Title: Distillability and positivity of partial transposes in general quantum field systems

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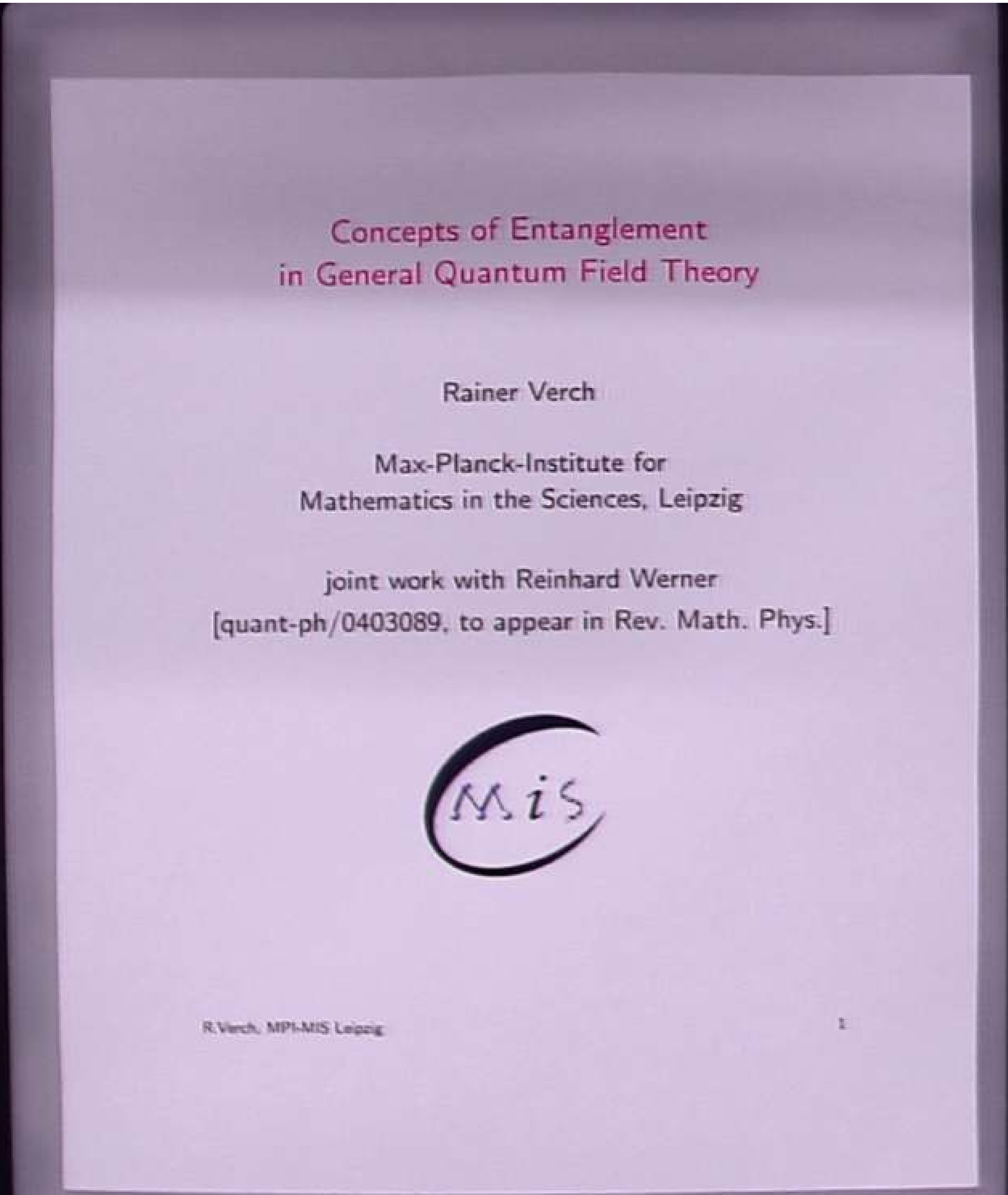
Abstract:



Concepts of Entanglement in General Quantum Field Theory

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joint work with Reinhard Werner
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1. GENERALITIES I: Algebraic description of general quantum systems and quantum field theories

A quantum system is described by specifying:

☐ $\mathcal{H}$: a Hilbertspace

☐ $\mathcal{R} \subset B(\mathcal{H})$: a $*$-algebra of operators, where:

- $A = A^* \in \mathcal{R}$ is interpreted as an observable
- For $\psi \in \mathcal{H}$ with $||\psi|| = 1$, the quantity

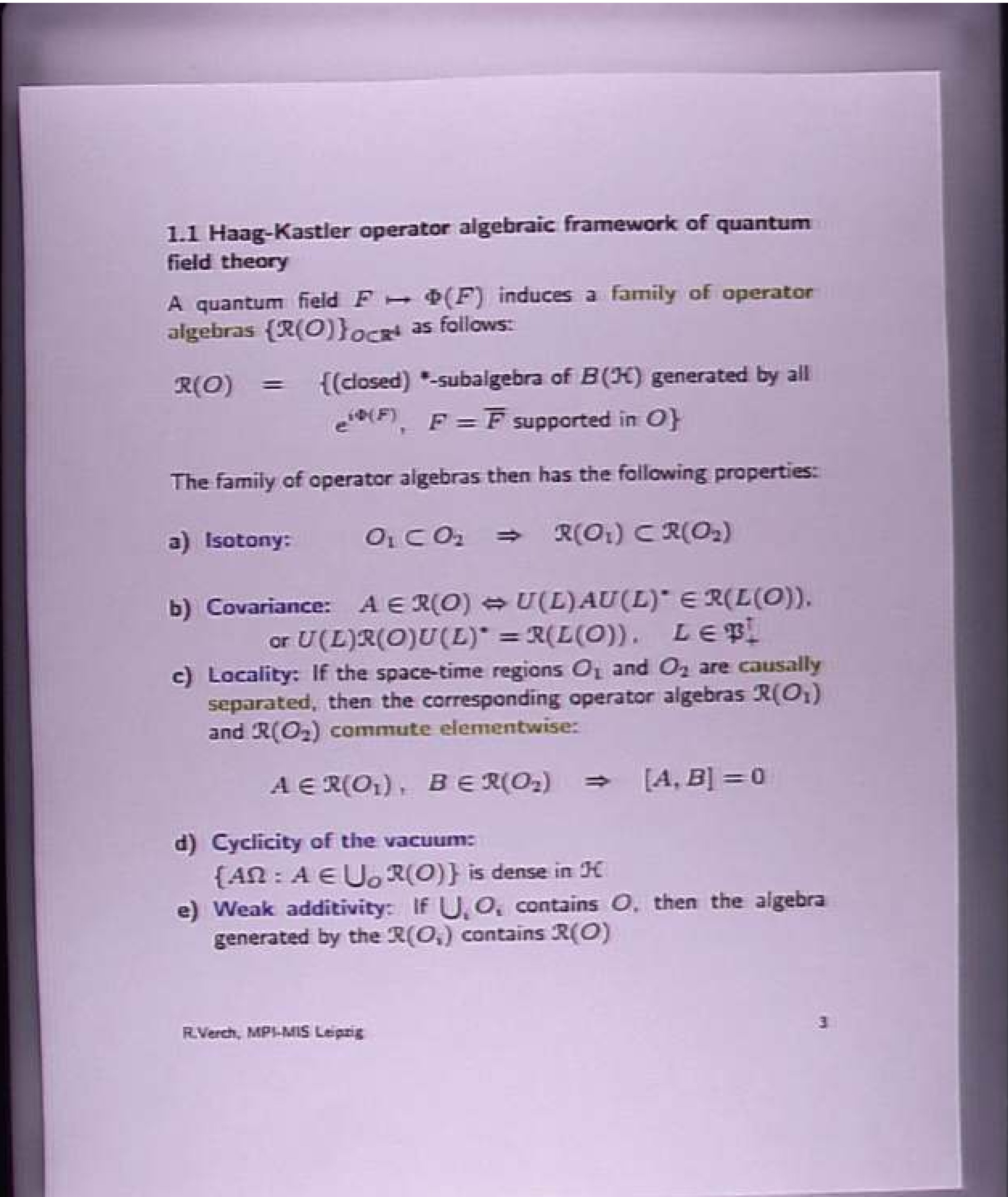
$$\langle A \rangle_\psi = \langle \psi | A | \psi \rangle$$

is interpreted as the expectation value of the observable A in the state given by ψ.

More generally: For $\rho =$ trace-class operator on $\mathcal{H}$ with $\rho \geq 0$, trace$(\rho) = 1$, we interpret $\langle A \rangle_\rho = \text{trace}(\rho\, A)$ as expectation value of A in the state given by ρ.

Remark i) Often we want to consider **unbounded** operators as observables which are *not* contained in $B(\mathcal{H})$ — and thus we would often like $\mathcal{R}$ also to contain unbounded operators. But these can be viewed as limits of bounded operators and thus $\mathcal{R} \subset B(\mathcal{H})$ is not a physically relevant restriction.

ii) Often, $\mathcal{R} = B(\mathcal{H})$, but it can happen that $\mathcal{R}$ is a proper subalgebra of $B(\mathcal{H})$ — e.g. if $\mathcal{R}$ models the observables of a subsystem of a larger (ambient) system, or in systems at finite temperature. Typical: $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$, $\quad \mathcal{R} = B(\mathcal{H}_a) \otimes \mathbb{1}$



1.1 Haag-Kastler operator algebraic framework of quantum field theory

A quantum field $F \mapsto \Phi(F)$ induces a family of operator algebras $\{\mathcal{R}(O)\}_{O \subset \mathbb{R}^4}$ as follows:

$$\mathcal{R}(O) \quad = \quad \{\text{(closed) }^*\text{-subalgebra of } B(\mathcal{H}) \text{ generated by all } e^{i\Phi(F)}, \ F = \overline{F} \text{ supported in } O\}$$

The family of operator algebras then has the following properties:

a) **Isotony:** $O_1 \subset O_2 \quad \Rightarrow \quad \mathcal{R}(O_1) \subset \mathcal{R}(O_2)$

b) **Covariance:** $A \in \mathcal{R}(O) \Leftrightarrow U(L)AU(L)^* \in \mathcal{R}(L(O))$,
or $U(L)\mathcal{R}(O)U(L)^* = \mathcal{R}(L(O))$, $\quad L \in \mathfrak{P}_+^\uparrow$

c) **Locality:** If the space-time regions O_1 and O_2 are causally separated, then the corresponding operator algebras $\mathcal{R}(O_1)$ and $\mathcal{R}(O_2)$ commute elementwise:

$$A \in \mathcal{R}(O_1), \ B \in \mathcal{R}(O_2) \quad \Rightarrow \quad [A,B] = 0$$

d) **Cyclicity of the vacuum:**
$\{A\Omega : A \in \bigcup_O \mathcal{R}(O)\}$ is dense in $\mathcal{H}$

e) **Weak additivity:** If $\bigcup_i O_i$ contains O, then the algebra generated by the $\mathcal{R}(O_i)$ contains $\mathcal{R}(O)$

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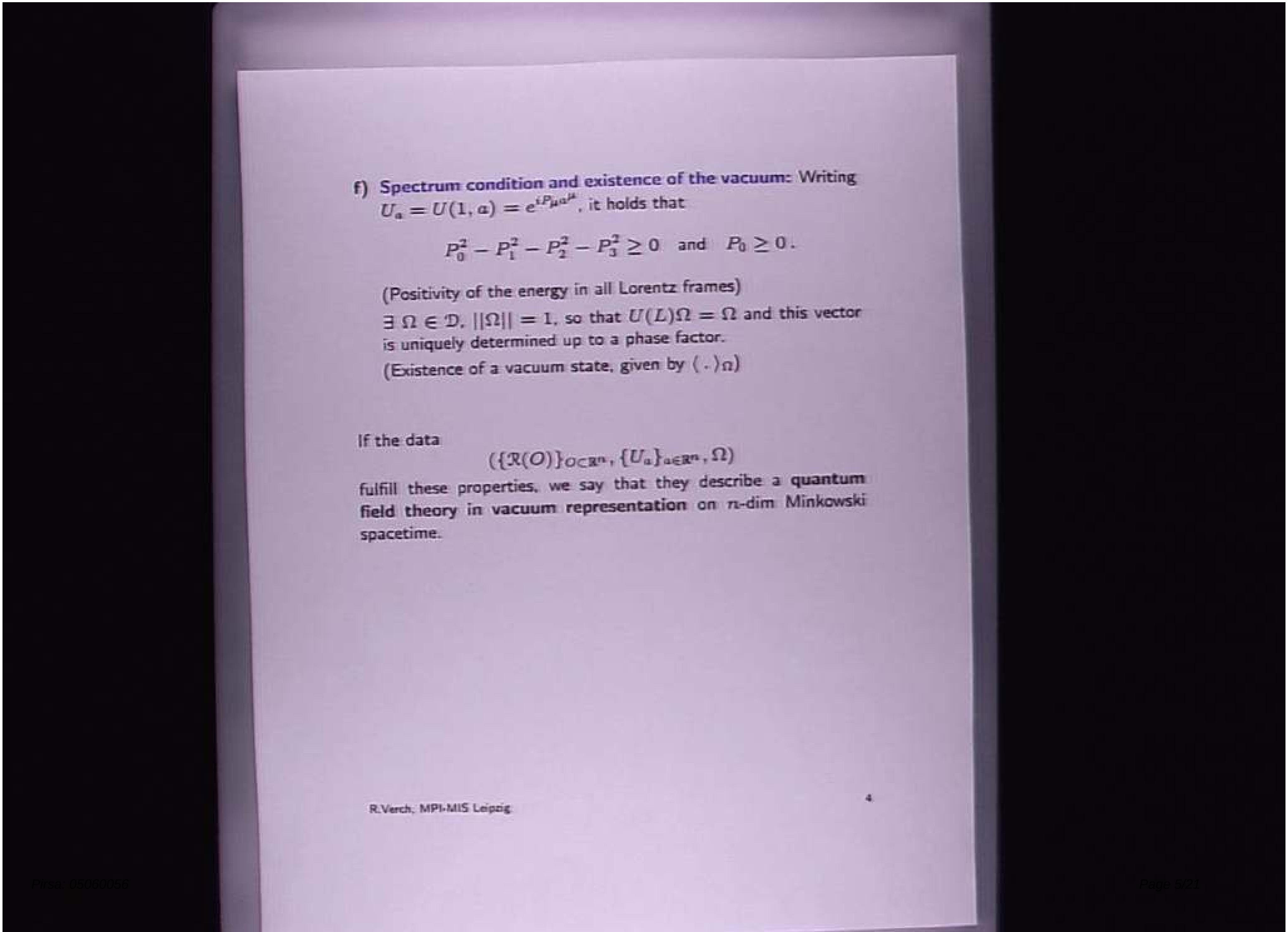


f) **Spectrum condition and existence of the vacuum:** Writing $U_a = U(1,a) = e^{iP_\mu a^\mu}$, it holds that

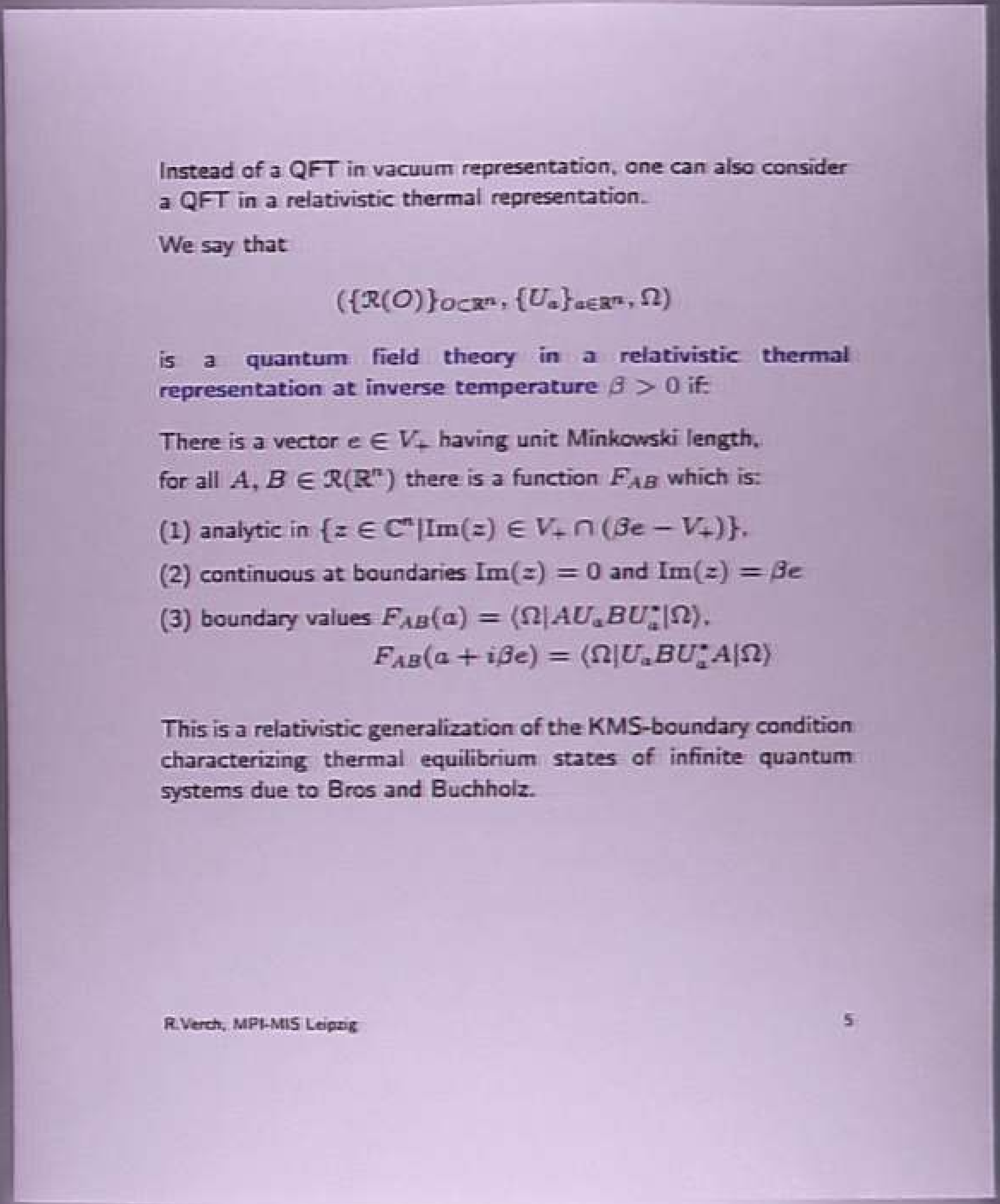
$$P_0^2 - P_1^2 - P_2^2 - P_3^2 \geq 0 \quad \text{and} \quad P_0 \geq 0 .$$

(Positivity of the energy in all Lorentz frames)
$\exists\, \Omega \in \mathcal{D}$, $||\Omega|| = 1$, so that $U(L)\Omega = \Omega$ and this vector is uniquely determined up to a phase factor.
(Existence of a vacuum state, given by $\langle\,.\,\rangle_\Omega$)

If the data

$$(\{\mathcal{R}(O)\}_{O\subset\mathbb{R}^n}, \{U_a\}_{a\in\mathbb{R}^n}, \Omega)$$

fulfill these properties, we say that they describe a **quantum field theory in vacuum representation** on n-dim Minkowski spacetime.



Instead of a QFT in vacuum representation, one can also consider a QFT in a relativistic thermal representation.

We say that

$$(\{\mathcal{R}(O)\}_{O\subset\mathbb{R}^n}, \{U_a\}_{a\in\mathbb{R}^n}, \Omega)$$

is a **quantum field theory in a relativistic thermal representation at inverse temperature** $\beta > 0$ if:

There is a vector $e \in V_+$ having unit Minkowski length,

for all $A, B \in \mathcal{R}(\mathbb{R}^n)$ there is a function F_{AB} which is:

(1) analytic in $\{z \in \mathbb{C}^n | \mathrm{Im}(z) \in V_+ \cap (\beta e - V_+)\}$,

(2) continuous at boundaries $\mathrm{Im}(z) = 0$ and $\mathrm{Im}(z) = \beta e$

(3) boundary values $F_{AB}(a) = \langle\Omega|AU_aBU_a^*|\Omega\rangle$,

$$F_{AB}(a + i\beta e) = \langle\Omega|U_aBU_a^*A|\Omega\rangle$$

This is a relativistic generalization of the KMS-boundary condition characterizing thermal equilibrium states of infinite quantum systems due to Bros and Buchholz.

2. GENERALITIES II: Entanglement in quantum physics and quantum field theory

A quantum system modelled by an algebra of observables $\mathcal{R}$ may possess two (or more) subsystems modelled by two subalgebras $\mathcal{A}$ and $\mathcal{B}$

$\mathcal{A} \leftrightarrow$ observables controlled by **"Alice"**

$\mathcal{B} \leftrightarrow$ observables controlled by **"Bob"**

Standing assumptions:

- $\mathcal{A}$ and $\mathcal{B}$ are C^*-subalgebras (with $\mathbb{1}$) of $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$
- can take
$$\mathcal{A} \overset{C^*}{\vee} \mathcal{B} \subset \mathcal{R} \subset \mathcal{B}(\mathcal{H})$$
according to situation
- $\mathcal{A} \subset \mathcal{B}'$, i.e. $AB = BA$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$

2.i Def $(\mathcal{A}, \mathcal{B}) \subset \mathcal{R}$ is called a **bipartite quantum system**

1.ii Def Let $(\mathcal{A},\mathcal{B}) \subset \mathcal{R}$ be a bipartite quantum system and $\omega(\,.\,) = \mathrm{trace}(\rho_\omega\,.\,)$ a state on $\mathcal{R}$

Then ω is called **entangled** with resp. to $(\mathcal{A},\mathcal{B})$ if it is **not separable**, i.e. not the (weak) limit of convex sums of product states over $(\mathcal{A},\mathcal{B})$.

- ω is a **product state** over $(\mathcal{A},\mathcal{B})$ iff
  $\exists$ states ω_a on $\mathcal{A}$, ω_b on $\mathcal{B}$ such that

$$\omega(AB) = \omega_a(A)\omega_b(B) \qquad \forall\, A \in \mathcal{A} \;\; B \in \mathcal{B}\,.$$

- ω is a **convex sum of product states** if

$$\omega = \sum_{k=1}^{N} \lambda_k \omega^{(k)}\,, \qquad \lambda_k > 0\,,\; \sum_k \lambda_k = 1$$

  and each $\omega^{(k)}$ is a product state over $(\mathcal{A},\mathcal{B})$.
- ω is **separable** if $\exists\, \omega_\alpha$, $\alpha \in \mathbb{J}$ so that

$$\omega(R) = \lim_\alpha \omega_\alpha(R) \qquad \forall\, R \in \mathcal{R}$$

  and each ω_α is a convex sum of product states over $(\mathcal{A},\mathcal{B})$.

Example: The **Bell state** for a bipartite quantum system, each part having 2 degrees of freedom

$$\begin{aligned} \mathcal{A} &= \mathcal{B}(\mathbb{C}^2)\,, \quad \mathcal{B} = \mathcal{B}(\mathbb{C}^2) \\ \mathcal{R} &= \mathcal{B}(\mathbb{C}^2) \otimes \mathcal{B}(\mathbb{C}^2) \equiv \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2) \end{aligned}$$

$$\omega_{|\mathrm{Bell}\rangle}(A \otimes B) = \langle \mathrm{Bell} | A \otimes B | \mathrm{Bell} \rangle\,,$$

$$|\mathrm{Bell}\rangle = \frac{1}{\sqrt{2}}\left(\binom{1}{0} \otimes \binom{0}{1} - \binom{0}{1} \otimes \binom{1}{0}\right)$$

In experimental realization, the $\mathbb{C}^2$-states correspond to linear polarization of photon states:

$$\begin{aligned} \binom{1}{0} &= |\updownarrow\rangle = \text{"vertically polarized photon state"} \\ \binom{0}{1} &= |\leftrightarrow\rangle = \text{"horizontally polarized photon state"} \end{aligned}$$

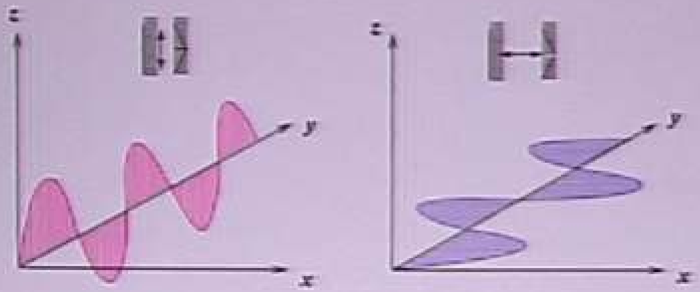
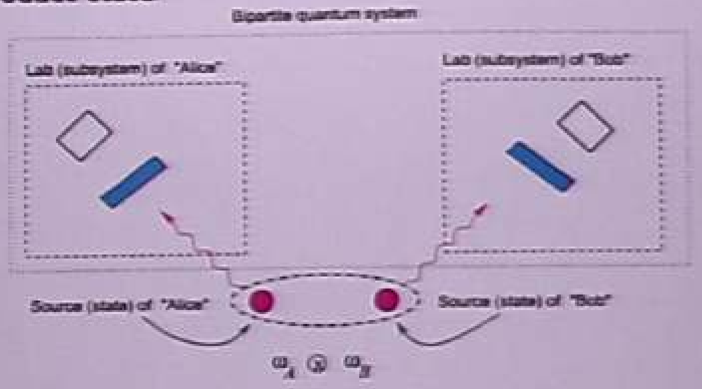


Illustration:

Product state



Bipartite quantum system

Lab (subsystem) of "Alice"

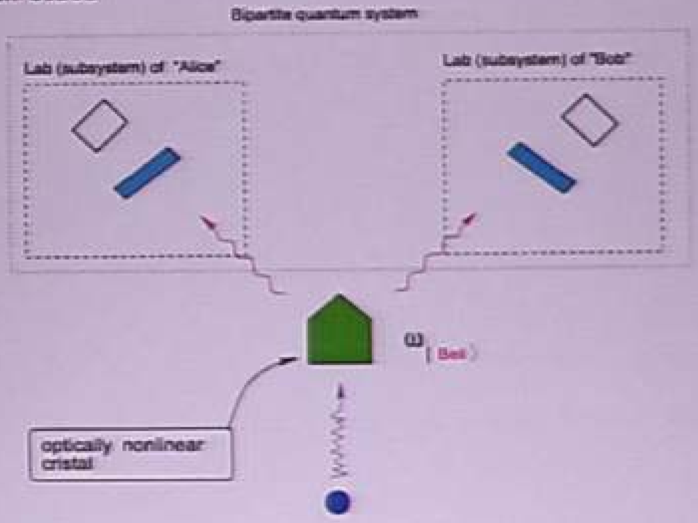
Lab (subsystem) of "Bob"

Source (state) of "Alice"

Source (state) of "Bob"

$\omega_A \otimes \omega_B$

Bell state



Bipartite quantum system

Lab (subsystem) of "Alice"

Lab (subsystem) of "Bob"

$\omega_{|\text{Bell}\rangle}$

optically nonlinear cristal

3. QUALIFYING ENTANGLEMENT

3.1 States with Positive Partial Transpose (ppt states)

The original definition of ppt states is due to Peres (1996) and applies to the case of a bipartite quantum system with $\mathcal{A}, \mathcal{B} = B(\mathbf{C}^N)$, $N \in \mathbb{N}$:

Let $\omega(A \otimes B) = \mathrm{trace}(\rho_\omega(A \otimes B))$ be a state on $B(\mathbf{C}^N) \otimes B(\mathbf{C}^N)$, then ω is called **ppt** if the **partial transpose** $\rho_\omega^{T_1}$ is **non-negative**, where

$$\langle e_k^{(a)} \otimes e_\ell^{(b)} | \rho_\omega^{T_1} | e_m^{(a)} \otimes e_n^{(b)} \rangle = \langle e_m^{(a)} \otimes e_\ell^{(b)} | \rho_\omega | e_k^{(a)} \otimes e_n^{(b)} \rangle$$

Remark $\rho_\omega^{T_1}$ depends on choice of bases, but the condition $\rho_\omega^{T_1} \geq 0$ doesn't.

For general quantum systems:

3.i Def Let $(\mathcal{A}, \mathcal{B}) \subset \mathcal{R}$ be a bipartite quantum system, and let ω be a state on $\mathcal{R}$. We call ω a **ppt state** if

$$\sum_{k,\ell} \omega(A_k A_\ell^* B_\ell^* B_k) \geq 0$$

holds for all choices of finitely many $A_1, \ldots, A_K \in \mathcal{A}$ and $B_1, \ldots, B_K \in \mathcal{B}$.

3.ii Lemma **For general bipartite systems:**

$$\omega \text{ separable} \Rightarrow \omega \text{ ppt}, \qquad \text{equivalently},$$

$$\omega \text{ non-ppt (npt)} \Rightarrow \omega \text{ entangled}.$$

3.2 Bell-CHSH Inequalities

3.iii Def Let $(\mathcal{A},\mathcal{B}) \subset \mathcal{R}$ be a bipartite quantum system, and let ω be a state on $\mathcal{R}$. One calls

$$\beta(\omega) = \sup_{A,A',B,B'} \omega(A(B'+B) + A'(B'-B)),$$

where $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{B}$ are hermitean and norm-bounded by 1, the **Bell-correlation** of ω.

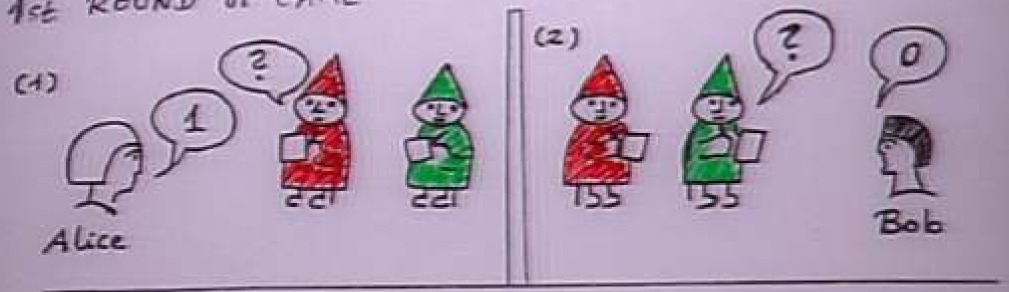
- If $\beta(\omega) = 2$, one says that ω **fulfills the Bell inequalities**
- If $\beta(\omega) > 2$, one says that ω **violates the Bell inequalities**
- If $\beta(\omega) = 2\sqrt{2}$, one says that ω **violates the Bell inequalities maximally**

3.iv The Bell state violates Bell's inqualities maximally

States violating Bell's inequalities contain non-classical correlations (entanglement). The entanglement of the Bell state cannot be reproduced by classical 'protocols', and this makes this state particularly useful for tasks of quantum communication, mainly for **secure key distribution** and **quantum teleportation**.

3.v Lemma A ppt state ω on a general bipartite quantum system fulfills Bell's inequalities.
Hence, a state violating Bell's inqualites is npt, therefore is entangled.

1st ROUND OF GAME



2nd ROUND

3rd ROUND

⋮

VALID ANSWERS:

(A ? RED) & (B ? RED) ⇒ [A=0] & [B=1] OR [A=1] & [B=0]

ALL OTHER COMBINATIONS:

(A ? RED) & (B ? GREEN)
(A ? GREEN) & (B ? RED)
(A ? GREEN) & (B ? GREEN)
} ⇒ [A=0] & [B=0] OR [A=1] & [B=1]

ALICE AND BOB MAY "PLOT A PROTOCOL" BEFORE BEGINNING OF THE GAME

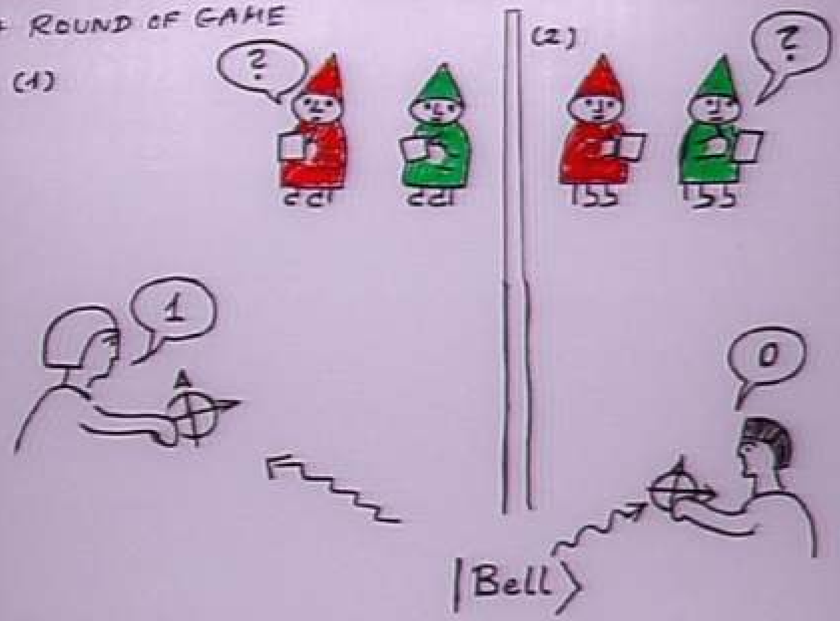
SUCCESS RATE

$$= \frac{\#\ \text{VALID ANSWERS}}{\#\ \text{ROUNDS}}$$

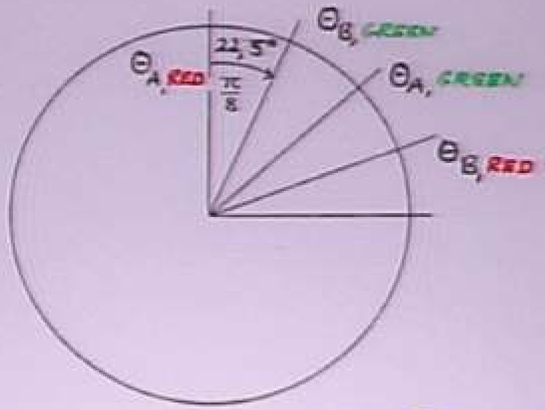
$$= \frac{3}{4} = 75\%$$

A	B
1. R=1, G=0	1. R=0, G=1
2. R=0, G=1	2. R=1, G=0
3 .	.
.	.

1st ROUND OF GAME



BEFORE BEGINNING OF THE GAME, ALICE AND BOB AGREE THAT:



SUCCESS RATE IS NOW : $\frac{1}{2} + \frac{\sqrt{2}}{4} \simeq 85{,}36\,\%$

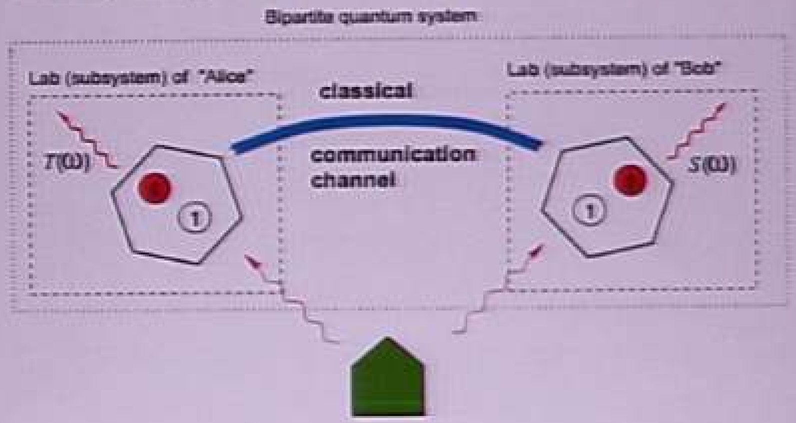
76

3.3 Distillability

For a class of entangled states the 'degree of entanglement' can be enhanced (up to maximal violation of Bell's inequalities) for a sub-ensemble of the original state by a process called **distillation**. States for which this is possible are generally called **distillable**. There is in principle a large class of distillation processes. The common underlying idea is that of **LOCC**,

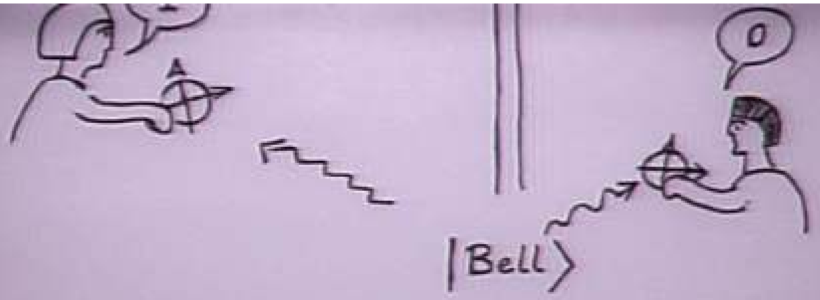
local operations and classical communication .

The simplest type is **1-distillability**:



$$T(\omega)(AB) = \omega(\tau(A)B), \quad S(\omega)(AB) = \omega(A\sigma(B))$$

with $\tau : \mathcal{A} \to \mathcal{A}$ and $\sigma : \mathcal{B} \to \mathcal{B}$ completely positive.



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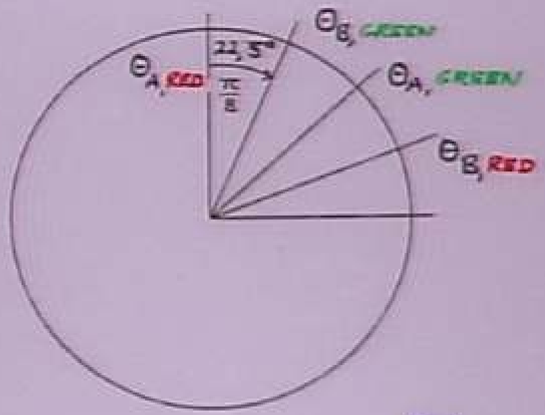

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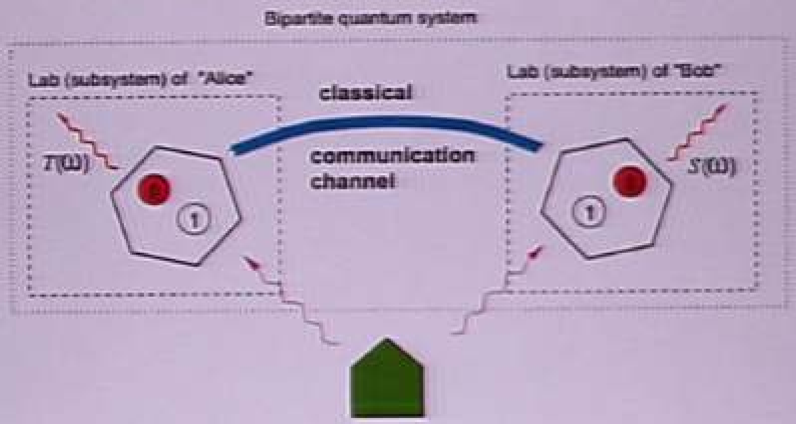


$$T(\omega)(AB) = \omega(\tau(A)B), \quad S(\omega)(AB) = \omega(A\sigma(B))$$

with $\tau : \mathcal{A} \to \mathcal{A}$ and $\sigma : \mathcal{B} \to \mathcal{B}$ completely positive.

3.vi Def Let ω be a state on a general bipartite quantum system $(\mathcal{A}, \mathcal{B}) \subset \mathcal{R}$.
ω is called **1-distillable** if for each $\varepsilon > 0$ there are

- a completely positive map $\tau : B(\mathbf{C}^2) \to \mathcal{A}$, $\tau(\mathbb{1}) \leq \mathbb{1}$, and
- a completely positive map $\sigma : B(\mathbf{C}^2) \to \mathcal{B}$, $\sigma(\mathbb{1}) \leq \mathbb{1}$,

so that

$$\left| \omega_{|\mathrm{Bell}\rangle}(X \otimes Y) - \omega(\tau(X)\sigma(Y))/\omega(\tau(\mathbb{1})\sigma(\mathbb{1})) \right| \leq \varepsilon \cdot ||X \otimes Y||$$

holds for all $X \otimes Y \in B(\mathbf{C}^2) \otimes B(\mathbf{C}^2)$.

Remark There are more general distillation processes, e.g. one could replace $\tau(\,.\,)\sigma(\,.\,)$ by $\sum_k \tau_k(\,.\,)\sigma_k(\,.\,)$.
It is important that operations of this type don't induce entanglement in non-entangled states. To this end, let completely positive maps

$$\tau_k : \mathcal{A}_{\mathrm{aux}} \to \mathcal{A}\,, \quad \sigma_k : \mathcal{B}_{\mathrm{aux}} \to \mathcal{B}\,, \quad k = 1, \ldots, K\,,$$

be given.

3.vii Prop Let ω be a state on the bipartite quantum system $(\mathcal{A}, \mathcal{B})$. If ω is ppt, then the positive functional on the bipartite system $(\mathcal{A}_{\mathrm{aux}}, \mathcal{B}_{\mathrm{aux}})$,

$$\omega_{\mathrm{aux}}(A_{\mathrm{aux}}B_{\mathrm{aux}}) = \omega(\sum_k \tau_k(A_{\mathrm{aux}})\sigma_k(B_{\mathrm{aux}}))$$

is also ppt.

In particular, a ppt state ω is not distillable by this kind of a LOCC process.

4. REEH-SCHLIEDER and DISTILLABILITY

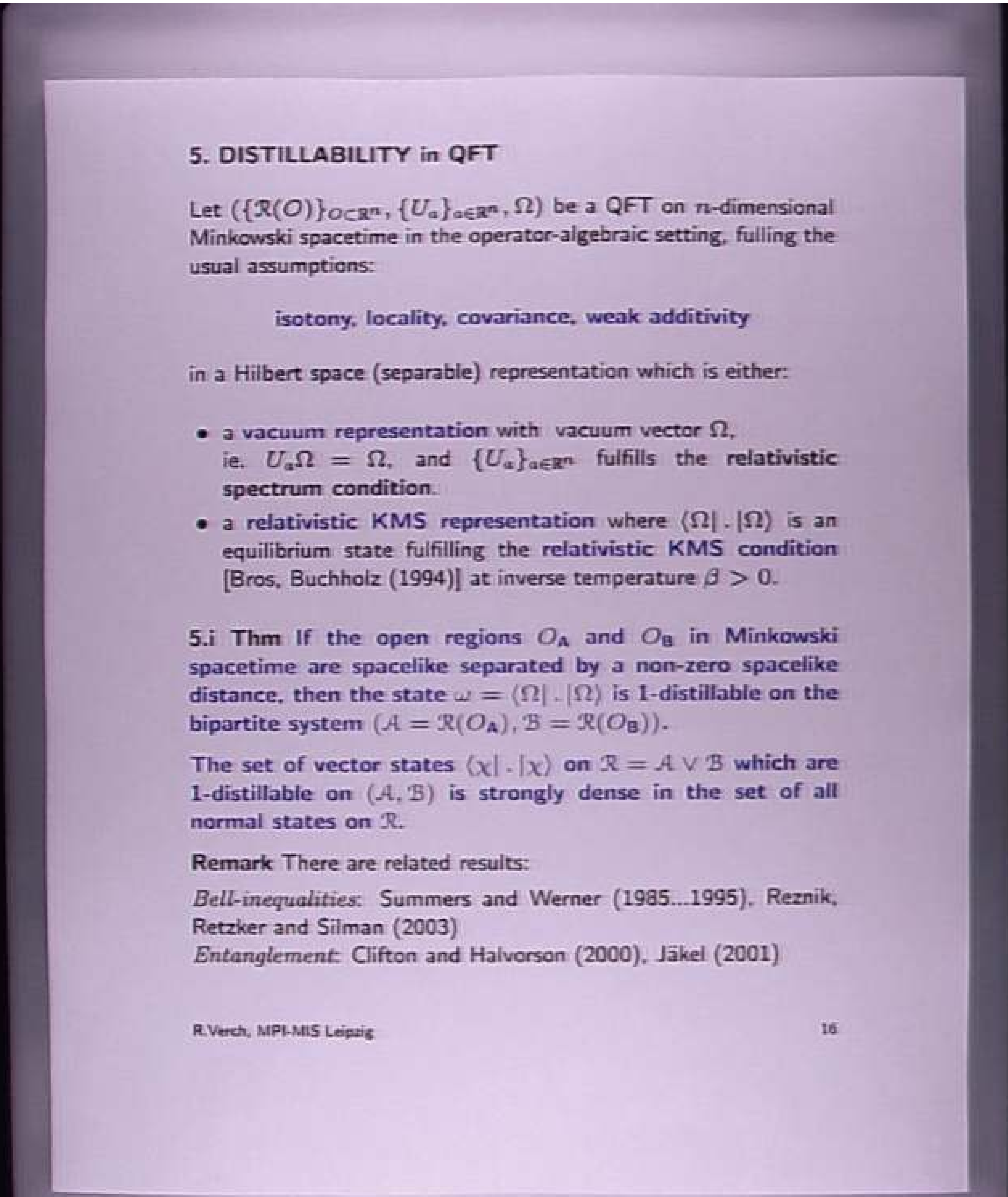
Standing assumptions: $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras on a common Hilbert space $\mathcal{H}$. The states considered are normal states.

4.i Def Let $(\mathcal{A}, \mathcal{B})$ be a bipartite quantum system. A vector $\psi \in \mathcal{H}$ is said to have the **Reeh-Schlieder property** with respect to $\mathcal{A}$ if ψ is **cyclic** for $\mathcal{A}$, ie.

$$\mathcal{A}\psi = \{A\psi | A \in \mathcal{A}\} \text{ is dense in } \mathcal{H}.$$

4.ii Thm Let $(\mathcal{A}, \mathcal{B})$ be a bipartite quantum system with both $\mathcal{A}$ and $\mathcal{B}$ non-abelian and let $\psi \in \mathcal{H}$ be a unit vector. Suppose that ψ is Reeh-Schlieder for $\mathcal{A}$. Then the state $\omega_\psi(\,.\,) = \langle\psi|\,.\,|\psi\rangle$ is 1-distillable.

$$\begin{tikzcd}
 & & \boxed{\text{violates Bell-ineqs}} \arrow[d, Rightarrow] & \\
\boxed{\text{Reeh-Schlieder property}} \arrow[r, Rightarrow] & \boxed{\text{1-distillable}} \arrow[d, Rightarrow] & & \\
 & \boxed{\text{distillable}} \arrow[r, Rightarrow] & \boxed{\text{not ppt}} \arrow[r, Rightarrow] & \boxed{\text{entangled}}
\end{tikzcd}$$



5. DISTILLABILITY in QFT

Let $(\{\mathcal{R}(O)\}_{O\subset\mathbb{R}^n}, \{U_a\}_{a\in\mathbb{R}^n}, \Omega)$ be a QFT on n-dimensional Minkowski spacetime in the operator-algebraic setting, fulling the usual assumptions:

isotony, locality, covariance, weak additivity

in a Hilbert space (separable) representation which is either:

- a **vacuum representation** with vacuum vector Ω, ie. $U_a\Omega = \Omega$, and $\{U_a\}_{a\in\mathbb{R}^n}$ fulfills the **relativistic spectrum condition**.
- a **relativistic KMS representation** where $\langle\Omega|\,.\,|\Omega\rangle$ is an equilibrium state fulfilling the **relativistic KMS condition** [Bros, Buchholz (1994)] at inverse temperature $\beta > 0$.

5.i Thm **If the open regions O_A and O_B in Minkowski spacetime are spacelike separated by a non-zero spacelike distance, then the state $\omega = \langle\Omega|\,.\,|\Omega\rangle$ is 1-distillable on the bipartite system $(\mathcal{A} = \mathcal{R}(O_A), \mathcal{B} = \mathcal{R}(O_B))$.**

The set of vector states $\langle\chi|\,.\,|\chi\rangle$ on $\mathcal{R} = \mathcal{A}\vee\mathcal{B}$ which are 1-distillable on $(\mathcal{A}, \mathcal{B})$ is strongly dense in the set of all normal states on $\mathcal{R}$.

Remark There are related results:

Bell-inequalities: Summers and Werner (1985...1995), Reznik, Retzker and Silman (2003)
Entanglement: Clifton and Halvorson (2000), Jäkel (2001)

5.ii Thm If the open regions O_A and O_B in M are causally separated by a non-zero spacelike distance, then the state $\omega = \langle \Omega | \, . \, | \Omega \rangle$ is 1-distillable on the bipartite system $(\mathcal{A} = \mathcal{R}(O_A), \mathcal{B} = \mathcal{R}(O_B))$.

The set of vector states $\langle \chi | \, . \, | \chi \rangle$ on $\mathcal{R} = \mathcal{A} \vee \mathcal{B}$ which are 1-distillable on $(\mathcal{A}, \mathcal{B})$ is strongly dense in the set of all normal states on $\mathcal{R}$.

Remark: "Distillability beyond spacetime horizons"

The regions O_A and O_B can be separated by a spacetime horizon (event horizon or cosmological horizon) — then one finds an abundance of distillable states on $(\mathcal{R}(O_A), \mathcal{R}(O_B))$ (but the actual distillation process requiring two-way classical communication between "Alice" localized in O_A and "Bob" localized in O_B cannot be carried out)

Example: Klein-Gordon field in representation of the Hartle-Hawking state on a static black hole spacetime.