Title: A Relational Formulation of Quantum Theory

Date: Jun 08, 2005 04:00 AM

URL: http://pirsa.org/05060062

Abstract:

A Relational Formulation of Quantum Theory

David Poulin

School of Physical Sciences, The University of Queensland

Goal

- Isolate the chief physical insight of quantum theory and general relativity, explore their consequences on simple models, and then try to generalize.

D. Poulin, quant-ph/0505081 (2005)

Goal

- Isolate the chief physical insight of quantum theory and general relativity, explore their consequences on simple models, and then try to generalize.
- We will assume that...
  - QM sets a mathematical framework to describe physical systems: Hilbert space, unitary representations, etc.
  - GR says that physical descriptions should be background independent.

D. Poulin, quant-ph/0505081 (2005)

Goal

- Isolate the chief physical insight of quantum theory and general relativity, explore their consequences on simple models, and then try to generalize.
- We will assume that...
  - QM sets a mathematical framework to describe physical systems: Hilbert space, unitary representations, etc.
  - GR says that physical descriptions should be background independent.
- We will also heavily rely on a Bayesian approach to quantum mechanics: Quantum states represent our knowledge about physical systems.

D. Poulin, quant-ph/0505081 (2005)

Goal

- To every "orthodox" physical description, we apply the following four rules:
  1. Treat everything quantum mechanically.
  2. Use Hamiltonians with appropriate symmetries.
  3. Introduce equivalence classes between quantum states related by an element of the symmetry group.
  4. Interpret diagonal entries of density operators as probability distributions.

Goal

- To every "orthodox" physical description, we apply the following four rules:
  1. Treat everything quantum mechanically.
  2. Use Hamiltonians with appropriate symmetries.
  3. Introduce equivalence classes between quantum states related by an element of the symmetry group.
  4. Interpret diagonal entries of density operators as probability distributions.
- In appropriate "macroscopic" limits, this description is equivalent to the orthodox description.
- Away from this limit, the relational description leads to new predictions.
  - The orthodox description is an approximation to the fundamental relational description.

Goal

- There is nothing really innovative about these rules…

Goal

- There is nothing really innovative about these rules...
- We will apply them with lots of zeal to a simple model.
- We will get a fully relational theory, that we can easily interpret.
- We will investigate the features of this theory.
  - New physical phenomenon.

Goal

- There is nothing really innovative about these rules…
- We will apply them with lots of zeal to a simple model.
- We will get a fully relational theory, that we can easily interpret.
- We will investigate the features of this theory.
  - New physical phenomenon.
  - Compare with more sophisticated relational theories: “experimental quantum gravity”.

Outline

- Give orthodox description of a simple quantum mechanical system.
- Gradually apply our rules to arrive at a fully relational description.
  - Measurements.
  - Dynamics.
  - Time.
- Discussion
  - Relational time.
  - Fundamental decoherence.
  - Spin networks.
  - Connexion to other programs.
- Summary

Outline

- Give orthodox description of a simple quantum mechanical system.
- Gradually apply our rules to arrive at a fully relational description.
  - Measurements.
  - Dynamics.
  - Time.
- Discussion
  - Relational time.
  - Fundamental decoherence.
  - Spin networks.
  - Connexion to other programs.
- Summary

Orthodox description

Spin-$\frac{1}{2}$ particle $\mathcal{S}$ immersed in a magnetic field.

- Choose $\hat{x}$ such that $\vec{B} = B\hat{x}$.
- Hamiltonian $H^{\mathcal{S}} = -B\sigma_x^{\mathcal{S}}$.
- System's initial state $|\psi(0)\rangle^{\mathcal{S}} = \alpha|\uparrow\rangle^{\mathcal{S}} + \beta|\downarrow\rangle^{\mathcal{S}}$ in σ_z basis.
  - At time t,

$$\begin{aligned} |\psi(t)\rangle^{\mathcal{S}} &= \alpha(t)|\uparrow\rangle^{\mathcal{S}} + \beta(t)|\downarrow\rangle^{\mathcal{S}}, \\ \alpha(t) &= \alpha\cos(Bt/2) + i\beta\sin(Bt/2) \\ \beta(t) &= i\alpha\sin(Bt/2) + \beta\cos(Bt/2). \end{aligned}$$

Orthodox description

To make a measurement at time τ, we need a measurement apparatus $\mathcal{A}$:

Orthodox description

To make a measurement at time τ, we need a measurement apparatus $\mathcal{A}$:

- Initialize it in state $(|\uparrow\rangle^{\mathcal{A}} + |\downarrow\rangle^{\mathcal{A}})/\sqrt{2}$.
- Coupling $H^{\mathcal{SA}}(t) = -g\delta(t-\tau)\sigma_z^{\mathcal{S}} \otimes \sigma_y^{\mathcal{A}}$ with $g = \pi/2$.
  - At time τ_+ immediately after τ, $\mathcal{S}$ and $\mathcal{A}$ are correlated:

$$|\Psi(\tau_+)\rangle^{\mathcal{SA}} = \alpha(\tau)|\uparrow\rangle^{\mathcal{S}} \otimes |\uparrow\rangle^{\mathcal{A}} + \beta(\tau)|\downarrow\rangle^{\mathcal{S}} \otimes |\downarrow\rangle^{\mathcal{A}}.$$

Orthodox description

To make a measurement at time τ, we need a measurement apparatus $\mathcal{A}$:

- Initialize it in state $(|\uparrow\rangle^{\mathcal{A}} + |\downarrow\rangle^{\mathcal{A}})/\sqrt{2}$.
- Coupling $H^{\mathcal{SA}}(t) = -g\delta(t-\tau)\sigma_z^{\mathcal{S}} \otimes \sigma_y^{\mathcal{A}}$ with $g = \pi/2$.
  - At time τ_+ immediately after τ, $\mathcal{S}$ and $\mathcal{A}$ are correlated:

$$|\Psi(\tau_+)\rangle^{\mathcal{SA}} = \alpha(\tau)|\uparrow\rangle^{\mathcal{S}} \otimes |\uparrow\rangle^{\mathcal{A}} + \beta(\tau)|\downarrow\rangle^{\mathcal{S}} \otimes |\downarrow\rangle^{\mathcal{A}}.$$

- $\mathcal{A}$ is “classical", so it collapses to either $|\uparrow\rangle^{\mathcal{A}}$ or $|\downarrow\rangle^{\mathcal{A}}$:

$$\rho^{\mathcal{SA}} = |\alpha(\tau)|^2 |\uparrow\rangle\langle\uparrow|^{\mathcal{S}} \otimes |\uparrow\rangle\langle\uparrow|^{\mathcal{A}} + |\beta(\tau)|^2 |\downarrow\rangle\langle\downarrow|^{\mathcal{S}} \otimes |\downarrow\rangle\langle\downarrow|^{\mathcal{A}}.$$

- $\mathcal{S}$ and $\mathcal{A}$ are either both in up or both in down state, with respective probabilities $|\alpha(\tau)|^2$ and $|\beta(\tau)|^2$.

Orthodox description

This description is not background independent.

- Both $\vec{B}$ and $\mathcal{A}$ are treated classically, and refer to (or define) an external coordinate system.
  - Magnetic field $\vec{B} \propto \hat{x}$.
  - The observable $\sigma_z^{\mathcal{A}}$ of $\mathcal{A}$ is superselected.
  - The coupling $H^{\mathcal{SA}}(t)$ is neither time independent or rotationally invariant.

... so this is a pretty good model to illustrate our approach.

Some notation

- Systems are labeled by a calligraphic capital letter, e.g. $\mathcal{A}$.

Some notation

- Systems are labeled by a calligraphic capital letter, e.g. $\mathcal{A}$.

Some notation

- Systems are labeled by a calligraphic capital letter, e.g. $\mathcal{A}$.
- Operators, states, and Hilbert spaces associated to this particle have the letter as a superscript.
- Quantum number associated to total angular momentum is the same capital letter in roman font.

Some notation

- Systems are labeled by a calligraphic capital letter, e.g. $\mathcal{A}$.
- Operators, states, and Hilbert spaces associated to this particle have the letter as a superscript.
- Quantum number associated to total angular momentum is the same capital letter in roman font.
- Quantum number associated to the angular momentum along z is the same lower case letter.
- For $\mathcal{A}$, a spin-A particle, this gives
  - $(J^{\mathcal{A}})^2|A,a\rangle^{\mathcal{A}} = A(A+1)|A,a\rangle^{\mathcal{A}}$
  - $J_z^{\mathcal{A}}|A,a\rangle^{\mathcal{A}} = a|A,a\rangle^{\mathcal{A}}$,
  - $|A,a\rangle^{\mathcal{A}} \in \mathcal{H}^{\mathcal{A}} = \mathbb{C}^{2A+1}$.

Measurement

- To measure angular momentum, we need a gyroscope $\mathcal{G}$.

Measurement

- To measure angular momentum, we need a gyroscope $\mathcal{G}$.
- Rule 1 says that it should be quantum mechanical, so to recover the orthodox result, we choose it to be in a coherent state $|G, g = G\rangle^{\mathcal{G}}$. We abbreviate this $|G, G\rangle^{\mathcal{G}}$.

Measurement

- To measure angular momentum, we need a gyroscope $\mathcal{G}$.
- Rule 1 says that it should be quantum mechanical, so to recover the orthodox result, we choose it to be in a coherent state $|G, g = G\rangle^{\mathcal{G}}$. We abbreviate this $|G, G\rangle^{\mathcal{G}}$.
- Define the TPCP map $\mathcal{E}^{\mathcal{SG}} : \mathcal{B}(\mathcal{H}^{\mathcal{SG}}) \rightarrow \mathcal{B}(\mathcal{H}^{\mathcal{SG}})$:

$$\mathcal{E}^{\mathcal{SG}}(\rho) = \int_{SO(3)} R^{\mathcal{SG}}(\Omega)\rho R^{\mathcal{SG}}(\Omega)^{\dagger} d\Omega,$$

  - $R^{\mathcal{SG}} = R^{\mathcal{S}} \otimes R^{\mathcal{G}}$ is the unitary representation of the rotation group on the pair $\mathcal{S} - \mathcal{G}$.

Measurement

- To measure angular momentum, we need a gyroscope $\mathcal{G}$.
- Rule 1 says that it should be quantum mechanical, so to recover the orthodox result, we choose it to be in a coherent state $|G, g = G\rangle^{\mathcal{G}}$. We abbreviate this $|G, G\rangle^{\mathcal{G}}$.
- Define the TPCP map $\mathcal{E}^{\mathcal{SG}} : \mathcal{B}(\mathcal{H}^{\mathcal{SG}}) \to \mathcal{B}(\mathcal{H}^{\mathcal{SG}})$:

$$\mathcal{E}^{\mathcal{SG}}(\rho) = \int_{SO(3)} R^{\mathcal{SG}}(\Omega)\rho R^{\mathcal{SG}}(\Omega)^{\dagger} d\Omega,$$

  - $R^{\mathcal{SG}} = R^{\mathcal{S}} \otimes R^{\mathcal{G}}$ is the unitary representation of the rotation group on the pair $\mathcal{S} - \mathcal{G}$.
  - $d\Omega$ is the invariant Haar measure on $SO(3)$.
  - For all $\rho^{\mathcal{SG}} \in \mathcal{B}(\mathcal{H}^{\mathcal{SG}})$, $\mathcal{E}(\rho^{\mathcal{SG}})$ is rotationally invariant.
  - Straightforward generalization to arbitrary number of systems.

Measurement

How do we justify the group average?

Measurement

How do we justify the group average?

- Denote $\rho_{|R}$ a quantum state expressed in a reference frame R.
    - We should think of $\rho_{|R}$ as the state given a reference frame R, just like $p(a|b)$ denotes the probability of a given the value of b.

Measurement

How do we justify the group average?

- Denote $\rho_{|R}$ a quantum state expressed in a reference frame R.
  - We should think of $\rho_{|R}$ as the state given a reference frame R, just like $p(a|b)$ denotes the probability of a given the value of b.
- In an other reference frame R', the same physical state is $\rho_{|R'} = R(\Omega)\rho_{|R}R(\Omega)^\dagger$ where Ω is the group element relating R to R'.

Measurement

How do we justify the group average?

- Denote $\rho_{|R}$ a quantum state expressed in a reference frame R.
  - We should think of $\rho_{|R}$ as the state given a reference frame R, just like $p(a|b)$ denotes the probability of a given the value of b.
- In an other reference frame R', the same physical state is $\rho_{|R'} = R(\Omega)\rho_{|R}R(\Omega)^\dagger$ where Ω is the group element relating R to R'.
- But what if we have no information about R, say because it doesn't exists?
  - Without knowledge of b, the probability of a is $p(a) = \sum_b p(a|b)p(b)$.
  - The group averaging procedure $\mathcal{E}$ is the exact analogue of this rule.

Measurement

How do we apply this map in practice?

Measurement

How do we apply this map in practice?

- $R^{\mathcal{SG}}$ is generated by the total angular momentum operator $\vec{J}^{\mathcal{SG}} = \vec{\sigma}^{\mathcal{S}} + \vec{J}^{\mathcal{G}}$.
  - Above, $\vec{\sigma}^{\mathcal{S}} = (\sigma_x^{\mathcal{S}}, \sigma_y^{\mathcal{S}}, \sigma_z^{\mathcal{S}})$ and $\vec{J}^{\mathcal{G}} = (J_x^{\mathcal{G}}, J_y^{\mathcal{G}}, J_z^{\mathcal{G}})$ are the system and gyroscope angular momentum operators.
- So it will be convenient to express the state of $\mathcal{S}$ and $\mathcal{G}$ $|\Psi\rangle^{\mathcal{SG}} = (\alpha|\uparrow\rangle^{\mathcal{S}} + \beta|\downarrow\rangle^{\mathcal{S}}) \otimes |G, G\rangle^{\mathcal{G}}$ in terms of $(J^{\mathcal{SG}})^2$ and $J_z^{\mathcal{SG}}$:

$$\alpha|G+\tfrac{1}{2}, G+\tfrac{1}{2}; \tfrac{1}{2}; G\rangle + \frac{\beta}{\sqrt{2G+1}}|G+\tfrac{1}{2}, G-\tfrac{1}{2}; \tfrac{1}{2}; G\rangle + \frac{\beta\sqrt{2G}}{\sqrt{2G+1}}|G-\tfrac{1}{2}, G-\tfrac{1}{2}; \tfrac{1}{2}; G\rangle.$$

Measurement

How do we apply this map in practice?

- $R^{\mathcal{SG}}$ is generated by the total angular momentum operator $\vec{J}^{\mathcal{SG}} = \vec{\sigma}^{\mathcal{S}} + \vec{J}^{\mathcal{G}}$.
  - Above, $\vec{\sigma}^{\mathcal{S}} = (\sigma_x^{\mathcal{S}}, \sigma_y^{\mathcal{S}}, \sigma_z^{\mathcal{S}})$ and $\vec{J}^{\mathcal{G}} = (J_x^{\mathcal{G}}, J_y^{\mathcal{G}}, J_z^{\mathcal{G}})$ are the system and gyroscope angular momentum operators.
- So it will be convenient to express the state of $\mathcal{S}$ and $\mathcal{G}$ $|\Psi\rangle^{\mathcal{SG}} = (\alpha|\uparrow\rangle^{\mathcal{S}} + \beta|\downarrow\rangle^{\mathcal{S}}) \otimes |G, G\rangle^{\mathcal{G}}$ in terms of $(J^{\mathcal{SG}})^2$ and $J_z^{\mathcal{SG}}$:

$$\alpha|G+\tfrac{1}{2},G+\tfrac{1}{2};\tfrac{1}{2};G\rangle+\frac{\beta}{\sqrt{2G+1}}|G+\tfrac{1}{2},G-\tfrac{1}{2};\tfrac{1}{2};G\rangle+\frac{\beta\sqrt{2G}}{\sqrt{2G+1}}|G-\tfrac{1}{2},G-\tfrac{1}{2};\tfrac{1}{2};G\rangle.$$

  - Quantum numbers: $(J^{\mathcal{SG}})^2$, $J_z^{\mathcal{SG}}$, $(\sigma^{\mathcal{S}})^2$, and $(J^{\mathcal{G}})^2$.
  - $(J^{\mathcal{SG}})^2$, $(\sigma^{\mathcal{S}})^2$, and $(J^{\mathcal{G}})^2$ are rotationally invariant as they commute with the generator $\vec{J}^{\mathcal{SG}}$.

Measurement

Since $J_z^{S\mathcal{G}}$ is the only operator depending on a coordinate system, the effect of $\mathcal{E}^{SS}$ can be readily anticipated: it randomizes the associated quantum number and leaves the other ones unchanged

$$\left[|\alpha|^2 + \frac{|\beta|^2}{2G+1}\right] |G+\tfrac{1}{2};\tfrac{1}{2};G\rangle\langle G+\tfrac{1}{2};\tfrac{1}{2};G| \otimes \frac{\mathbb{1}_{2G+2}}{2G+2} + \frac{2G|\beta|^2}{2G+G} |G-\tfrac{1}{2};\tfrac{1}{2};G\rangle\langle G-\tfrac{1}{2};\tfrac{1}{2};G| \otimes \frac{\mathbb{1}_{2G}}{2G},$$

Measurement

How do we apply this map in practice?

- $R^{\mathcal{SG}}$ is generated by the total angular momentum operator $\vec{J}^{\mathcal{SG}} = \vec{\sigma}^{\mathcal{S}} + \vec{J}^{\mathcal{G}}$.
  - Above, $\vec{\sigma}^{\mathcal{S}} = (\sigma_x^{\mathcal{S}}, \sigma_y^{\mathcal{S}}, \sigma_z^{\mathcal{S}})$ and $\vec{J}^{\mathcal{G}} = (J_x^{\mathcal{G}}, J_y^{\mathcal{G}}, J_z^{\mathcal{G}})$ are the system and gyroscope angular momentum operators.
- So it will be convenient to express the state of $\mathcal{S}$ and $\mathcal{G}$ $|\Psi\rangle^{\mathcal{SG}} = (\alpha|\uparrow\rangle^{\mathcal{S}} + \beta|\downarrow\rangle^{\mathcal{S}}) \otimes |G, G\rangle^{\mathcal{G}}$ in terms of $(J^{\mathcal{SG}})^2$ and $J_z^{\mathcal{SG}}$:

$$\alpha|G+\tfrac{1}{2}, G+\tfrac{1}{2}; \tfrac{1}{2}; G\rangle + \frac{\beta}{\sqrt{2G+1}}|G+\tfrac{1}{2}, G-\tfrac{1}{2}; \tfrac{1}{2}; G\rangle + \frac{\beta\sqrt{2G}}{\sqrt{2G+1}}|G-\tfrac{1}{2}, G-\tfrac{1}{2}; \tfrac{1}{2}; G\rangle.$$

  - Quantum numbers: $(J^{\mathcal{SG}})^2$, $J_z^{\mathcal{SG}}$, $(\sigma^{\mathcal{S}})^2$, and $(J^{\mathcal{G}})^2$.
  - $(J^{\mathcal{SG}})^2$, $(\sigma^{\mathcal{S}})^2$, and $(J^{\mathcal{G}})^2$ are rotationally invariant as they commute with the generator $\vec{J}^{\mathcal{SG}}$.

Measurement

Since $J_z^{\mathcal{SG}}$ is the only operator depending on a coordinate system, the effect of $\mathcal{E}^{SS}$ can be readily anticipated: it randomizes the associated quantum number and leaves the other ones unchanged

$$\left[|\alpha|^2 + \frac{|\beta|^2}{2G+1}\right] |G+\tfrac{1}{2};\tfrac{1}{2};G\rangle\langle G+\tfrac{1}{2};\tfrac{1}{2};G| \otimes \frac{\mathbb{1}_{2G+2}}{2G+2} + \frac{2G|\beta|^2}{2G+G} |G-\tfrac{1}{2};\tfrac{1}{2};G\rangle\langle G-\tfrac{1}{2};\tfrac{1}{2};G| \otimes \frac{\mathbb{1}_{2G}}{2G},$$

Measurement

Since J_z^{SG} is the only operator depending on a coordinate system, the effect of $\mathcal{E}^{SS}$ can be readily anticipated: it randomizes the associated quantum number and leaves the other ones unchanged

$$\left[|\alpha|^2 + \frac{|\beta|^2}{2G+1}\right] |G+\tfrac{1}{2};\tfrac{1}{2};G\rangle\langle G+\tfrac{1}{2};\tfrac{1}{2};G| \otimes \frac{\mathbb{1}_{2G+2}}{2G+2} + \frac{2G|\beta|^2}{2G+G}|G-\tfrac{1}{2};\tfrac{1}{2};G\rangle\langle G-\tfrac{1}{2};\tfrac{1}{2};G| \otimes \frac{\mathbb{1}_{2G}}{2G},$$

- We can remove the quantum number associated to J_z^{SG} from the physical description as it is always in a maximally mixed state, and hence carries no information

$$\rho^{SG}_{\mathbf{physical}} = \left[|\alpha|^2 + \frac{|\beta|^2}{2G+1}\right] |G+\tfrac{1}{2};\tfrac{1}{2};G\rangle\langle G+\tfrac{1}{2};\tfrac{1}{2};G| + \frac{2G|\beta|^2}{2G+1}|G-\tfrac{1}{2};\tfrac{1}{2};G\rangle\langle G-\tfrac{1}{2};\tfrac{1}{2};G|.$$

Measurement

Since $J_z^{\mathcal{SG}}$ is the only operator depending on a coordinate system, the effect of $\mathcal{E}^{SS}$ can be readily anticipated: it randomizes the associated quantum number and leaves the other ones unchanged

$$\left[|\alpha|^2 + \frac{|\beta|^2}{2G+1}\right] |G+\tfrac{1}{2};\tfrac{1}{2};G\rangle\langle G+\tfrac{1}{2};\tfrac{1}{2};G| \otimes \frac{1\!\!1_{2G+2}}{2G+2} + \frac{2G|\beta|^2}{2G+G} |G-\tfrac{1}{2};\tfrac{1}{2};G\rangle\langle G-\tfrac{1}{2};\tfrac{1}{2};G| \otimes \frac{1\!\!1_{2G}}{2G},$$

- We can remove the quantum number associated to $J_z^{\mathcal{SG}}$ from the physical description as it is always in a maximally mixed state, and hence carries no information

$$\rho^{\mathcal{SG}}_{\mathbf{physical}} = \left[|\alpha|^2 + \frac{|\beta|^2}{2G+1}\right] |G+\tfrac{1}{2};\tfrac{1}{2};G\rangle\langle G+\tfrac{1}{2};\tfrac{1}{2};G| + \frac{2G|\beta|^2}{2G+1} |G-\tfrac{1}{2};\tfrac{1}{2};G\rangle\langle G-\tfrac{1}{2};\tfrac{1}{2};G|.$$

  - Rule 4 gives the desired interpretation.
  - When $G \to \infty$, we recover the orthodox result.
  - This description is fully relational.

Dynamics

There is no symmetric single particle Hamiltonian: dynamics results from interactions.

Dynamics

There is no symmetric single particle Hamiltonian: dynamics results from interactions.

- We need a magnet $\mathcal{M}$ pointing in the $\hat{x}$ direction:

$$|M, M\rangle_x^{\mathcal{M}} = \frac{1}{2^M} \sum_{m=-M}^{M} \binom{2M}{M+m}^{1/2} |M, m\rangle^{\mathcal{M}}.$$

Dynamics

- The solution to Schrödinger's equation of motion is

$$|\Psi(t)\rangle^{\mathcal{S}\mathcal{M}} = |M,M\rangle_x^{\mathcal{M}} \otimes |\psi(t)\rangle^{\mathcal{S}} + C(t)\left[\frac{1}{\sqrt{2M}}|M,M\rangle_x^{\mathcal{M}} \otimes |\downarrow\rangle^{\mathcal{S}} + |M,M-1\rangle_x^{\mathcal{M}} \otimes |\uparrow\rangle^{\mathcal{S}}\right]$$

  - $|\psi(t)\rangle^{\mathcal{S}}$ is the solution in the orthodox description with $B = \lambda(2M+1)$.
  - $C(t) = i\sqrt{M}2(\alpha-\beta)\sin(Bt/2)/(2M+1)$ vanishes as $M \to \infty$.

Dynamics

We reintroduce the gyroscope and apply the map $\mathcal{E}^{\mathcal{SMG}}$ to obtain

$$\rho_{\text{physical}}^{\mathcal{SMG}} \approx \frac{1}{2^{2M}} \sum_{n=-M}^{M-1} \binom{2M}{M+n} |\Psi_n(t)\rangle\langle\Psi_n(t)|^{\mathcal{SMG}}$$

Dynamics

We reintroduce the gyroscope and apply the map $\mathcal{E}^{\mathcal{SMG}}$ to obtain

$$\rho^{\mathcal{SMG}}_{\text{physical}} \approx \frac{1}{2^{2M}} \sum_{n=-M}^{M-1} \binom{2M}{M+n} |\Psi_n(t)\rangle\langle\Psi_n(t)|^{\mathcal{SMG}}$$

where

$$|\Psi_n(t)\rangle^{\mathcal{SMG}} = \alpha(t)|G+\tfrac{1}{2}+n; G+\tfrac{1}{2}\rangle^{\mathcal{SMG}} + \beta(t)\sqrt{\frac{M-n}{M+n+1}}|G+\tfrac{1}{2}+n; G-\tfrac{1}{2}\rangle^{\mathcal{SMG}}$$

refering to the rotationally invariant quantum numbers $(J^{\mathcal{SMG}})^2$ and $(J^{\mathcal{SG}})^2$.

- For clarity, the quantum numbers $(\sigma^{\mathcal{S}})^2 = S(S+1)$, $(J^{\mathcal{M}})^2 = M(M+1)$, and $(J^{\mathcal{G}})^2 = G(G+1)$ have been omitted.

Dynamics

We reintroduce the gyroscope and apply the map $\mathcal{E}^{\mathcal{SMG}}$ to obtain

$$\rho^{\mathcal{SMG}}_{\text{physical}} \approx \frac{1}{2^{2M}} \sum_{n=-M}^{M-1} \binom{2M}{M+n} |\Psi_n(t)\rangle\langle\Psi_n(t)|^{\mathcal{SMG}}$$

where

$$|\Psi_n(t)\rangle^{\mathcal{SMG}} = \alpha(t)|G+\tfrac{1}{2}+n; G+\tfrac{1}{2}\rangle^{\mathcal{SMG}} + \beta(t)\sqrt{\frac{M-n}{M+n+1}}|G+\tfrac{1}{2}+n; G-\tfrac{1}{2}\rangle^{\mathcal{SMG}}$$

refering to the rotationally invariant quantum numbers $(J^{\mathcal{SMG}})^2$ and $(J^{\mathcal{SG}})^2$.

- For clarity, the quantum numbers $(\sigma^{\mathcal{S}})^2 = S(S+1)$, $(J^{\mathcal{M}})^2 = M(M+1)$, and $(J^{\mathcal{G}})^2 = G(G+1)$ have been omitted.
  - The binomian distribution is peaked around $n = 0$, with fluctuations of size $\Delta n \sim \sqrt{M}$.

Dynamics

We reintroduce the gyroscope and apply the map $\mathcal{E}^{\mathcal{SMG}}$ to obtain

$$\rho^{\mathcal{SMG}}_{\text{physical}} \approx \frac{1}{2^{2M}} \sum_{n=-M}^{M-1} \binom{2M}{M+n} |\Psi_n(t)\rangle\langle\Psi_n(t)|^{\mathcal{SMG}}$$

where

$$|\Psi_n(t)\rangle^{\mathcal{SMG}} = \alpha(t)|G+\tfrac{1}{2}+n; G+\tfrac{1}{2}\rangle^{\mathcal{SMG}} + \beta(t)\sqrt{\frac{M-n}{M+n+1}}|G+\tfrac{1}{2}+n; G-\tfrac{1}{2}\rangle^{\mathcal{SMG}}$$

refering to the rotationally invariant quantum numbers $(J^{\mathcal{SMG}})^2$ and $(J^{\mathcal{SG}})^2$.

- For clarity, the quantum numbers $(\sigma^{\mathcal{S}})^2 = S(S+1)$, $(J^{\mathcal{M}})^2 = M(M+1)$, and $(J^{\mathcal{G}})^2 = G(G+1)$ have been omitted.
  - The binomian distribution is peaked around $n = 0$, with fluctuations of size $\Delta n \sim \sqrt{M}$.
  - In this range, the term under the square root is $1 + O(1/\sqrt{M})$.

Dynamics

Thus, with probability approaching one as $M \to \infty$,

$$|\Psi_n(t)\rangle \approx \alpha(t)|G+\tfrac{1}{2}+n; G+\tfrac{1}{2}\rangle^{\mathcal{SMG}} + \beta(t)|G+\tfrac{1}{2}+n; G-\tfrac{1}{2}\rangle^{\mathcal{SMG}}$$

for some random $n \in [-\sqrt{M}, \sqrt{M}]$.

Dynamics

Thus, with probability approaching one as $M \to \infty$,

$$|\Psi_n(t)\rangle \approx \alpha(t)|G+\tfrac{1}{2}+n; G+\tfrac{1}{2}\rangle^{\mathcal{SMG}} + \beta(t)|G+\tfrac{1}{2}+n; G-\tfrac{1}{2}\rangle^{\mathcal{SMG}}$$

for some random $n \in [-\sqrt{M}, \sqrt{M}]$.

- The state of $\mathcal{S}$ and $\mathcal{G}$ obtained from tracing out the magnet is

$$\rho^{\mathcal{SG}}_{\text{physical}} \approx |\alpha(t)|^2|G+\tfrac{1}{2}; G; \tfrac{1}{2}\rangle\langle G+\tfrac{1}{2}; G; \tfrac{1}{2}| + |\beta(t)|^2|G-\tfrac{1}{2}; G; \tfrac{1}{2}\rangle\langle G-\tfrac{1}{2}; G; \tfrac{1}{2}|.$$

Dynamics

Thus, with probability approaching one as $M \to \infty$,

$$|\Psi_n(t)\rangle \approx \alpha(t)|G + \tfrac{1}{2} + n; G + \tfrac{1}{2}\rangle^{\mathcal{SMG}} + \beta(t)|G + \tfrac{1}{2} + n; G - \tfrac{1}{2}\rangle^{\mathcal{SMG}}$$

for some random $n \in [-\sqrt{M}, \sqrt{M}]$.

- The state of $\mathcal{S}$ and $\mathcal{G}$ obtained from tracing out the magnet is

$$\rho^{\mathcal{SG}}_{\text{physical}} \approx |\alpha(t)|^2 |G + \tfrac{1}{2}; G; \tfrac{1}{2}\rangle\langle G + \tfrac{1}{2}; G; \tfrac{1}{2}| + |\beta(t)|^2 |G - \tfrac{1}{2}; G; \tfrac{1}{2}\rangle\langle G - \tfrac{1}{2}; G; \tfrac{1}{2}|.$$

  - Can be established through direct calculations.
  - Can use the fact that $[Tr_{\mathcal{B}}, \mathcal{E}^{\mathcal{AB}}] = 0$ when the symmetry group acts unitarilly on $\mathcal{B}$, combined with the result concerning measurement.

Time

Our description still uses an external time coordinate.

Dynamics

Thus, with probability approaching one as $M \to \infty$,

$$|\Psi_n(t)\rangle \approx \alpha(t)|G+\frac{1}{2}+n; G+\frac{1}{2}\rangle^{\mathcal{SMG}} + \beta(t)|G+\frac{1}{2}+n; G-\frac{1}{2}\rangle^{\mathcal{SMG}}$$

for some random $n \in [-\sqrt{M}, \sqrt{M}]$.

- The state of $\mathcal{S}$ and $\mathcal{G}$ obtained from tracing out the magnet is

$$\rho^{\mathcal{SG}}_{\text{physical}} \approx |\alpha(t)|^2 |G+\frac{1}{2}; G; \frac{1}{2}\rangle\langle G+\frac{1}{2}; G; \frac{1}{2}| + |\beta(t)|^2 |G-\frac{1}{2}; G; \frac{1}{2}\rangle\langle G-\frac{1}{2}; G; \frac{1}{2}|.$$

  - Can be established through direct calculations.
  - Can use the fact that $[Tr_{\mathcal{B}}, \mathcal{E}^{\mathcal{AB}}] = 0$ when the symmetry group acts unitarilly on $\mathcal{B}$, combined with the result concerning measurement.

Time

Our description still uses an external time coordinate.

Time

Our description still uses an external time coordinate.

- To keep tract of time, we need a clock C.
- Rule 1 says that it needs to be quantum mechanical $|C, C\rangle^{C}$.

Time

Our description still uses an external time coordinate.

- To keep tract of time, we need a clock $\mathcal{C}$.
- Rule 1 says that it needs to be quantum mechanical $|C, C\rangle^{\mathcal{C}}$.
- We need a magnet $\mathcal{N}$ to power this clock $|N, N\rangle_x^{\mathcal{N}}$.
  - Heisenberg coupling $H^{\mathcal{CN}} = -2\lambda \vec{J}^{\mathcal{C}} \cdot \vec{J}^{\mathcal{N}}$.
  - $\Lambda = M/N$ should be an integer greater than 1, so that the clock's period is longer than the system's period.

Time

This group average is given the same Bayesian justification as above.

Time

Our description still uses an external time coordinate.

- To keep tract of time, we need a clock $\mathcal{C}$.
- Rule 1 says that it needs to be quantum mechanical $|C, C\rangle^{\mathcal{C}}$.
- We need a magnet $\mathcal{N}$ to power this clock $|N, N\rangle_x^{\mathcal{N}}$.
  - Heisenberg coupling $H^{\mathcal{CN}} = -2\lambda \vec{J}^{\mathcal{C}} \cdot \vec{J}^{\mathcal{N}}$.
  - $\Lambda = M/N$ should be an integer greater than 1, so that the clock's period is longer than the system's period.
- We eliminate time using Rule 3 exactly as we did for orientation:

$$\mathcal{T}(\rho) = \frac{1}{T^{\mathcal{C}}} \int_0^{T^{\mathcal{C}}} U(t)\rho U(t)^\dagger dt.$$

Time

This group average is given the same Bayesian justification as above.

Time

This group average is given the same Bayesian justification as above.

- Denote $\rho_{|T}(t)$ a quantum state at time t, where the time refers to an external time coordinate system T.
  - We should think of $\rho_{|T}(t)$ as the state given a coordinate system T.

Time

This group average is given the same Bayesian justification as above.

- Denote $\rho_{|T}(t)$ a quantum state at time t, where the time refers to an external time coordinate system T.
  - We should think of $\rho_{|T}(t)$ as the state given a coordinate system T.
- In an other coordinate system T', the same physical state is $\rho_{|T'}(t) = e^{-iH\Delta}\rho_{|T}e^{iH\Delta}$ where Δ is the time translation relating T to T'.

Time

We apply the same procedure as above:

Time

We apply the same procedure as above:

- Solve equations of motion for $\mathcal{S}$, $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{N}$.
- Introduce gyroscope into the picture and perform rotation group average.
- Perform time average $\mathcal{T}$ and trace our magnets, or vice and versa.

Time

We apply the same procedure as above:

- Solve equations of motion for $\mathcal{S}$, $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{N}$.
- Introduce gyroscope into the picture and perform rotation group average.
- Perform time average $\mathcal{T}$ and trace our magnets, or vice and versa.
- Remove maximally mixed non-relational degrees of freedom from physical description.

$$\sum_c \sum_{s,s',r,r'} \frac{a_s a^*_{s'}}{2} \frac{1}{2^{2C}} \sqrt{\binom{2C}{C+c+\Lambda s}\binom{2C}{C+c+\Lambda s'}} (-1)^{(r-1/2)(s-1/2)+(r'-1/2)(s'-1/2)}$$
$$\times \sum_u d^C_{u-r,c+\Lambda s} d^C_{u-r',c+\Lambda s'} |G+u; G+u-r\rangle\langle G+u; G+u-r'|^{\mathcal{SCG}}$$

- Quantum numbers $(J^{\mathcal{CG}})^2$ and $(J^{\mathcal{SCG}})^2$.

Time

- To “read time", we must measure $(J^{\mathcal{CG}})^2$: this yields an outcome $G + u$.
  - Interpretation: Clock’s needle is at an angle $\theta = \cos^{-1}(u/C)$ with respect to $\mathcal{G}$, so it’s θ o’clock.

Time

We apply the same procedure as above:

- Solve equations of motion for $\mathcal{S}$, $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{N}$.
- Introduce gyroscope into the picture and perform rotation group average.
- Perform time average $\mathcal{T}$ and trace our magnets, or vice and versa.
- Remove maximally mixed non-relational degrees of freedom from physical description.

$$\sum_c \sum_{s,s',r,r'} \frac{a_s a^*_{s'}}{2} \frac{1}{2^{2C}} \sqrt{\binom{2C}{C+c+\Lambda s}\binom{2C}{C+c+\Lambda s'}} (-1)^{(r-1/2)(s-1/2)+(r'-1/2)(s'-1/2)}$$
$$\times \sum_u d^C_{u-r,c+\Lambda s} d^C_{u-r',c+\Lambda s'} |G+u; G+u-r\rangle\langle G+u; G+u-r'|^{\mathcal{SCG}}$$

- Quantum numbers $(J^{\mathcal{CG}})^2$ and $(J^{\mathcal{SCG}})^2$.

Time

- To “read time", we must measure $(J^{\mathcal{CG}})^2$: this yields an outcome $G+u$.
  - Interpretation: Clock’s needle is at an angle $\theta = \cos^{-1}(u/C)$ with respect to $\mathcal{G}$, so it’s θ o’clock.

Time

- To "read time", we must measure $(J^{\mathcal{CG}})^2$: this yields an outcome $G+u$.
  - Interpretation: Clock's needle is at an angle $\theta = \cos^{-1}(u/C)$ with respect to $\mathcal{G}$, so it's θ o'clock.
- Given this outcome, the state $\rho^{\mathcal{SGC}}$ updates according to von Neumann postulate, which in this case is equivalent to a classical Bayesian update.

Time

- To “read time", we must measure $(J^{\mathcal{C}\mathcal{G}})^2$: this yields an outcome $G+u$.
  - Interpretation: Clock’s needle is at an angle $\theta = \cos^{-1}(u/C)$ with respect to $\mathcal{G}$, so it’s θ o’clock.
- Given this outcome, the state $\rho^{\mathcal{S}\mathcal{G}\mathcal{C}}$ updates according to von Neumann postulate, which in this case is equivalent to a classical Bayesian update.
- Conditioned on this outcome u, we measure $(J^{\mathcal{S}\mathcal{C}\mathcal{G}})^2$ and obtain the outcome $G+u+s$ with probability $P(s|\theta)$.
  - $s = \pm 1/2$ is directly interpreted as the system’s spin relative to the gyroscope.

Time

We apply the same procedure as above:

- Solve equations of motion for $\mathcal{S}$, $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{N}$.
- Introduce gyroscope into the picture and perform rotation group average.
- Perform time average $\mathcal{T}$ and trace our magnets, or vice and versa.
- Remove maximally mixed non-relational degrees of freedom from physical description.

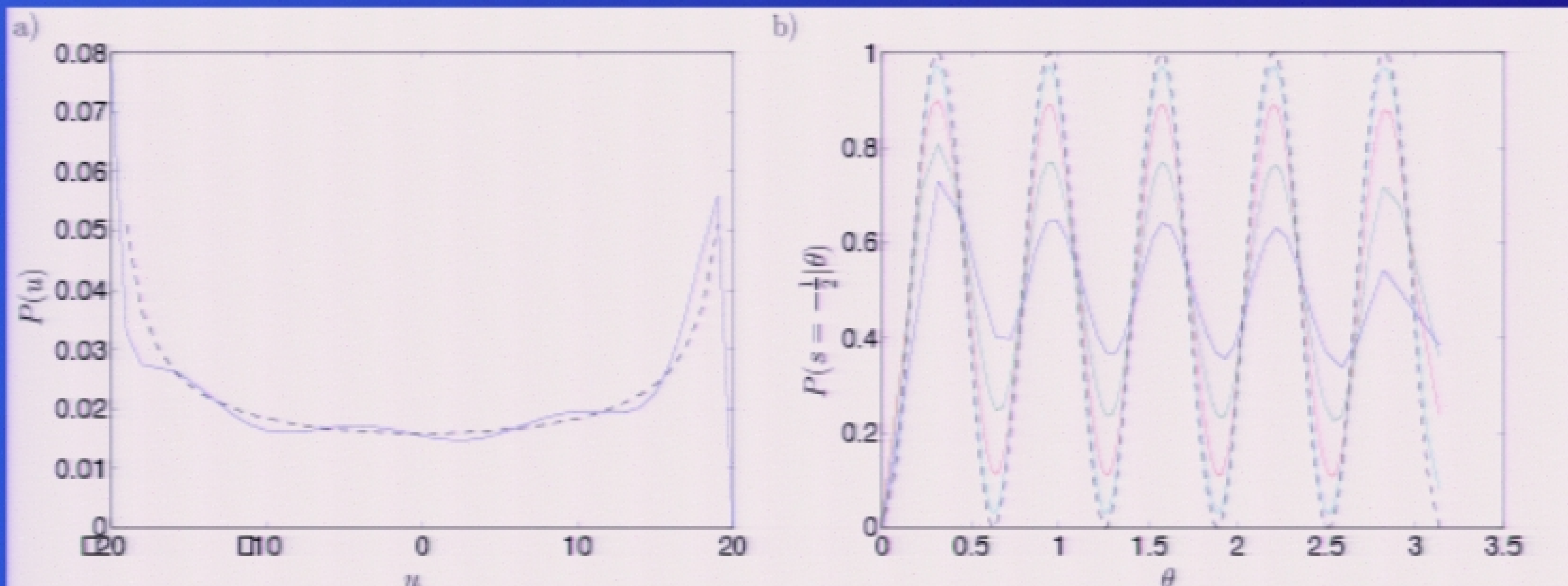
$$\sum_c \sum_{s,s',r,r'} \frac{a_s a^*_{s'}}{2} \frac{1}{2^{2C}} \sqrt{\binom{2C}{C+c+\Lambda s}\binom{2C}{C+c+\Lambda s'}} (-1)^{(r-1/2)(s-1/2)+(r'-1/2)(s'-1/2)}$$
$$\times \sum_u d^C_{u-r,c+\Lambda s} d^C_{u-r',c+\Lambda s'} |G+u; G+u-r\rangle\langle G+u; G+u-r'|^{\mathcal{SCG}}$$

- Quantum numbers $(J^{\mathcal{CG}})^2$ and $(J^{\mathcal{SCG}})^2$.

Time

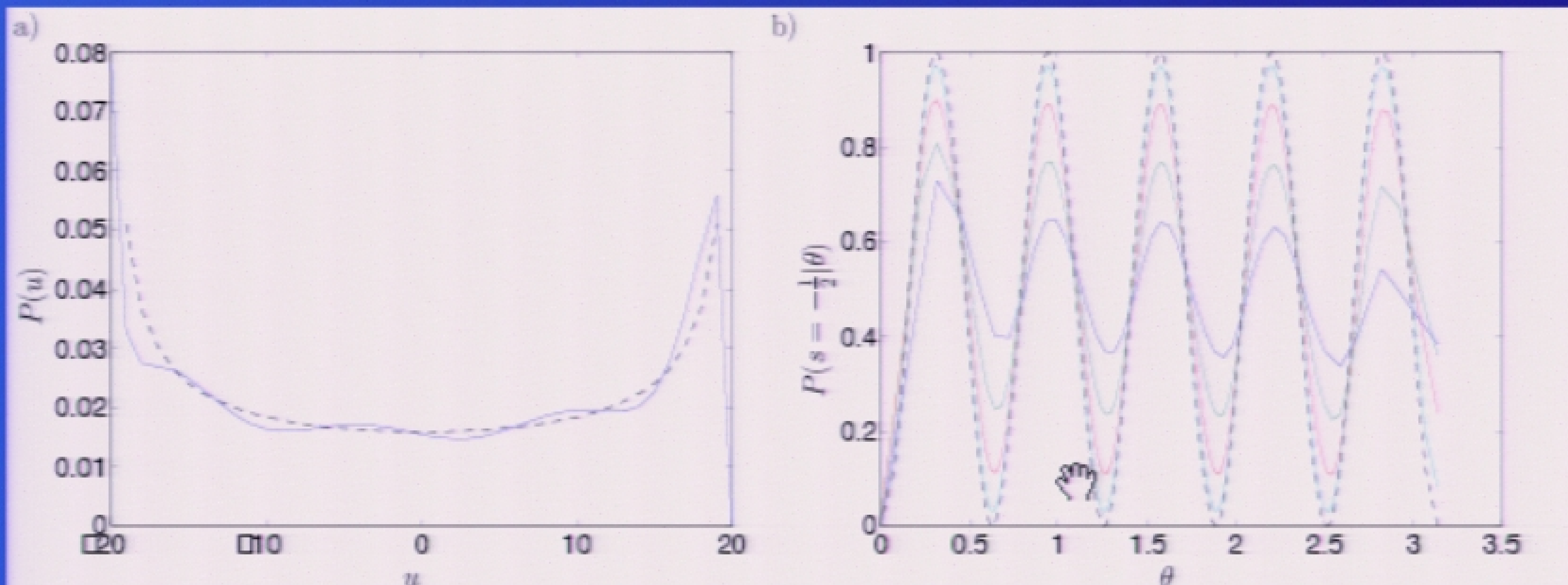
- To “read time", we must measure $(J^{\mathcal{CG}})^2$: this yields an outcome $G+u$.
  - Interpretation: Clock’s needle is at an angle $\theta = \cos^{-1}(u/C)$ with respect to $\mathcal{G}$, so it’s θ o’clock.
- Given this outcome, the state $\rho^{\mathcal{SGC}}$ updates according to von Neumann postulate, which in this case is equivalent to a classical Bayesian update.
- Conditioned on this outcome u, we measure $(J^{\mathcal{SCG}})^2$ and obtain the outcome $G+u+s$ with probability $P(s|\theta)$.
  - $s = \pm 1/2$ is directly interpreted as the system’s spin relative to the gyroscope.

Time



a) Probability distribution for the measurement outcome of $(J^{\mathcal{CG}})^2$ for clock size $C = 20$. Dash line is $1/\pi\sqrt{C^2 - u^2}$ corresponding to a flat distribution of θ.

Time



a) Probability distribution for the measurement outcome of $(J^{\mathcal{CG}})^2$ for clock size $C = 20$. Dash line is $1/\pi\sqrt{C^2 - u^2}$ corresponding to a flat distribution of θ.

b) Conditional probability of $(J^{\mathcal{SCG}})^2$ indicating $s = -1/2$ for clock size $C = 20$, 40, 100, and 400. Dash line indicate orthodox prediction.

What have we done so far?

- We have applied our four rules to the simple example. This forced us to...
  - Quantize the spacial and temporal reference frame, by introducing a quantum mechanical gyroscope and clock.
  - Quantize the external fields generating dynamics.

What have we done so far?

- We have applied our four rules to the simple example. This forced us to...
  - Quantize the spacial and temporal reference frame, by introducing a quantum mechanical gyroscope and clock.
  - Quantize the external fields generating dynamics.
- All physical quantities described in this theory are background independent.
- In the appropriate macroscopic limits, the predictions are the same as those of the orthodox theory.

What have we done so far?

- We have applied our four rules to the simple example. This forced us to...
  - Quantize the spacial and temporal reference frame, by introducing a quantum mechanical gyroscope and clock.
  - Quantize the external fields generating dynamics.
- All physical quantities described in this theory are background independent.
- In the appropriate macroscopic limits, the predictions are the same as those of the orthodox theory.
- But this limit is an approximation to reality...
- We will now explore some features of this relational theory.

Relational time

- As a consequence of the time average, $[\rho_{\mathrm{physical}}, H] = 0$.

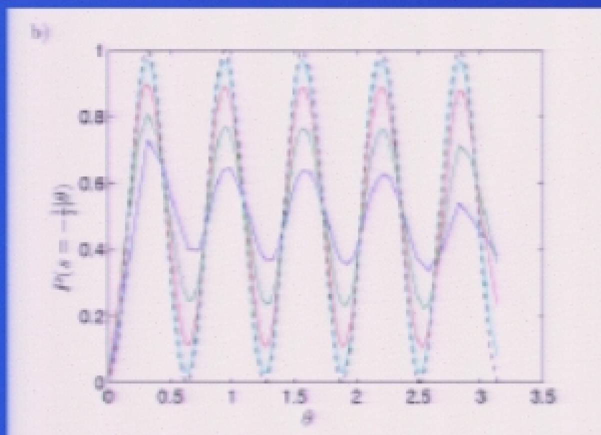
Relational time

- As a consequence of the time average, $[\rho_{\text{physical}}, H] = 0$.
  - This differs from the usual Wheeler-DeWitt equation $H|\Psi\rangle = 0$, which is a special case of our constraint.

Relational time

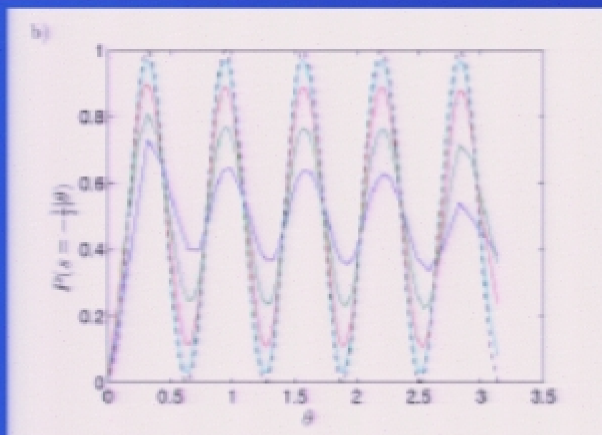
- As a consequence of the time average, $[\rho_{\text{physical}}, H] = 0$.
  - This differs from the usual Wheeler-DeWitt equation $H|\Psi\rangle = 0$, which is a special case of our constraint.
  - Should the constraint be applied at the level of $\mathcal{B}(\mathcal{H})$ or $\mathcal{H}$?
  - The mixed state solution is more natural in a Bayesian approach. (How could we know which energy eigenstate is realized by the universe from within the universe?)

Fundamental decoherence



Arrow of time?

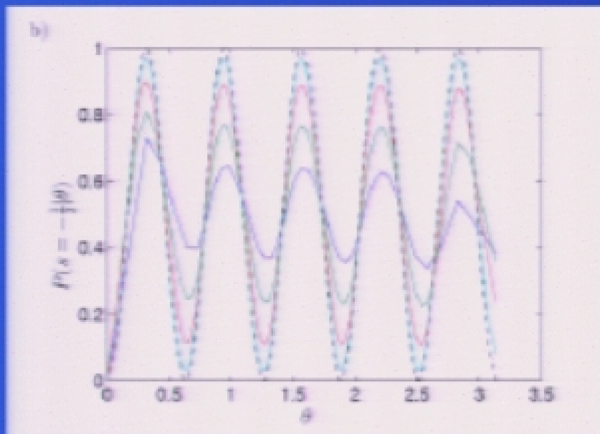
Fundamental decoherence



Arrow of time?

- Solutions all agree at $\theta = 0$, but deteriorate as θ increase.
- Fundamental decoherence due to quantum fluctuations of the clock, equivalent to fluctuations of H.

Fundamental decoherence

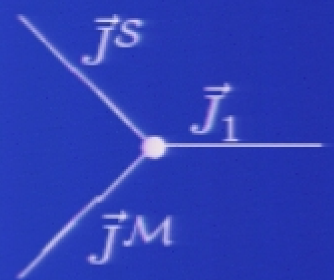


Arrow of time?

- Solutions all agree at $\theta = 0$, but deteriorate as θ increase.
- Fundamental decoherence due to quantum fluctuations of the clock, equivalent to fluctuations of H.
- Bayesian approach: clock measurement used to estimate the coordinate time $p(t|\theta) = p(\theta|t)p(t)/p(\theta)$.
- Given clock measurement outcome θ, the state of S is $\rho^S(\theta) = \int p(t|\theta)\rho(t)dt = \int p(t|\theta)e^{-iHt}\rho(0)e^{iHt}dt$

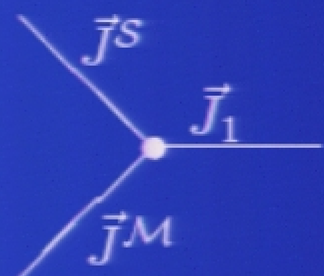
Spin networks

- To solve equations of motion, we first introduced two new operators $\vec{J}_1$ and $\vec{J}_2$ satisfying $\vec{J}^{\mathcal{S}} + \vec{J}^{\mathcal{M}} + \vec{J}_1 = 0$ and $\vec{J}^{\mathcal{C}} + \vec{J}^{\mathcal{N}} - \vec{J}_2 = 0$:



Spin networks

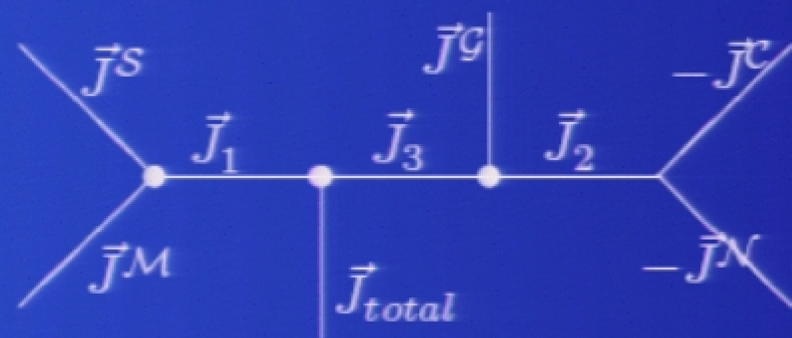
- To solve equations of motion, we first introduced two new operators $\vec{J}_1$ and $\vec{J}_2$ satisfying $\vec{J}^{\mathcal{S}} + \vec{J}^{\mathcal{M}} + \vec{J}_1 = 0$ and $\vec{J}^{\mathcal{C}} + \vec{J}^{\mathcal{N}} - \vec{J}_2 = 0$:



- To read the time, we introduced an other operator $\vec{J}_3$ satisfying $\vec{J}^{\mathcal{G}} + \vec{J}_2 + \vec{J}_3 = 0$, or in other words $\vec{J}_3 = -(\vec{J}^{\mathcal{C}} + \vec{J}^{\mathcal{N}} + \vec{J}^{\mathcal{G}})$.
- To measure the system's state relative to the gyroscope and clock, we introduced $\vec{J}_{total}$ satisfying $\vec{J}_1 + \vec{J}_3 + \vec{J}_{total} = 0$, or equivalently $\vec{J}_{total} = \vec{J}^{\mathcal{S}} + \vec{J}^{\mathcal{M}} + \vec{J}^{\mathcal{C}} + \vec{J}^{\mathcal{N}} + \vec{J}^{\mathcal{G}}$.

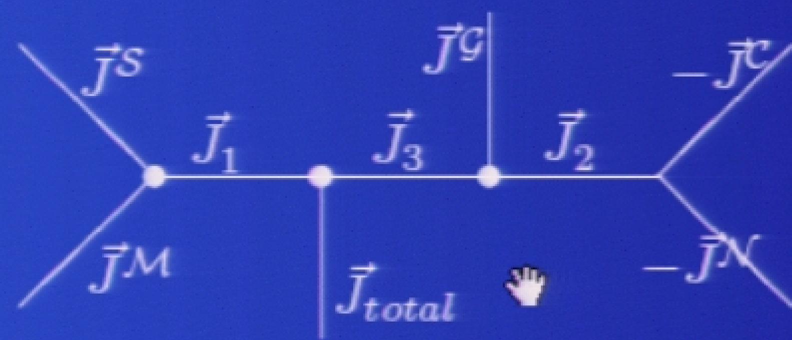
Spin networks

- Putting all this together yields the diagram



Spin networks

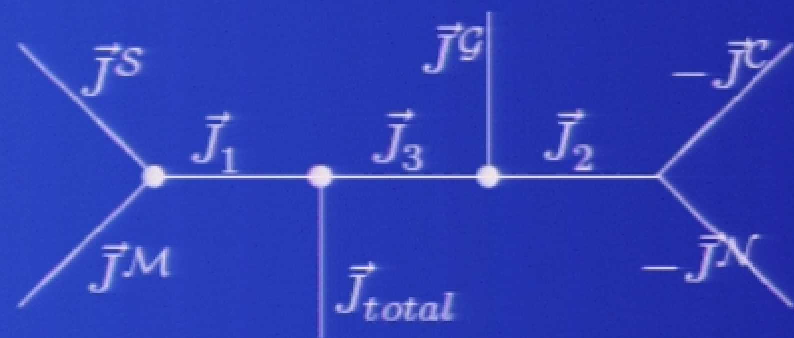
- Putting all this together yields the diagram



- The next step was the group average. On the diagram, this essentially boils down to removing the arrows!!!

Spin networks

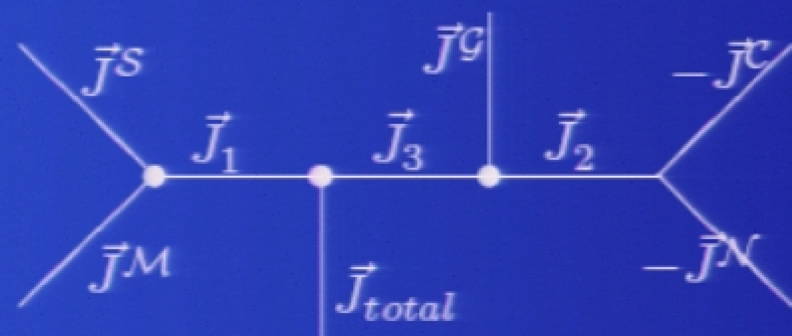
- Putting all this together yields the diagram



- The next step was the group average. On the diagram, this essentially boils down to removing the arrows!!!
  - $\vec{J} \to j$ such that $(\vec{J})^2 = j(j+1)$.
  - Not all edges are in an eigenstate of the operator $(\vec{J})^2$, so we need superposition of graphs.

Spin networks

- Putting all this together yields the diagram

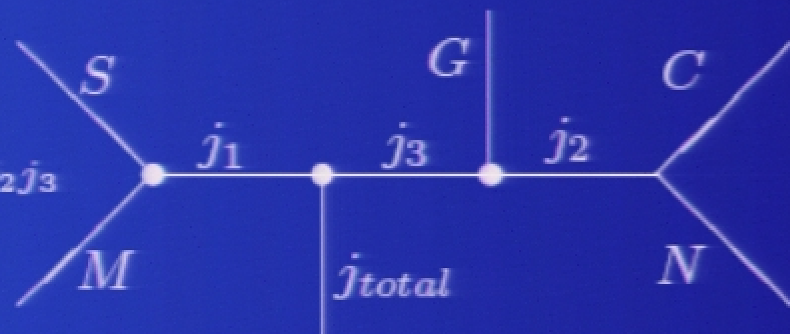


- The next step was the group average. On the diagram, this essentially boils down to removing the arrows!!!
  - $\vec{J} \to j$ such that $(\vec{J})^2 = j(j+1)$.
  - Not all edges are in an eigenstate of the operator $(\vec{J})^2$, so we need superposition of graphs.
  - The amplitudes are linear functions of the non-relational amplitudes and Clebsch-Gordan coefficients.

Spin networks

- This yields the diagram

$$\sum_{j_{total}} P(j_{total}) \left[\sum_{j_1 j_2 j_3} \alpha_{j_1 j_2 j_3} \right.$$

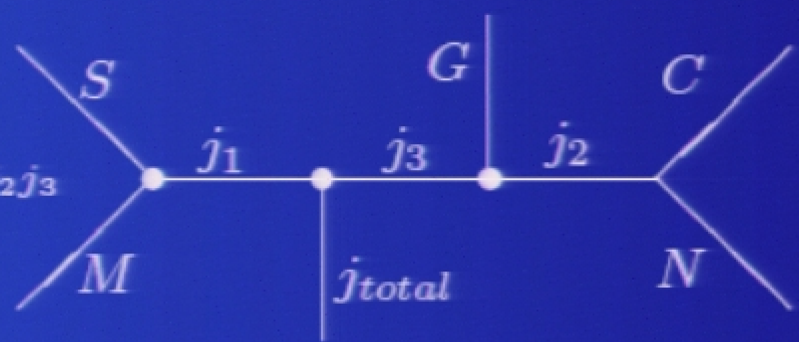


$$\left. \right]$$

Spin networks

- This yields the diagram

$$\sum_{j_{total}} P(j_{total}) \Big[\sum_{j_1 j_2 j_3} \alpha_{j_1 j_2 j_3}$$



$$\Big]$$

- Where $\sum_i p_i [|\Gamma_i\rangle]$ stands for $\sum_i p_i |\Gamma_i\rangle\langle\Gamma_i|$.
- The final step was to perform a time average which imposed an energy superselection rule.
  - This implies superselection of $j_1(j_1+1) + j_2(j_2+1)$, which can be imposed by a Kronecker delta in the previous sum.

Spin networks

- Each decorated graph Γ is a spin network corresponding to a basis state of the relational theory.

Spin networks

- Each decorated graph Γ is a spin network corresponding to a basis state of the relational theory.
- Vertex with edges j_1, j_2, and j_3 carries an intertwining operator $\mathbb{C}^{j_1(j_1+1)} \otimes \mathbb{C}^{j_2(j_2+1)} \otimes \mathbb{C}^{j_3(j_3+1)} \rightarrow \mathbb{C}$, which in our simple model, give the Clebsch-Gordan coefficients required to remove the arrows.

Spin networks

- Each decorated graph Γ is a spin network corresponding to a basis state of the relational theory.
- Vertex with edges j_1, j_2, and j_3 carries an intertwining operator $\mathbb{C}^{j_1(j_1+1)} \otimes \mathbb{C}^{j_2(j_2+1)} \otimes \mathbb{C}^{j_3(j_3+1)} \to \mathbb{C}$, which in our simple model, give the Clebsch-Gordan coefficients required to remove the arrows.
- The "sum over histories" $\mathcal{T}$, usually performed using spin foams, is here implemented at the level of $\mathcal{B}(\mathcal{H})$ rather than $\mathcal{H}$.

Connexions to other programs

- The program was motivated by the noiseless subsystems method of quantum information science.

Summary (if you buy my rules...)

- Quantum theory is fundamentally relational.
  - The "orthodox" description is a semi-classical limit, so it is approximate.

Summary (if you buy my rules...)

- Quantum theory is fundamentally relational.
  - The "orthodox" description is a semi-classical limit, so it is approximate.
- Elementary quantum mechanics with no classical approximations allowed us to recover interesting results:
  - Relational time of Page and Wootters.
  - Fundamental decoherence of Gambini et al.
  - Spin network representation.

Summary (if you buy my rules...)

- Quantum theory is fundamentally relational.
  - The “orthodox" description is a semi-classical limit, so it is approximate.
- Elementary quantum mechanics with no classical approximations allowed us to recover interesting results:
  - Relational time of Page and Wootters.
  - Fundamental decoherence of Gambini et al.
  - Spin network representation.
- But our results were slightly different.
  - Hamiltonian constraint $[\rho_{\text{physical}}, H] = 0$ rather than WDW.
  - Relational time rises from classical correlations rather than entanglement.