Title: Complexity of elections: how to measure your voting power?

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Abstract: Observers agree that a citizen of Ohio had much larger voting power than a citizen of Texas or California in the recent US presidential election. Why is it so? A brief introduction to the theory of voting will be provided. We analyze the voting power of a member of a voting body, or of a person which elects his representative, who will take part in the voting on her behalf. The notion of voting power is illustrated by examples of the systems of voting in the European Council. We propose a representative voting system based on the square root law of Penrose. Using statistical approach and considering fictitious countries with randomly chosen populations we study the problem of selecting an optimal quota.
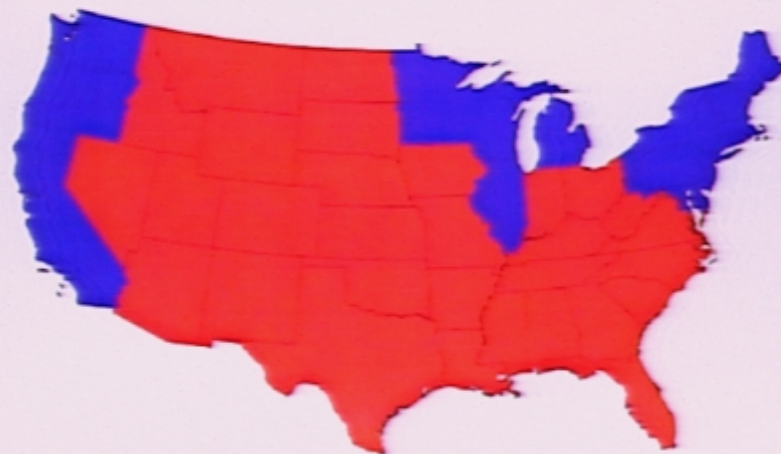
Complexity of elections:

How to measure your voting power?

Karol Życzkowski

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and

Wojciech Słomczyński
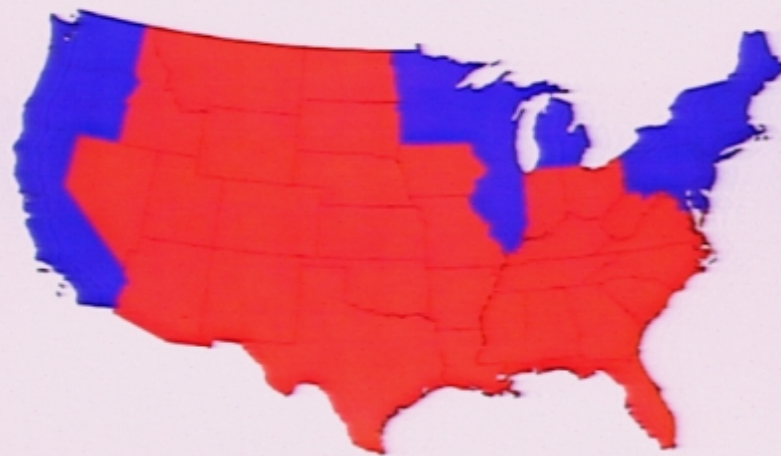
(Jagiellonian University,
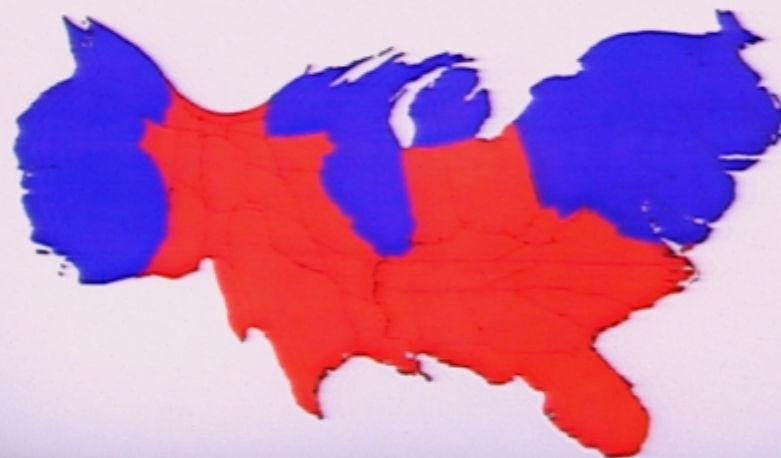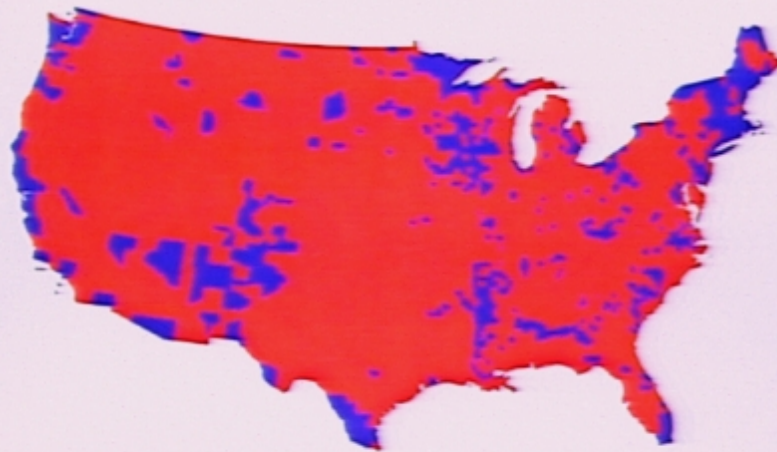Cracow, Poland)

2004 US Presidential Elections

Results by state

2004 US Presidential Elections

Results by state

2004 US Presidential Elections

Results by state

Population cartogram

2004 US Presidential Elections

Results by county

Population cartogram
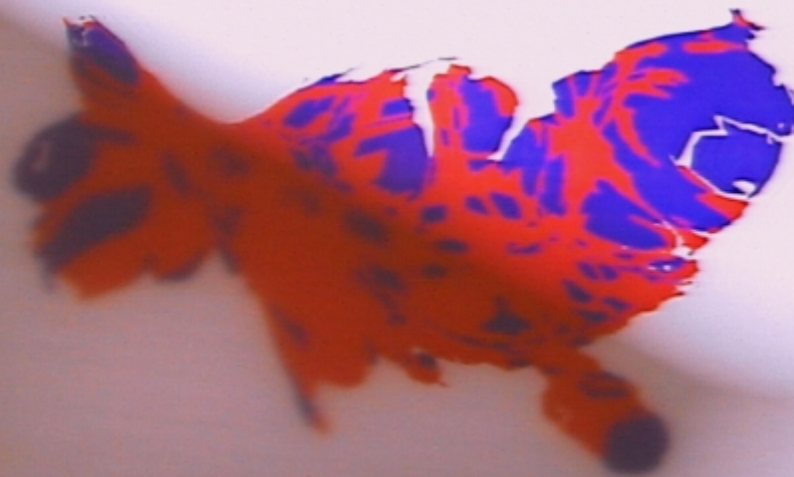
2004 US Presidential Elections
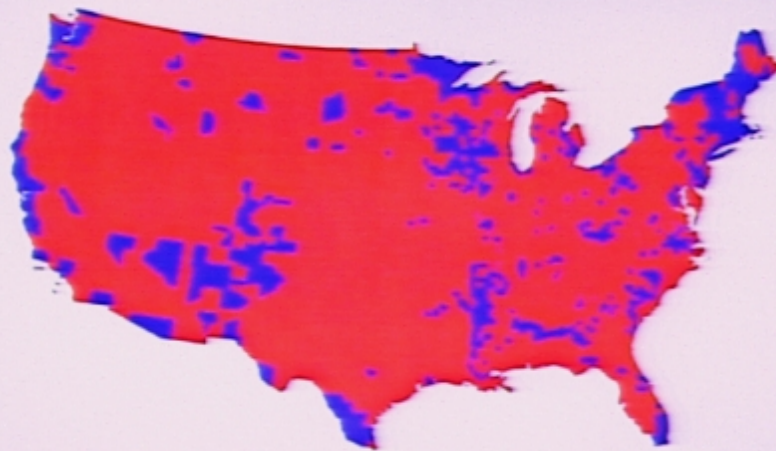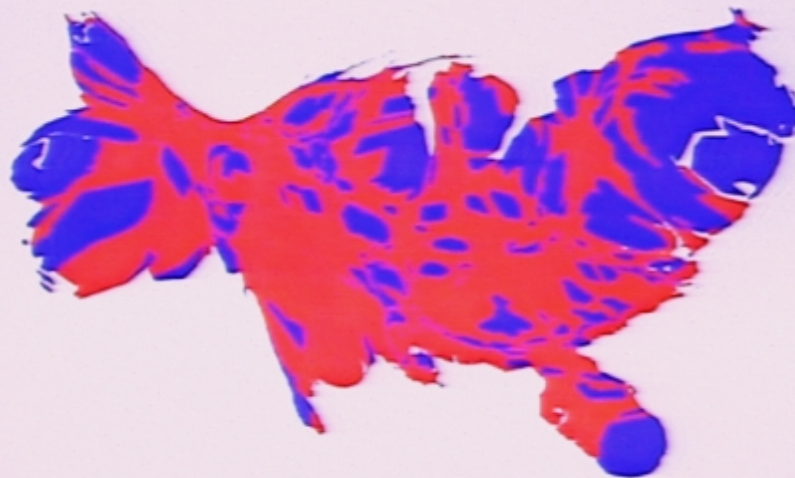
Results by county

Population cartogram

2004 US Presidential Elections

Results by county
red > 70% R;   blue >70% D
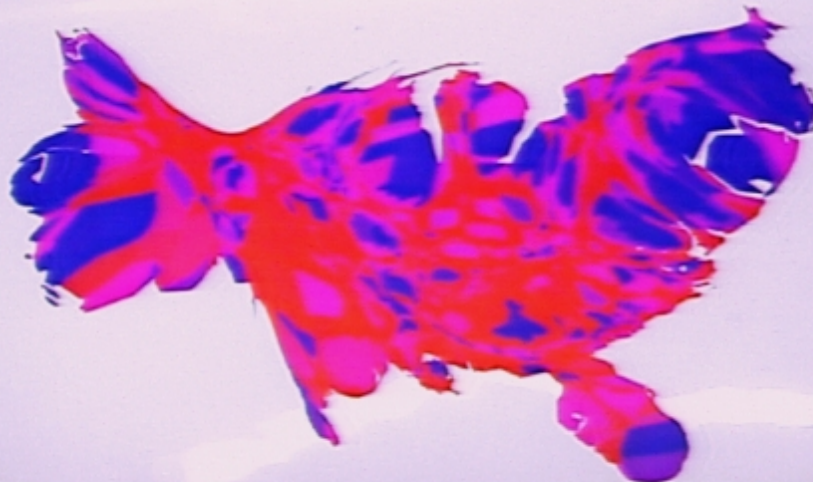
2004 Presidential Election
Purple America
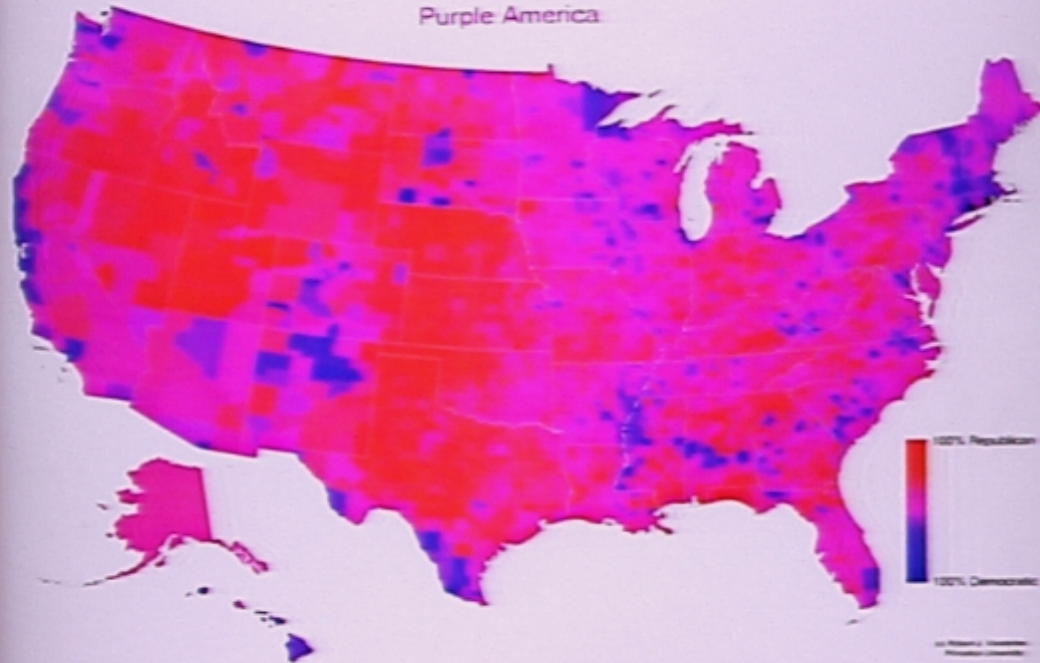
Election 2004 Results

Robert J. Vanderbei

2004 US
Presidential Elections
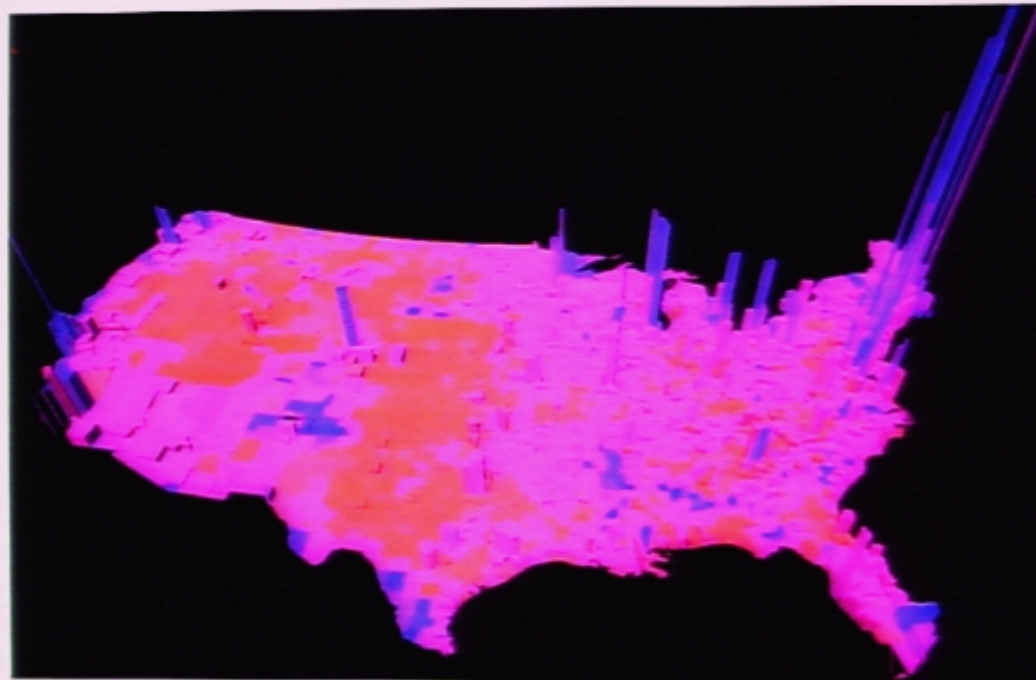
Height ~ population density
Volume ~ total number of votes

2004 US
Presidential Elections

Height ~ population density
Volume ~ total number of votes

Qualified majority voting:

A decision of a voting body is taken, if it is approved by a qualified majority

Examples:

- Parliaments,
- Bodies of international organisations,
- **European Council,**
- ....

Observation

The voting power needs not to be proportional to the number of votes:

Example

Shareholder A has **52%** of stocks,
~~Shareholder B has~~ **48%** of stocks

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Shareholder A has **52%** of stocks,
Shareholder B has **48%** of stocks,
Decisions are taken with 50% majority...

How to define the *voting power* of a given party/country/shareholder/voter?

...

Lionel Sharples Penrose (1898-1972)

a British geneticist, psychiatrist, mathematician and chess theorist.

3

Theory of **Penrose** (1946):

A priori **voting power** of **X** =

 probability *that a vote cast by* **X** *in a hypothetical ballot will* **be decisive**:

winning coalition would fail to satisfy the qualified majority condition without **X** .

(Various assumptions concerning voting and probability measures lead to various indeces defining the **voting power**:

- Shapley-Shubik   (1954)
- **Banzhaf**   **(1965) (Penrose)**
- Coleman   (1971)
- Deegan-Packel   (1979)
- Hoede-Bakker   (1982)
- Holler   (1982)
- ...

How to compute the Banzhaf index ?

(*Banzhaf*, 1965): number of players $\quad\quad\quad\quad\quad$ n

\# of coalitions $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ 2^n

\# of winning coalitions $\quad\quad\quad\quad\quad\quad\quad\quad$ w

\# of coalitions with i-th player $\quad\quad\quad\quad\quad$ 2^{n-1}

\# of wining coalitions with i-th player X_i $\quad$ w_i

\# of coalitions, for which the vote of X_i is critical

$$c_i := w_i - (w - w_i) = 2 \cdot w_i - w$$

Banzhaf index $= c_i / 2^{n-1}$

probability that vote of X_i will be decisive

Penrose-Banzhaf index (normalised)

$$\beta_i = \boxed{\dfrac{c_i}{\sum\limits_{i=1}^{n} c_i}}$$

(*Penrose*, 1946): probability, that player X_i is going to winn

$$p_i = (1 + \beta_i)/2$$

Example

Council of Ministers of **European Economic Community** 1958-1972

# of countries:	$n = 6$
sum of all votes (weights):	$S = 17$
quota:	$q = 12$
# of coalitions	$T = 2^6 = 64$
# of coalitions with state X	32
# of winning coalitions:	$w = 14$

State	votes	Winning coal. with X	Winning coal. Without	Diffe-rence	Banzhaf index	Banzhaf Normalis. Index
		w_i	$W - w_i$	c_i	$c_i/2^{n-1}$	β_i
Germany	4	12	2	10	5/16	$5/21 = 0.24$
France	4	12	2	10	5/16	$5/21 = 0.24$
Italy	4	12	2	10	5/16	$5/21 = 0.24$
Holland	2	10	4	6	3/16	$3/21 = 0.14$
Belgium	2	10	4	6	3/16	$3/21 = 0.14$
Luxemb.	1	7	7	0	0	0

Efficiency Index of Coleman:
$$A = w/T = 7/32 \approx 0.22$$

Voting in the European Council

Council adopts a piece of legislation if:

1) Current system (Treaty of Nice)

321 'votes' are distributed among 25 member states, e.g.:

D, F, I, GB – **29** weights („votes")
E, PL – **27** weights („votes"), etc.

a) Sum of weights of Member States voting in favour exceeds **232** (**~72% of 321**);

b) Member States represent at least **62%** of the total population of the Union;

c) Number of States voting 'yes' constitues a standard majority (at least **13** out of **25**).

2) Future system (European Constitution) – 'double majority'

b) Member States represent at least **65%** of the total population of the Union;

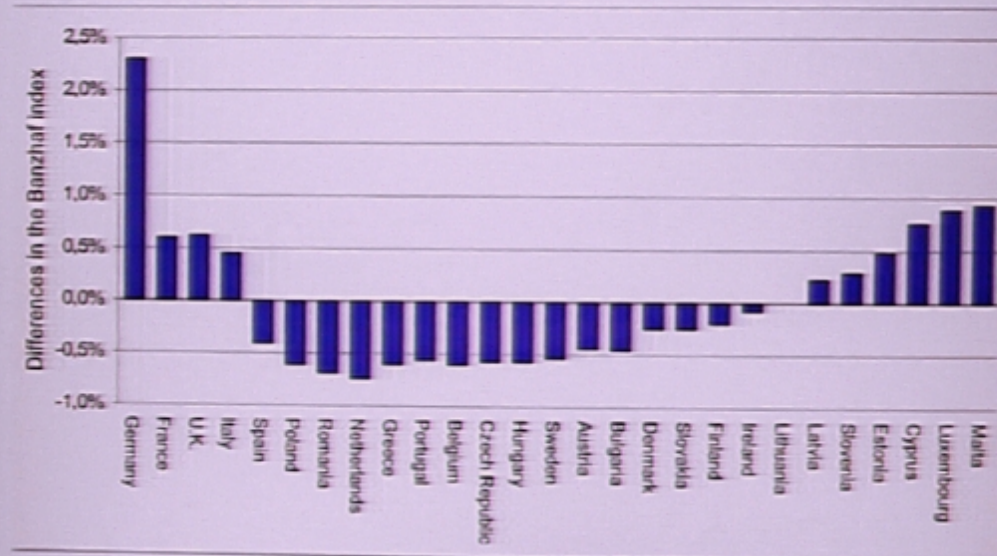c) Number of States voting 'yes' satisfies* **55%** majority (at least **15** out of **25**).

* **72%** in exceptional circumstances
** blocking minority must include at least **4** members

Voting Power in *European Council*:

The difference of the voting power
(**Penrose**-**Banzhaf index**)

European Constitution
versus
System of Nice

(...) if two votings were required for every decision, one on a *per capita* basis and the other upon the basis of a **single vote** for **each country**, the system would be inaccurate in that it would tend to favour large countries.

(L. S. Penrose, 1952)

A voting system, in which weights of a state is proportional to its population is not representative !

Square root law of Penrose
(Penrose, 1946)

Each **citizen** of the Union has the same influence on the decisions taken by the **Council**, if number of votes of a state is proportional to **square root** of its population

1. Voting power **W** of a citizen in indirect voting equals to the product of his voting power **L** in his country times the voting power β of his Minister in the Council.
2. Voting power **L** of a citizen in a country with population **N** is proportional to $1/\sqrt{N}$
   so scales the probability that all voives split into two equal parts, so his vote will be decisive,

Bernoulli scheme + Stirling expansion

$$P_k = p^k q^{1-k} \binom{N}{k}$$

$$P_k = \left(\tfrac{1}{2}\right)^{N/2} \left(\tfrac{1}{2}\right)^{N/2} \frac{N!}{(N/2)!(N/2)!} \approx$$

$$\approx \frac{1}{2^N} \frac{(N/e)^N \sqrt{2\pi N}}{\left[(N/2e)^{N/2}\sqrt{2\pi N/2}\right]^2} = \sqrt{\frac{2}{\pi N}} \sim \frac{1}{\sqrt{N}}$$

3. Hence voting power of every citizen is equal

$$W = \beta \cdot L \sim M/\sqrt{N} \sim 1$$

if weight M *of his country behaves as* $\sqrt{N}$

(additional assumption: voting weight M of every Member State is equal to its voting power β).

System of voting in European Council based on Penrose' law

3) One criterion system 'P-62'

- A State with population **N** gets the number of votes proportional to **sqrt(N)**

- Council takes a decision if the sum of votes in favour exceeds **62%** of all votes

(this quota optimizes representativeness

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One criterion voting system P-62

State	Popu-lation [mln]	Square root	Weight [%]	Power index β [%]
Germany	82,54	9,09	10,36	10,35
France	59,63	7,72	8,81	8,82
UK	59,09	7,69	8,77	8,78
Italy	57,07	7,55	8,62	8,63
Spain	40,68	6,38	7,27	7,28
Poland	38,21	6,18	7,05	7,06
Holland	16,19	4,02	4,59	4,59
Greece	11,02	3,32	3,79	3,79
Portugal	10,41	3,23	3,68	3,68
Belgium	10,34	3,22	3,67	3,67
Czech Rep.	10,20	3,19	3,64	3,64
Hungary	10,15	3,19	3,63	3,63
Sweden	8,94	2,99	3,41	3,41
Austria	8,06	2,84	3,24	3,24
Denmark	5,38	2,32	2,65	2,65
Slovakia	5,38	2,32	2,65	2,65
Finland	5,21	2,28	2,60	2,60
Irland	3,96	1,99	2,27	2,27
Lithuania	3,46	1,86	2,12	2,12
Latvia	2,33	1,53	1,74	1,74
Slovenia	2,00	1,41	1,61	1,61
Estonia	1,36	1,17	1,33	1,33
Cyprus	0,80	0,89	1,02	1,02
Luxembourg	0,45	0,67	0,76	0,76
Malta				

How to choose the optimal quota R_{opt}?

$$\frac{\text{Banzhaf power index}}{\text{voting weight}}$$
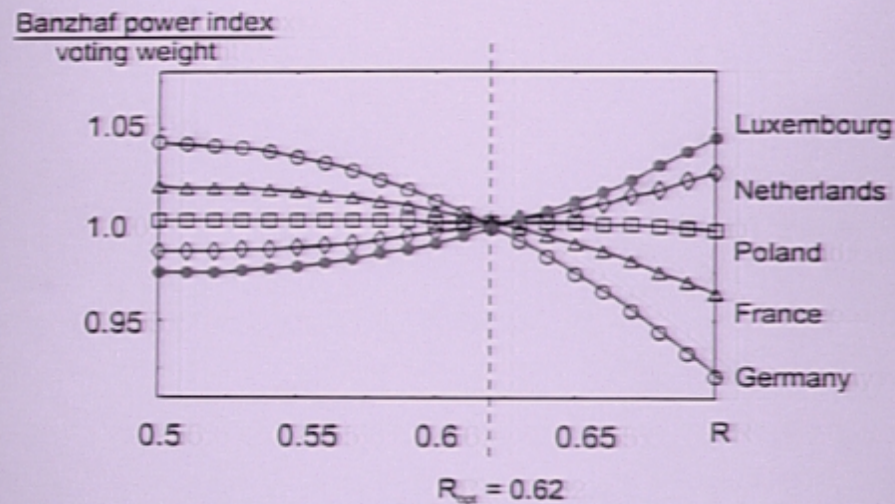
Ratio of **voting power** (Penrose-Banzhaf index) to **voting weight** as a function of the **quota** R for five exemplary states of EU-25:

Luxembourg, the Netherlands, Poland, United Kingdom, and Germany;

all functions cross near the critical point $R_{opt} = 62\%$.
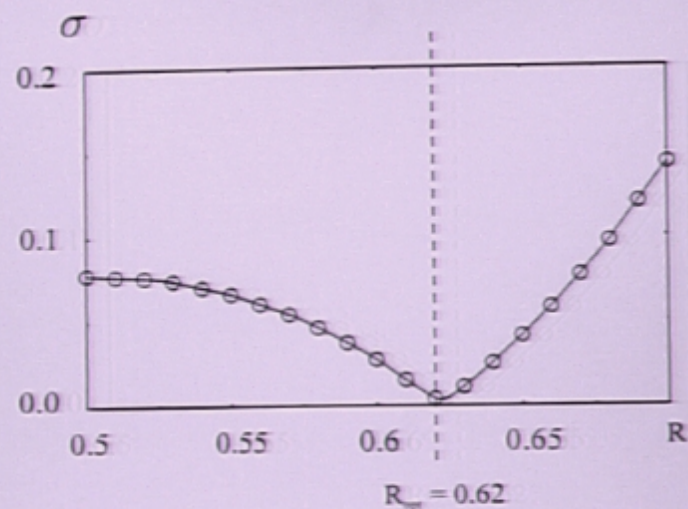
Square root of the sum of square residuals σ

$$\sigma^2 = \Sigma_i (\beta_i - \text{sqrt}\{N_i\})^2$$

between PBI values and voting weights (for the entire EU-25) as a function of the threshold R

Minimal deviation singles out a critical quota

$R_{opt} = 62\%$,

at which voting power and weights coincide, so the system becomes optimally representative.

Statistical approach

Optimal quota for 'unions' of T people living in M states

of random populations $N_i = T\, x_i$ where random numbers x_i are distributed according to:
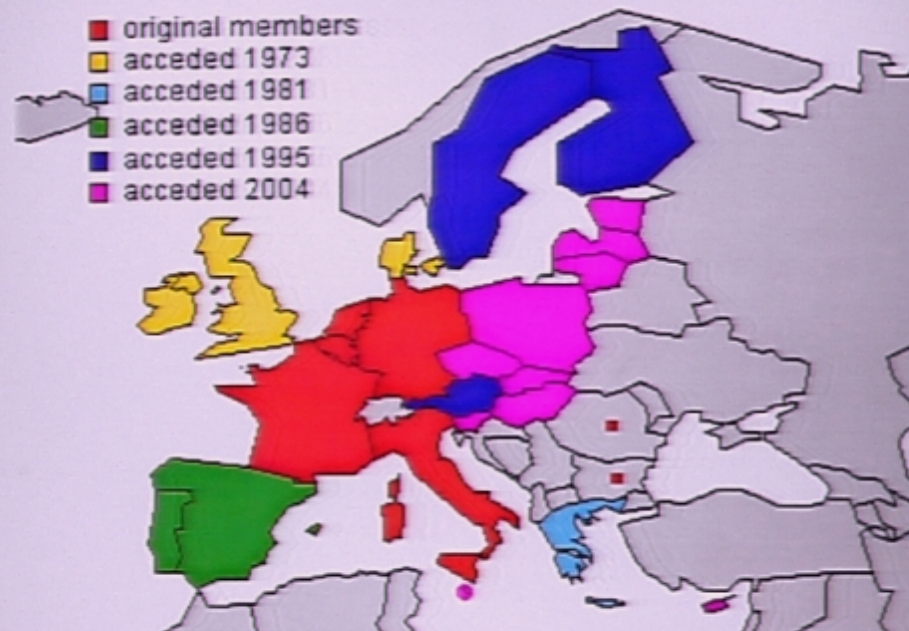
a) uniform measure on the simplex

b) Fisher statistical measure

$$P(x_i) \sim (x_i)^{1/2} \quad \text{and} \quad x_i \geq 0,\ \sum_{i=1}^{M} x_i = 1$$

Average optimal quota

M	10	12	14	16	18	20	22	24	26
$R_{opt}(U)$ (in %)	65.5	65.2	63.6	63.2	62.9	62.2	61.7	61.2	60.6
$R_{opt}(F)$ (in %)	66.0	65.8	64.6	64.4	63.4	63.1	62.6	62.0	61.4

In the limit $M \to \infty$ R_{opt} tends to **1**

EU-M	6	9	10	12	15	25	27
R_{opt}	73.0%	67.4%	65.5%	64.5%	64.4%	62,0%	61.4%

Value of the critical quota R_{opt} as a function of the number M of members of the EU.

Conclusions:

1) Voting system adopted by the European Constitution is not representative.

2) The 'double criterion' system is favourable for small and large states at the expense of all European states with intermediate population.

3) The system **P-62** based on the law of Penrose is simple, representative, efficient, transparent and modifiable

4) Numerical analysis shows that the optimal quota R_{opt} decreases with the number **M** of states in the Union, (but is almost independent on the distribution of population).

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