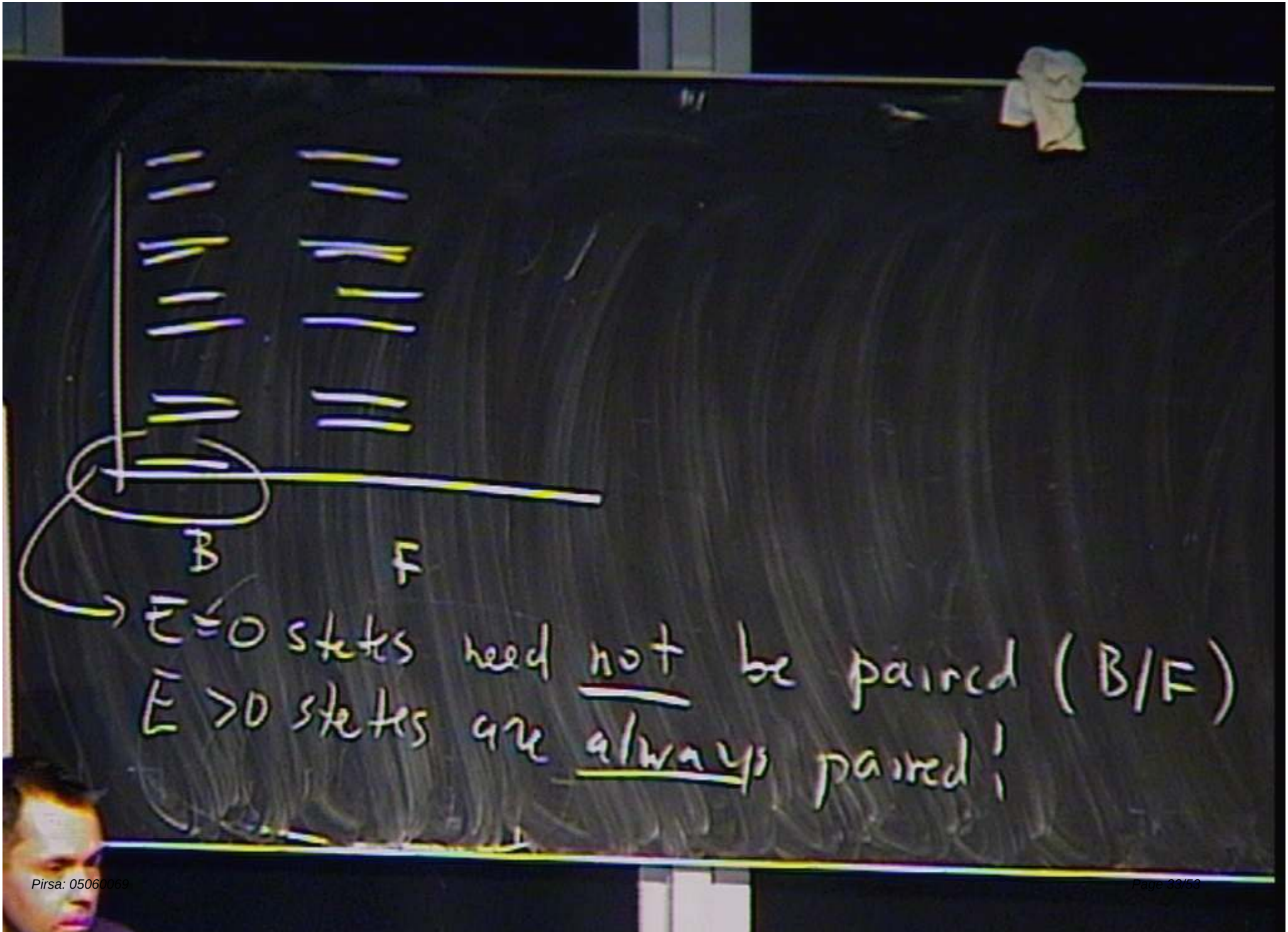
then $|E,1\rangle = \frac{Q_1|E\rangle}{\sqrt{E}}$ is also an eigenstate of same E & opposite $(-)^F$


B
F
E=0 states need not be paired (B/F)
E>0 states are always paired!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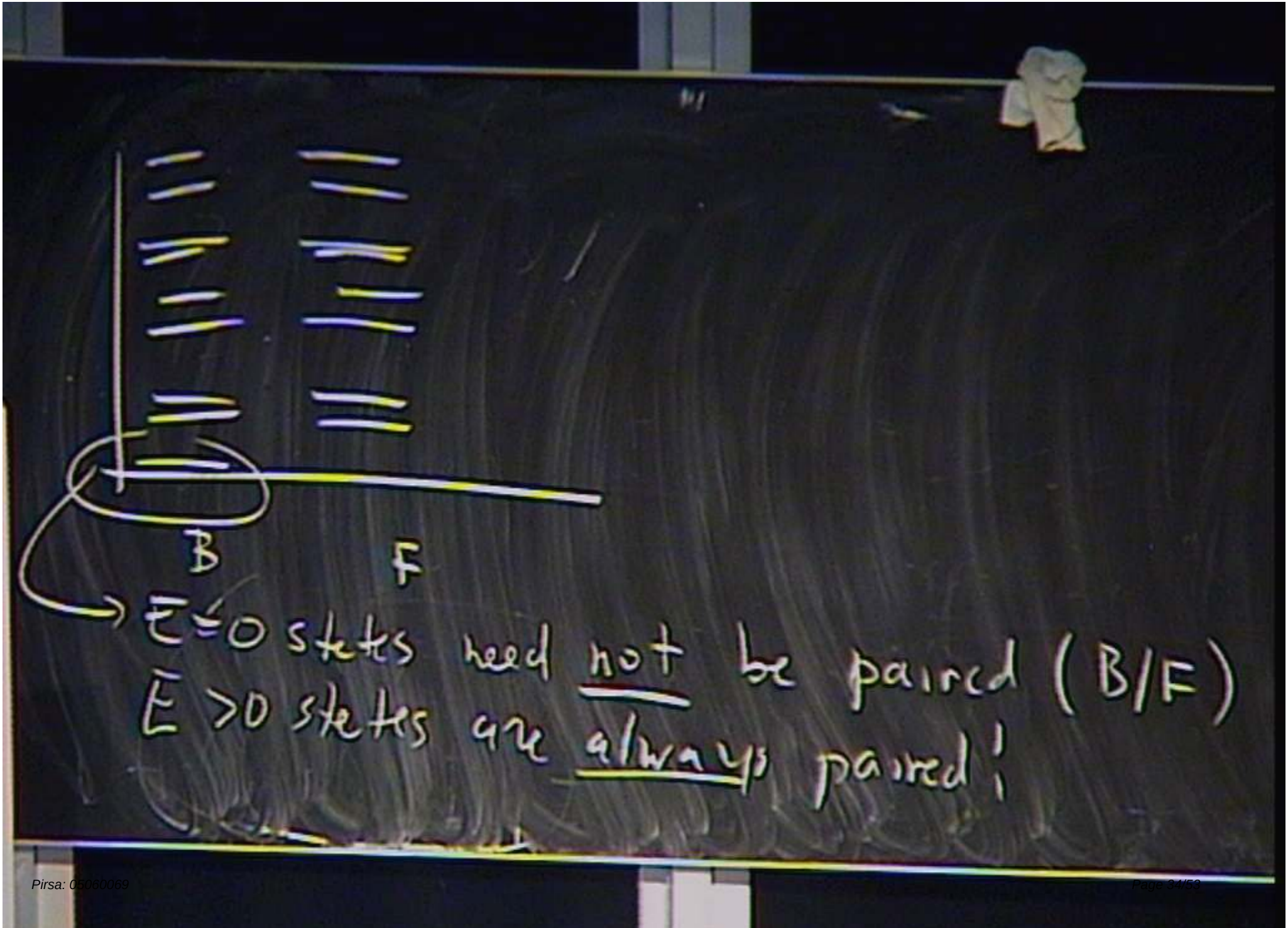
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(B/F)

For a system with given H (W(x), th
about the properties of its ground state

this case) can we say sm
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– is the ground state

For a system with given H (W(x), ...)
about the properties of its ground state

(turns out we can ... SUSY theories ... a general

$H|\psi\rangle = 0 \longrightarrow$ since $Q_1^2 = Q_2^2 = H$ (shown) $\Rightarrow H|\psi\rangle = 0$ iff $Q_i|\psi\rangle = 0$

(if $|\psi\rangle$ is a SUSY ground state, above must hold, $i = 1, 2$)

... these case) can we say something (w/out solving the system)

? (i.e. does it have $E_0 = 0$ or not —

— is the ground state SUSY ?)

criterion can be given based on W, its index ?

For a system with given H ($W(x)$, ch

about the properties of its ground state

? (i.e. does it have $E_0 = 0$ or not —

— is the ground state SUSY, ?)

(turns out we can [illegible] SUSY theorems a general criterion can be given based on $W(x)$ [illegible] index)

$H|\psi\rangle = 0 \longrightarrow$ since $Q_1^2 = Q_2^2 = H$ (SHOW) $\Rightarrow$ $H|\psi\rangle = 0$ iff $Q_1|\psi\rangle = 0$ — 2nd order $\rightarrow$ 1st order !

(if $|\psi\rangle$ is a SUSY ground state, above must hold!)

$$Q_1|\Psi\rangle = 0 \Rightarrow \left(i\sigma_1 \frac{d}{dx} + \sigma_2 W'(x)\right)\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} = 0$$

SHOW:

$$\psi_0(x) = \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} = e^{\sigma_3 W(x)}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \begin{pmatrix}e^{W(x)} & \\ & e^{-W(x)}\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}$$

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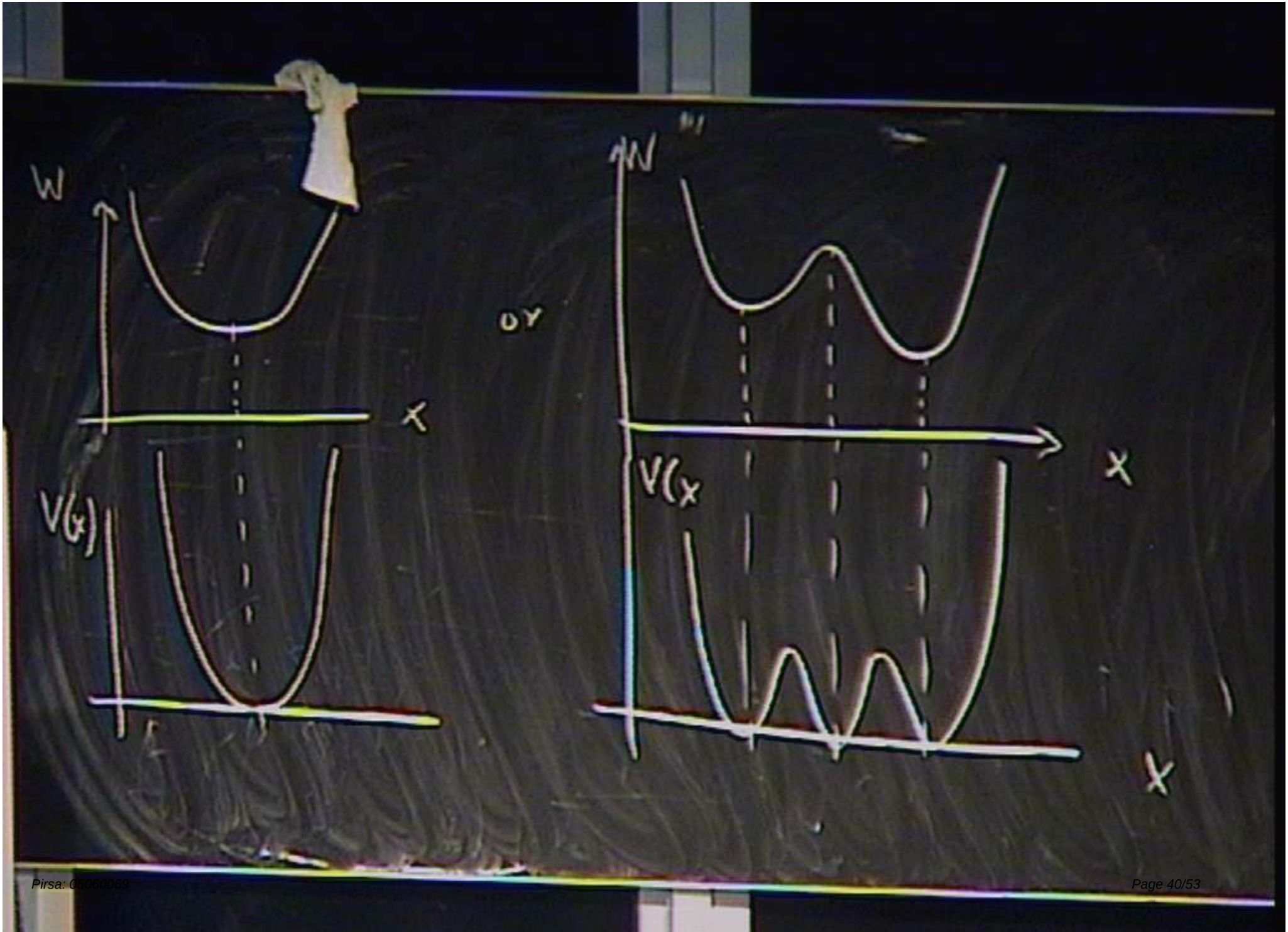
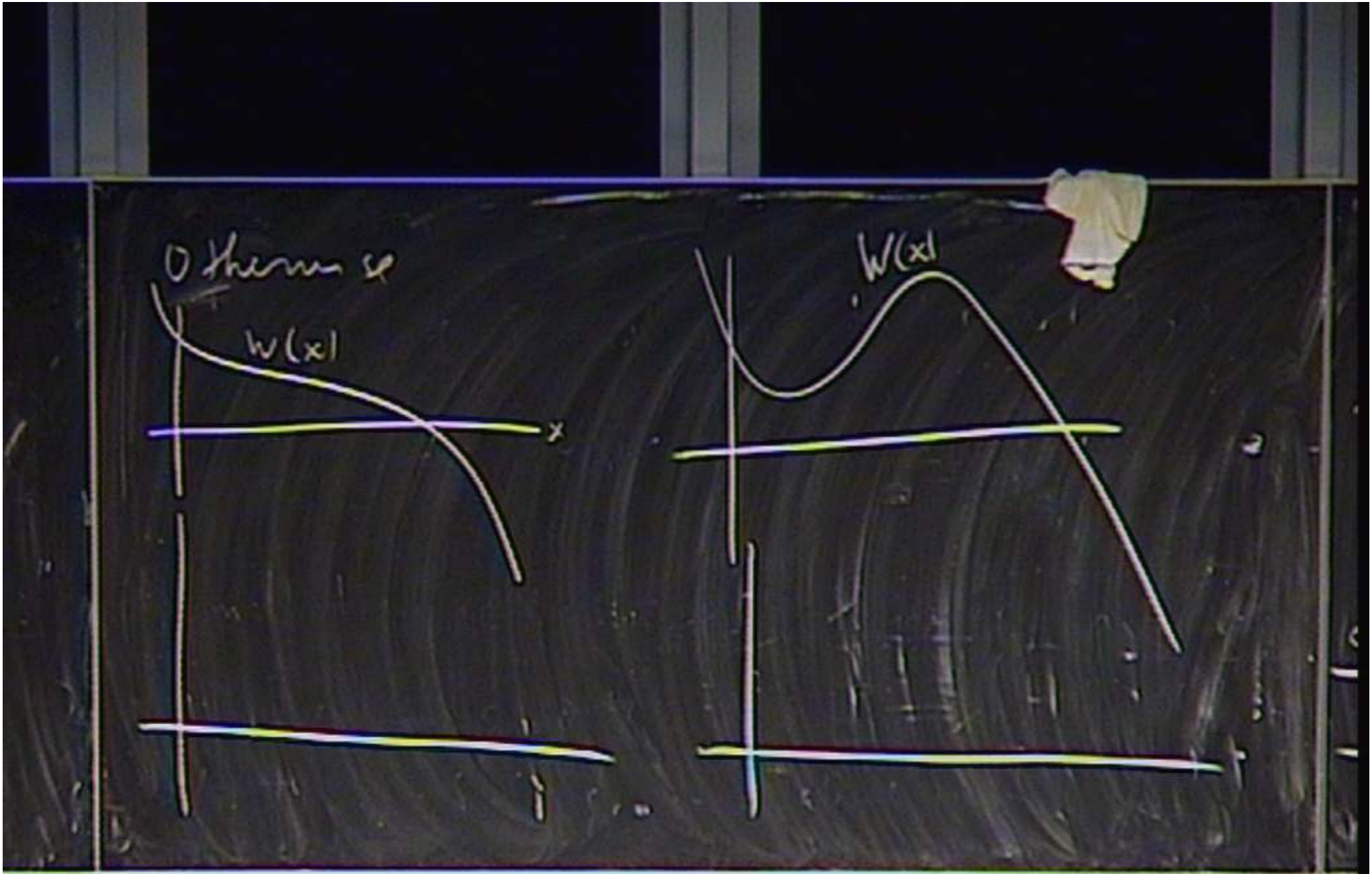
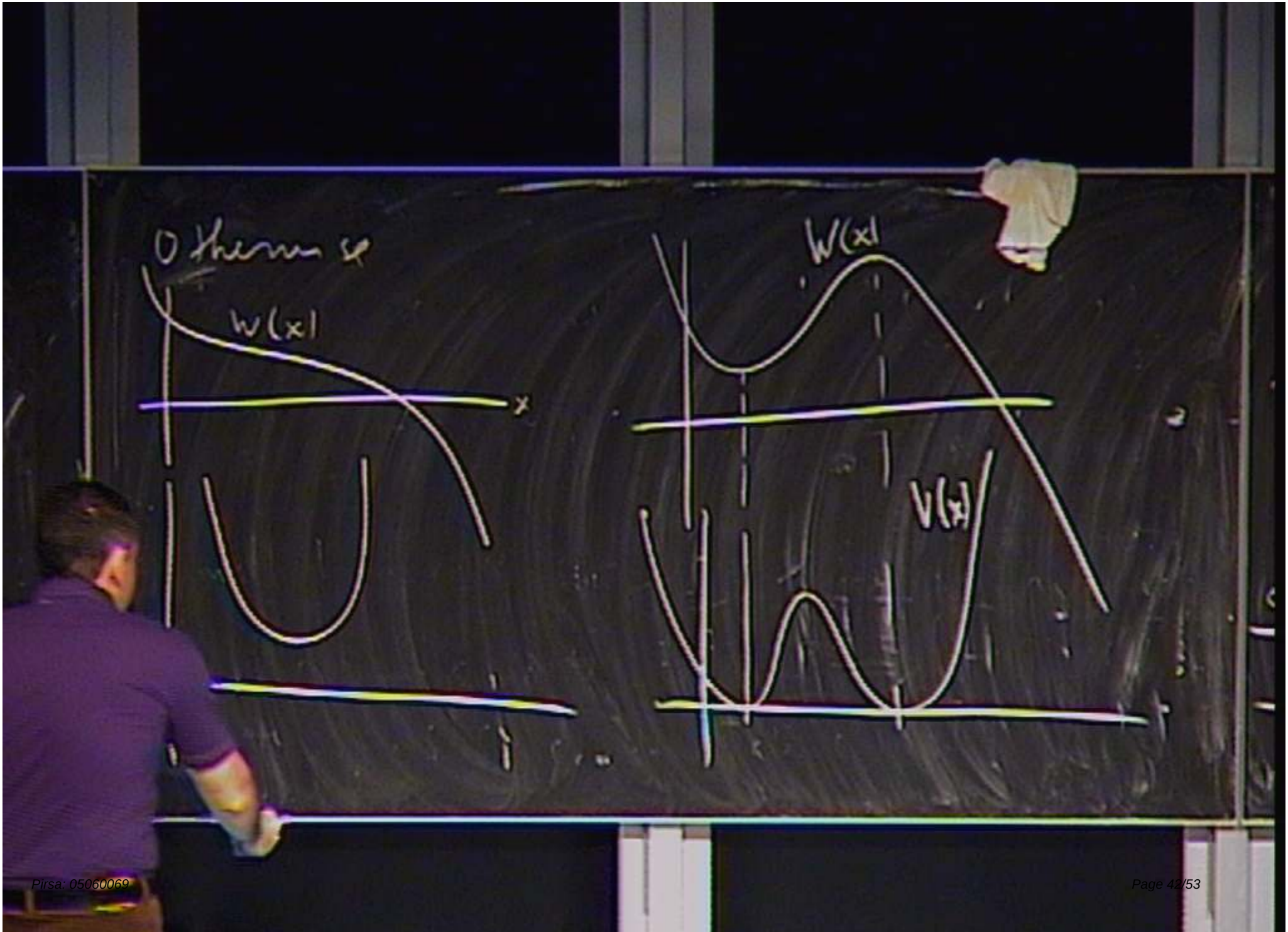
Is it normalizable? — (1) if $c_1 = 0$, then $W(x) \to +\infty$ as $|x| \to \infty$

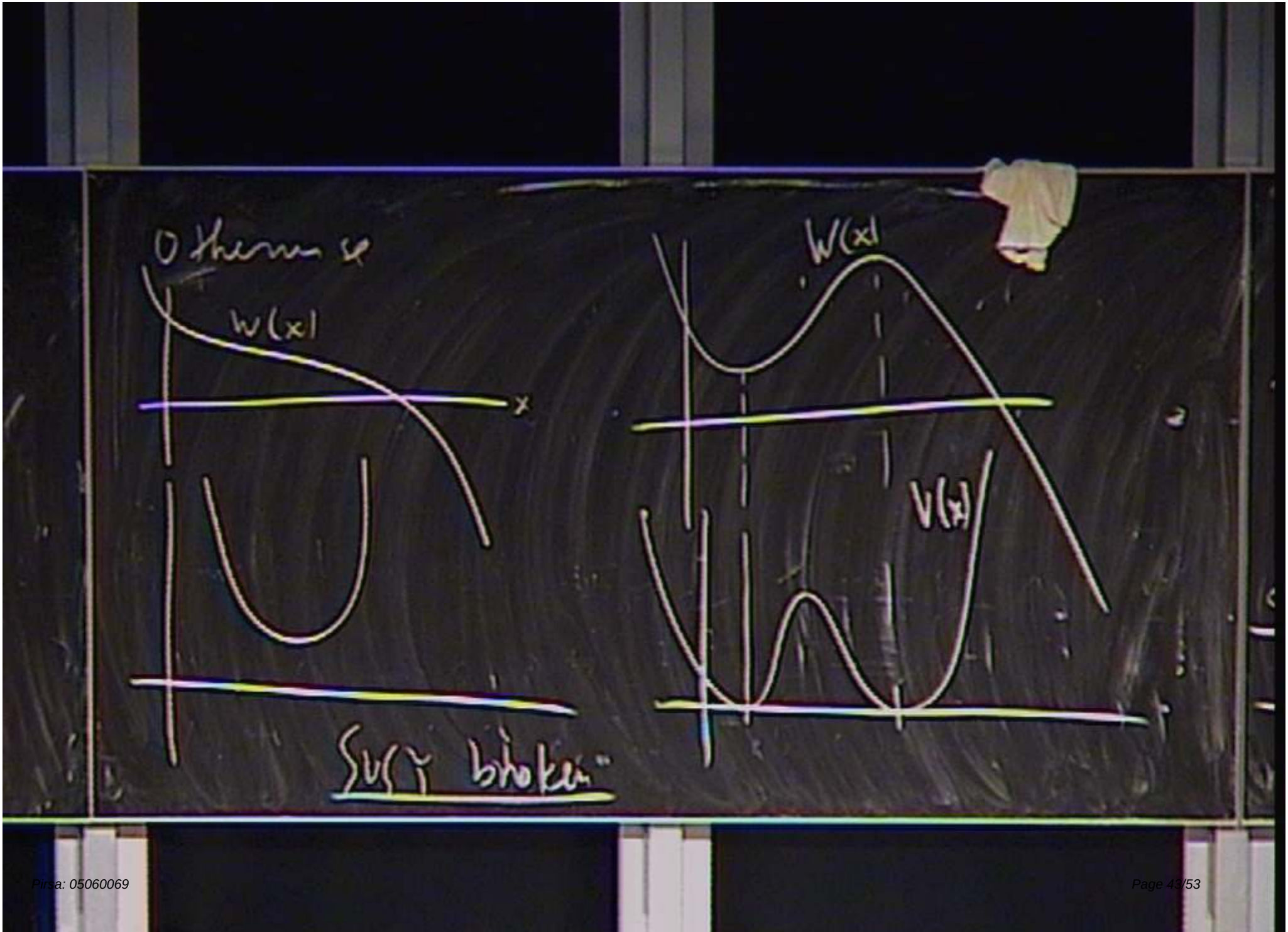
(2) if $c_2 = 0$, then $W(x) \to -\infty$ as $|x| \to \infty$

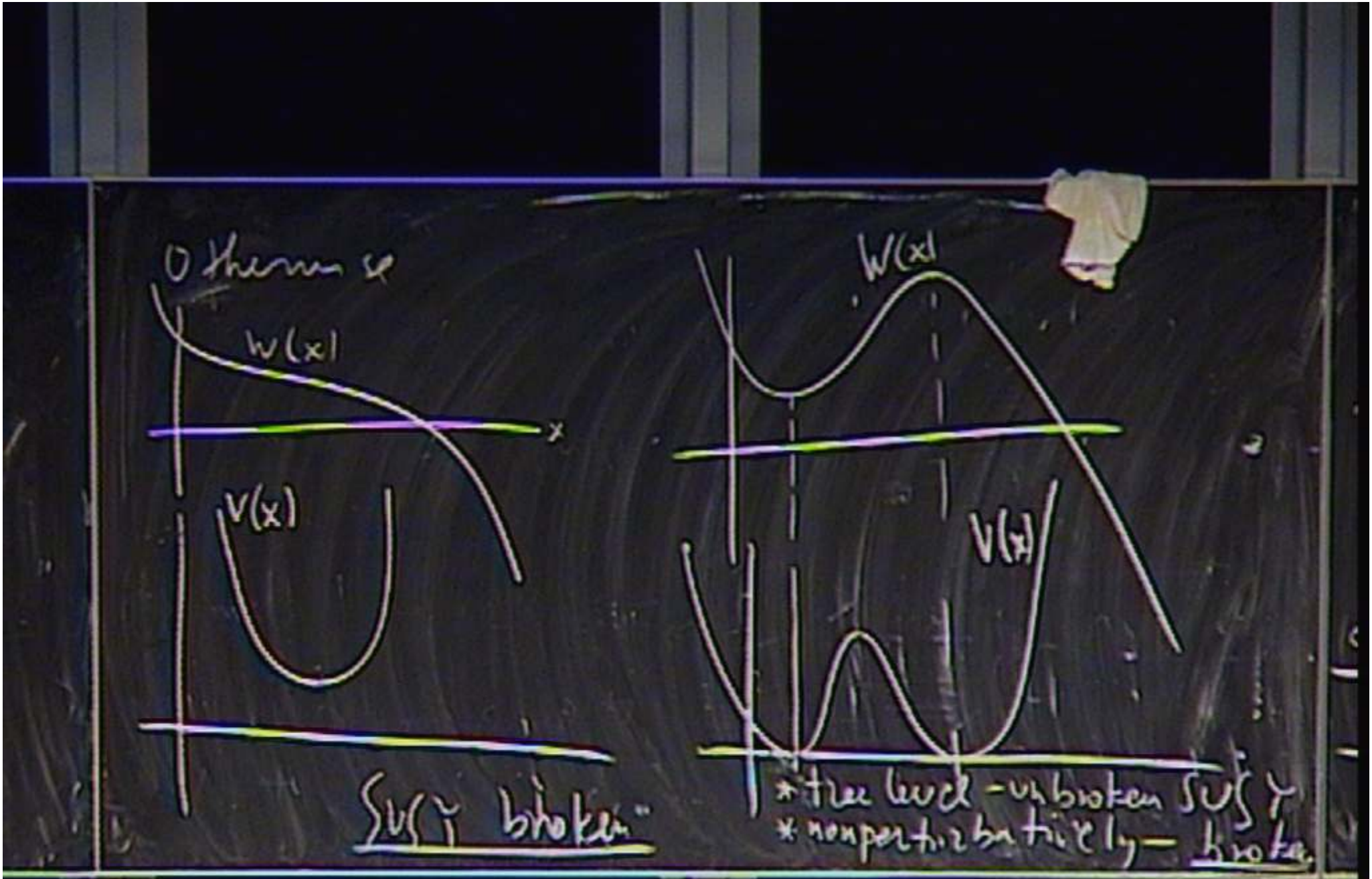
$\Rightarrow$ Summary: $W(x)$ is "even at ∞"


W
x
V(x)
OY
W
V(x
x
x


Otherwise
W(x)
x
W(x)


W(x)
x
W(x)
V(x)


Otherwise
W(x)
x
W(x)
V(x)
Symmetry broken


Otherwise
W(x)
x
V(x)
SUSY broken"
W(x)
V(x)
* tree level - unbroken SUSY
* nonperturbatively - broken

Witten index

$$\mathrm{tr}(-1)^F =$$

Witten index

$$\mathrm{tr}(-1)^F = \sum_E n_B(E) - n_F(E) =$$

$$= n_B(0) - n_F(0)$$

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$$\mathrm{tr}(-1)^F = \sum_E n_B(E) - n_F(E) =$$

$$= n_B(0) - n_F(0) \Rightarrow$$ if $\mathrm{tr}(-1)^F \neq 0$, then SUSY is unbroken

Witten index

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$$= n_B(0) - n_F(0) \Rightarrow \underline{\text{if } \mathrm{tr}(-1)^F \neq 0, \text{ then SUSY is unbroken}}$$

$$\Rightarrow \text{if } \mathrm{tr}(-)^F = 0$$

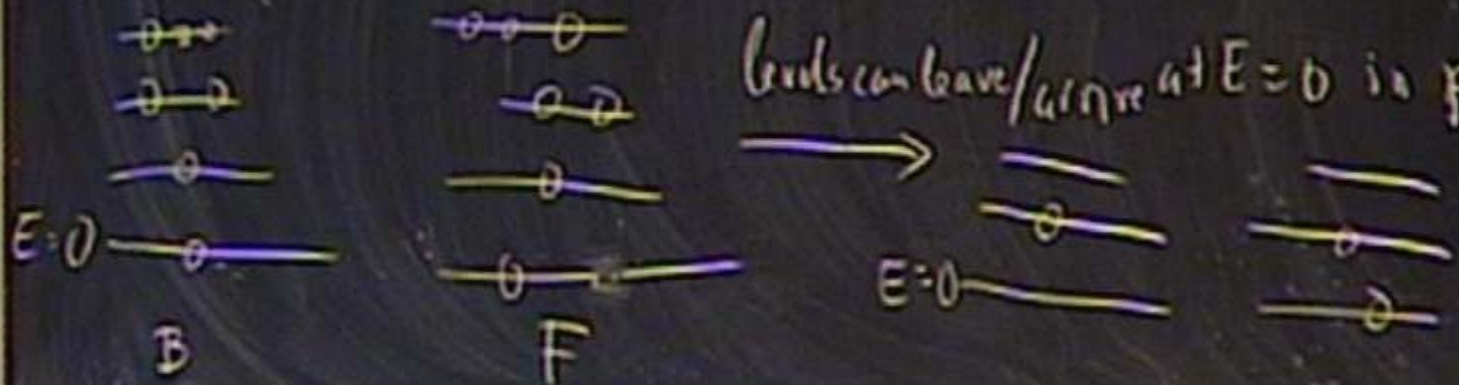
Witten index

$$\mathrm{tr}(-1)^F = \sum_E n_B(E) - n_F(E) =$$

$$= n_B(0) - n_F(0) \Rightarrow \underline{\text{if } \mathrm{tr}(-)^F \neq 0, \text{ then SUSY is unbroken}}$$

$\Rightarrow$ if $\mathrm{tr}(-)^F = 0$ SUSY can break.

$\mathrm{tr}(-)^F$ is invariant under continuous deformations of the theory.



Witten index

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In 4d QFT

pure SYM

$\Rightarrow$ if $\mathrm{tr}(-)^F = 0$ SUSY can break

SUSY broke

Witten index

$$\mathrm{tr}(-1)^F = \sum_E n_B(E) - n_F(E) =$$

$$= n_B(0) - n_F(0) \Rightarrow \underline{\text{if } \mathrm{tr}(-)^F \neq 0, \text{ then SUSY is unbroken}}$$

In 4d QFT

pure SYM $\mathrm{tr}(-)^F = r+1$ (r: rank of the group)

$\Rightarrow$ if $\mathrm{tr}(-)^F = 0$ susy can break