Title: Distillability and positivity of partial transposes in general quantum field systems

Date: Jun 01, 2005 04:00 PM

URL: http://pirsa.org/05060056

Abstract:

# Concepts of Entanglement in General Quantum Field Theory

Rainer Verch

Max-Planck-Institute for Mathematics in the Sciences, Leipzig

joint work with Reinhard Werner
[quant-ph/0403089, to appear in Rev. Math. Phys.]

**1. GENERALITIES I: Algebraic description of general quantum systems and quantum field theories**

A quantum system is described by specifying:

- $\mathcal{H}$ : a Hilbertspace
- $\mathcal{R} \subset B(\mathcal{H})$ : a $*$-algebra of operators, where:
  - $A = A^* \in \mathcal{R}$ is interpreted as an observable
  - For $\psi \in \mathcal{H}$ with $||\psi|| = 1$, the quantity

$$\langle A \rangle_\psi = \langle \psi | A | \psi \rangle$$

is interpreted as the expectation value of the observable $A$ in the state given by $\psi$.

More generally: For $\rho =$ trace-class operator on $\mathcal{H}$ with $\rho \geq 0$, $\mathrm{trace}(\rho) = 1$, we interpret $\langle A \rangle_\rho = \mathrm{trace}(\rho\, A)$ as expectation value of $A$ in the state given by $\rho$.

**Remark i)** Often we want to consider **unbounded** operators as observables which are *not* contained in $B(\mathcal{H})$ — and thus we would often like $\mathcal{R}$ also to contain unbounded operators. But these can be viewed as limits of bounded operators and thus $\mathcal{R} \subset B(\mathcal{H})$ is not a physically relevant restriction.

**ii)** Often, $\mathcal{R} = B(\mathcal{H})$, but it can happen that $\mathcal{R}$ is a proper subalgebra of $B(\mathcal{H})$ — e.g. if $\mathcal{R}$ models the observables of a subsystem of a larger (ambient) system, or in systems at finite temperature. Typical: $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$, $\quad \mathcal{R} = B(\mathcal{H}_a) \otimes 1\!\!1$

## 1.1 Haag-Kastler operator algebraic framework of quantum field theory

A quantum field $F \mapsto \Phi(F)$ induces a family of operator algebras $\{\mathcal{R}(O)\}_{O\subset\mathbb{R}^4}$ as follows:

$$\mathcal{R}(O) = \{\text{(closed) *-subalgebra of } B(\mathcal{H}) \text{ generated by all } e^{i\Phi(F)}, \ F = \overline{F} \text{ supported in } O\}$$

The family of operator algebras then has the following properties:

a) **Isotony:** $O_1 \subset O_2 \Rightarrow \mathcal{R}(O_1) \subset \mathcal{R}(O_2)$

b) **Covariance:** $A \in \mathcal{R}(O) \Leftrightarrow U(L)AU(L)^* \in \mathcal{R}(L(O))$, or $U(L)\mathcal{R}(O)U(L)^* = \mathcal{R}(L(O))$, $L \in \mathfrak{P}_+^\uparrow$

c) **Locality:** If the space-time regions $O_1$ and $O_2$ are causally separated, then the corresponding operator algebras $\mathcal{R}(O_1)$ and $\mathcal{R}(O_2)$ commute elementwise:

$$A \in \mathcal{R}(O_1), \ B \in \mathcal{R}(O_2) \Rightarrow [A, B] = 0$$

d) **Cyclicity of the vacuum:** $\{A\Omega : A \in \bigcup_O \mathcal{R}(O)\}$ is dense in $\mathcal{H}$

e) **Weak additivity:** If $\bigcup_i O_i$ contains $O$, then the algebra generated by the $\mathcal{R}(O_i)$ contains $\mathcal{R}(O)$

f) **Spectrum condition and existence of the vacuum:** Writing $U_a = U(1, a) = e^{iP_\mu a^\mu}$, it holds that

$$P_0^2 - P_1^2 - P_2^2 - P_3^2 \geq 0 \quad \text{and} \quad P_0 \geq 0 .$$

(Positivity of the energy in all Lorentz frames)
$\exists\, \Omega \in \mathcal{D}$, $||\Omega|| = 1$, so that $U(L)\Omega = \Omega$ and this vector is uniquely determined up to a phase factor.
(Existence of a vacuum state, given by $\langle\,.\,\rangle_\Omega$)

If the data

$$(\{\mathcal{R}(O)\}_{O\subset\mathbb{R}^n}, \{U_a\}_{a\in\mathbb{R}^n}, \Omega)$$

fulfill these properties, we say that they describe a **quantum field theory in vacuum representation** on $n$-dim Minkowski spacetime.

Instead of a QFT in vacuum representation, one can also consider a QFT in a relativistic thermal representation.

We say that

$$(\{\mathcal{R}(O)\}_{O\subset\mathbb{R}^n}, \{U_a\}_{a\in\mathbb{R}^n}, \Omega)$$

is a **quantum field theory in a relativistic thermal representation at inverse temperature** $\beta > 0$ if:

There is a vector $e \in V_+$ having unit Minkowski length, for all $A, B \in \mathcal{R}(\mathbb{R}^n)$ there is a function $F_{AB}$ which is:

(1) analytic in $\{z \in \mathbb{C}^n | \mathrm{Im}(z) \in V_+ \cap (\beta e - V_+)\}$,

(2) continuous at boundaries $\mathrm{Im}(z) = 0$ and $\mathrm{Im}(z) = \beta e$

(3) boundary values $F_{AB}(a) = \langle\Omega|AU_a BU_a^*|\Omega\rangle$,
$$F_{AB}(a + i\beta e) = \langle\Omega|U_a BU_a^* A|\Omega\rangle$$

This is a relativistic generalization of the KMS-boundary condition characterizing thermal equilibrium states of infinite quantum systems due to Bros and Buchholz.



## 2. GENERALITIES II: Entanglement in quantum physics and quantum field theory

A quantum system modelled by an algebra of observables $\mathcal{R}$ may possess two (or more) subsystems modelled by two subalgebras $\mathcal{A}$ and $\mathcal{B}$

$\mathcal{A} \leftrightarrow$ observables controlled by **"Alice"**

$\mathcal{B} \leftrightarrow$ observables controlled by **"Bob"**

Standing assumptions:

- $\mathcal{A}$ and $\mathcal{B}$ are $C^*$-subalgebras (with $\mathbb{1}$) of $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$
- can take
$$\mathcal{A} \overset{C^*}{\vee} \mathcal{B} \subset \mathcal{R} \subset \mathcal{B}(\mathcal{H})$$
according to situation
- $\mathcal{A} \subset \mathcal{B}'$, i.e. $AB = BA$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$

**2.i Def** $(\mathcal{A}, \mathcal{B}) \subset \mathcal{R}$ is called a **bipartite quantum system**

**1.ii Def** Let $(\mathcal{A}, \mathcal{B}) \subset \mathcal{R}$ be a bipartite quantum system and $\omega(\,.\,) = \text{trace}(\rho_\omega\,.\,)$ a state on $\mathcal{R}$

Then $\omega$ is called **entangled** with resp. to $(\mathcal{A}, \mathcal{B})$ if it is **not separable**, i.e. not the (weak) limit of convex sums of product states over $(\mathcal{A}, \mathcal{B})$.

- $\omega$ is a **product state** over $(\mathcal{A}, \mathcal{B})$ iff
  $\exists$ states $\omega_a$ on $\mathcal{A}$, $\omega_b$ on $\mathcal{B}$ such that

$$\omega(AB) = \omega_a(A)\omega_b(B) \quad \forall\, A \in \mathcal{A} \;\; B \in \mathcal{B}.$$

- $\omega$ is a **convex sum of product states** if

$$\omega = \sum_{k=1}^{N} \lambda_k \omega^{(k)}\,, \qquad \lambda_k > 0\,,\ \sum_k \lambda_k = 1$$

  and each $\omega^{(k)}$ is a product state over $(\mathcal{A}, \mathcal{B})$.

- $\omega$ is **separable** if $\exists\ \omega_\alpha,\ \alpha \in \mathbb{J}$ so that

$$\omega(R) = \lim_\alpha \omega_\alpha(R) \quad \forall\, R \in \mathcal{R}$$

  and each $\omega_\alpha$ is a convex sum of product states over $(\mathcal{A}, \mathcal{B})$.

**Example:** The Bell state for a bipartite quantum system, each part having 2 degrees of freedom

$$\mathcal{A} = \mathcal{B}(\mathbb{C}^2)\,, \quad \mathcal{B} = \mathcal{B}(\mathbb{C}^2)$$
$$\mathcal{R} = \mathcal{B}(\mathbb{C}^2) \otimes \mathcal{B}(\mathbb{C}^2) \equiv \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)$$

$$\omega_{|\mathrm{Bell}\rangle}(A \otimes B) = \langle \mathrm{Bell}| A \otimes B |\mathrm{Bell}\rangle\,,$$

$$|\mathrm{Bell}\rangle = \frac{1}{\sqrt{2}}\left(\binom{1}{0} \otimes \binom{0}{1} - \binom{0}{1} \otimes \binom{1}{0}\right)$$

In experimental realization, the $\mathbb{C}^2$-states correspond to linear polarization of photon states:

$$\binom{1}{0} = |\updownarrow\rangle = \text{"vertically polarized photon state"}$$
$$\binom{0}{1} = |\leftrightarrow\rangle = \text{"horizontally polarized photon state"}$$

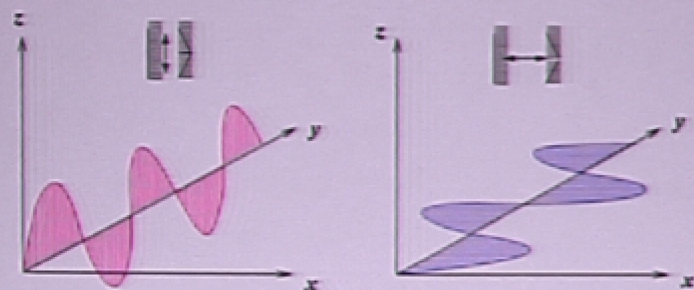
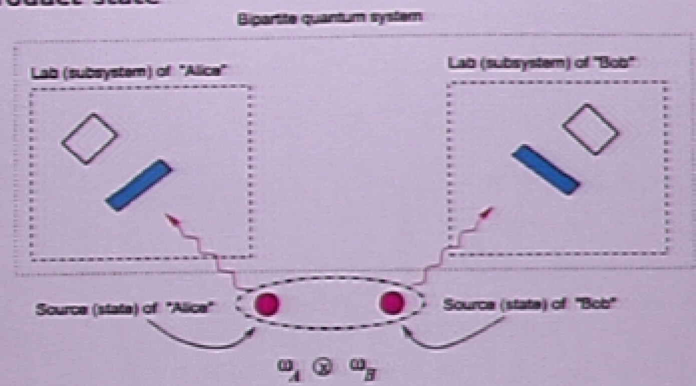
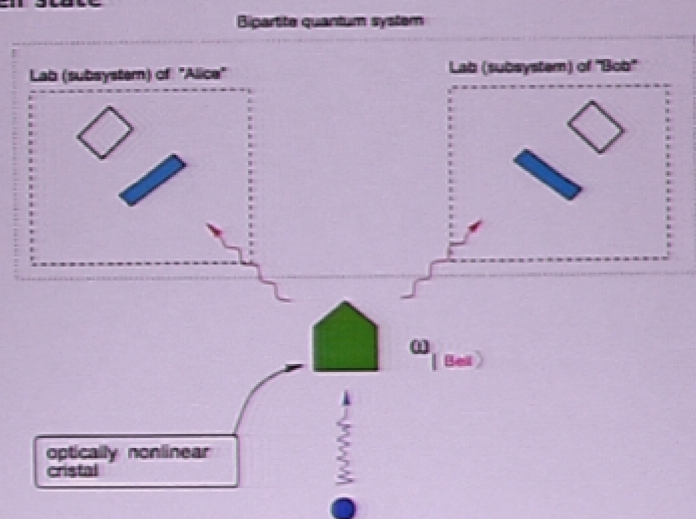


**Illustration:**

**Product state**




**Bell state**



## 3. QUALIFYING ENTANGLEMENT

### 3.1 States with Positive Partial Transpose (ppt states)

The original definition of ppt states is due to Peres (1996) and applies to the case of a bipartite quantum system with $\mathcal{A}, \mathcal{B} = B(\mathbb{C}^N)$, $N \in \mathbb{N}$:

Let $\omega(A \otimes B) = \text{trace}(\rho_\omega(A \otimes B))$ be a state on $B(\mathbb{C}^N) \otimes B(\mathbb{C}^N)$, then $\omega$ is called **ppt** if the **partial transpose** $\rho_\omega^{T_1}$ is **non-negative**, where

$$\langle e_k^{(a)} \otimes e_\ell^{(b)} | \rho_\omega^{T_1} | e_m^{(a)} \otimes e_n^{(b)} \rangle = \langle e_m^{(a)} \otimes e_\ell^{(b)} | \rho_\omega | e_k^{(a)} \otimes e_n^{(b)} \rangle$$

**Remark** $\rho_\omega^{T_1}$ depends on choice of bases, but the condition $\rho_\omega^{T_1} \geq 0$ doesn't.

For general quantum systems:

**3.i Def** Let $(\mathcal{A}, \mathcal{B}) \subset \mathcal{R}$ be a bipartite quantum system, and let $\omega$ be a state on $\mathcal{R}$. We call $\omega$ a **ppt state** if

$$\sum_{k,\ell} \omega(A_k A_\ell^* B_\ell^* B_k) \geq 0$$

holds for all choices of finitely many $A_1, \ldots, A_K \in \mathcal{A}$ and $B_1, \ldots, B_K \in \mathcal{B}$.

**3.ii Lemma** **For general bipartite systems:**

$\omega$ **separable** $\Rightarrow \omega$ **ppt** , **equivalently** ,

$\omega$ **non-ppt (npt)** $\Rightarrow \omega$ **entangled** .

## 3.2 Bell-CHSH Inequalities

**3.iii Def** Let $(\mathcal{A}, \mathcal{B}) \subset \mathcal{R}$ be a bipartite quantum system, and let $\omega$ be a state on $\mathcal{R}$. One calls

$$\beta(\omega) = \sup_{A,A',B,B'} \omega(A(B' + B) + A'(B' - B)),$$

where $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{B}$ are hermitean and norm-bounded by 1, the **Bell-correlation** of $\omega$.

- If $\beta(\omega) = 2$, one says that $\omega$ **fulfills the Bell inequalities**
- If $\beta(\omega) > 2$, one says that $\omega$ **violates the Bell inequalities**
- If $\beta(\omega) = 2\sqrt{2}$, one says that $\omega$ **violates the Bell inequalities maximally**

**3.iv The Bell state violates Bell's inqualities maximally**

States violating Bell's inequalities contain non-classical correlations (entanglement). The entanglement of the Bell state cannot be reproduced by classical 'protocols', and this makes this state particularly useful for tasks of quantum communication, mainly for **secure key distribution** and **quantum teleportation**.

**3.v Lemma A ppt state $\omega$ on a general bipartite quantum system fulfills Bell's inequalities.**
**Hence, a state violating Bell's inqualites is npt, therefore is entangled.**

1st ROUND OF GAME


2nd ROUND

3rd ROUND

⋮

VALID ANSWERS:

(A ? RED) & (B ? RED) ⇒ [A=0] & [B=1] OR [A=1] & [B=0]

ALL OTHER COMBINATIONS:

(A ? RED) & (B ? GREEN)
(A ? GREEN) & (B ? RED)
(A ? GREEN) & (B ? GREEN)

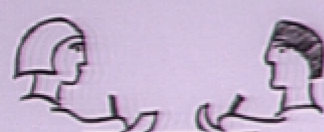
⇒ [A=0] & [B=0] OR [A=1] & [B=1]

ALICE AND BOB MAY "PLOT A PROTOCOL" BEFORE BEGINNING OF THE GAME

SUCCESS RATE

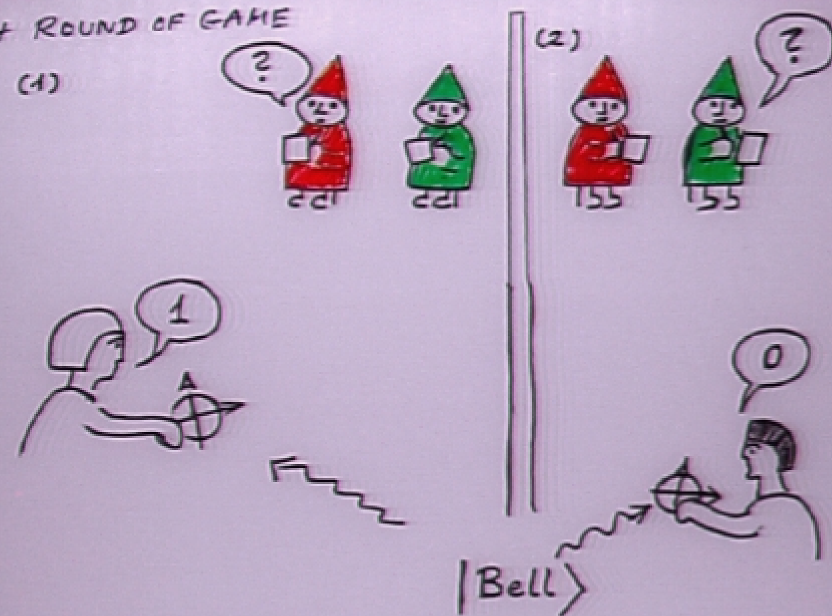
$$= \frac{\#\ \text{VALID ANSWERS}}{\#\ \text{ROUNDS}}$$

$$= \frac{3}{4} = 75\%$$

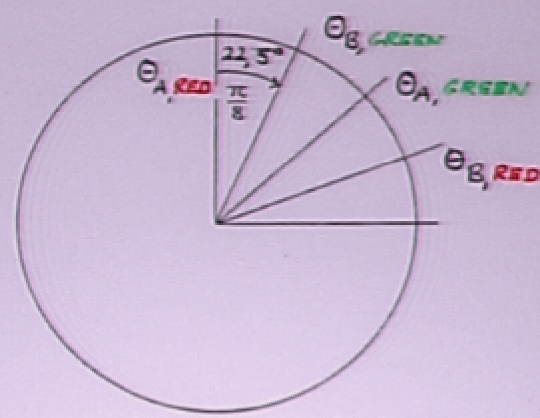


| A | B |
|---|---|
| 1. R=1, G=0 | 1. R=0, G=1 |
| 2. R=0, G=1 | 2. R=1, G=0 |
| 3. . | . |
| 4. . | . |

1st ROUND OF GAME




BEFORE BEGINNING OF THE GAME, ALICE AND BOB AGREE THAT:



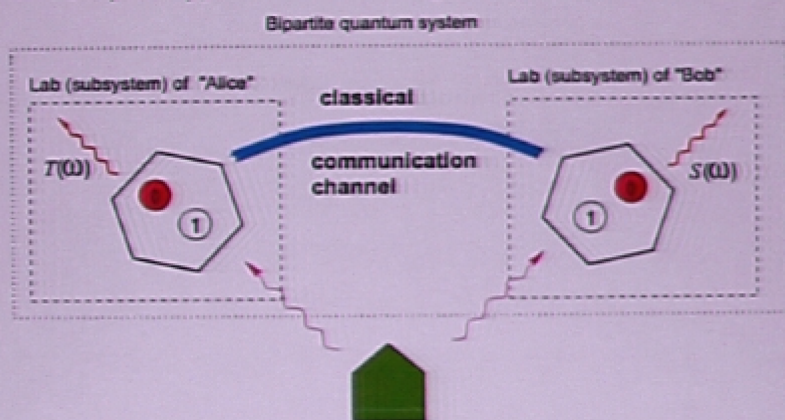

SUCCESS RATE IS NOW : $\frac{1}{2} + \frac{\sqrt{2}}{4} \approx 85{,}36\,\%$

## 3.3 Distillability

For a class of entangled states the 'degree of entanglement' can be enhanced (up to maximal violation of Bell's inequalities) for a sub-ensemble of the original state by a process called **distillation**. States for which this is possible are generally called **distillable**. There is in principle a large class of distillation processes. The common underlying idea is that of **LOCC**,

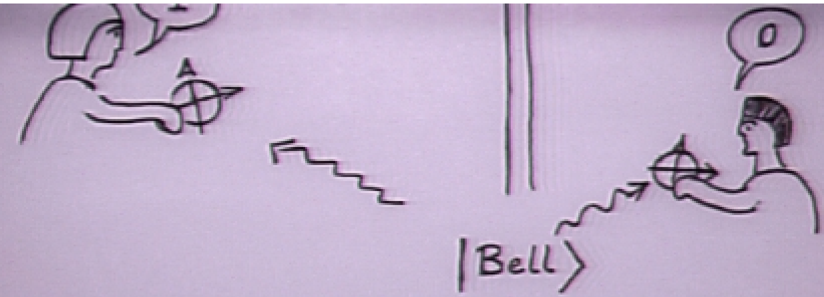
**local operations and classical communication** .

The simplest type is **1-distillability**:

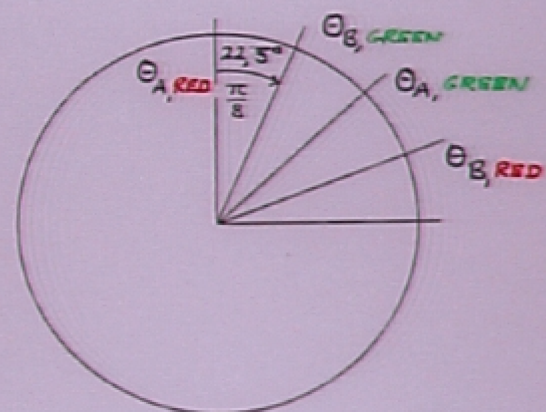



$$T(\omega)(AB) = \omega(\tau(A)B), \quad S(\omega)(AB) = \omega(A\sigma(B))$$

with $\tau : \mathcal{A} \to \mathcal{A}$ and $\sigma : \mathcal{B} \to \mathcal{B}$ completely positive.




BEFORE BEGINNING OF THE GAME, ALICE AND BOB AGREE THAT:




SUCCESS RATE IS NOW : $\frac{1}{2} + \frac{\sqrt{2}}{4} \approx 85{,}36\,\%$

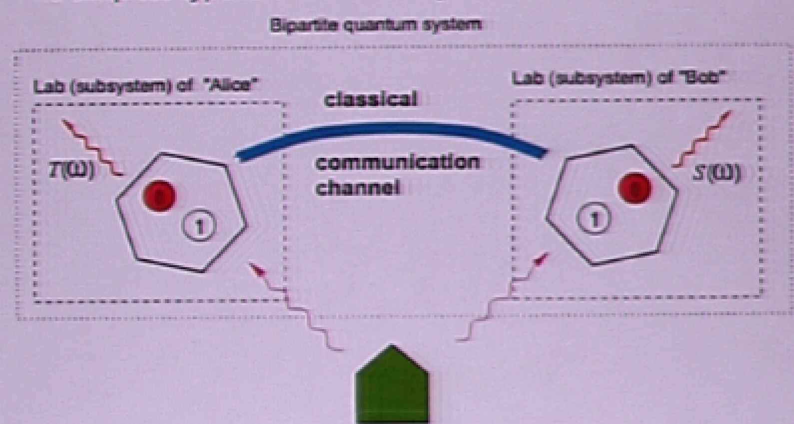
76

### 3.3 Distillability

For a class of entangled states the 'degree of entanglement' can be enhanced (up to maximal violation of Bell's inequalities) for a sub-ensemble of the original state by a process called **distillation**. States for which this is possible are generally called **distillable**. There is in principle a large class of distillation processes. The common underlying idea is that of **LOCC**,

**local operations and classical communication** .

The simplest type is **1-distillability**:




$$T(\omega)(AB) = \omega(\tau(A)B), \quad S(\omega)(AB) = \omega(A\sigma(B))$$

with $\tau : \mathcal{A} \to \mathcal{A}$ and $\sigma : \mathcal{B} \to \mathcal{B}$ completely positive.

**3.vi Def** Let $\omega$ be a state on a general bipartite quantum system $(\mathcal{A}, \mathcal{B}) \subset \mathcal{R}$.
$\omega$ is called **1-distillable** if for each $\varepsilon > 0$ there are

- a completely positive map $\tau : B(\mathbb{C}^2) \to \mathcal{A}$, $\tau(\mathbb{1}) \leq \mathbb{1}$, and
- a completely positive map $\sigma : B(\mathbb{C}^2) \to \mathcal{B}$, $\sigma(\mathbb{1}) \leq \mathbb{1}$,

so that

$$\left| \omega_{|\mathrm{Bell}\rangle}(X \otimes Y) - \omega(\tau(X)\sigma(Y))/\omega(\tau(\mathbb{1})\sigma(\mathbb{1})) \right| \leq \varepsilon \cdot ||X \otimes Y||$$

holds for all $X \otimes Y \in B(\mathbb{C}^2) \otimes B(\mathbb{C}^2)$.

**Remark** There are more general distillation processes, e.g. one could replace $\tau(\,.\,)\sigma(\,.\,)$ by $\sum_k \tau_k(\,.\,)\sigma_k(\,.\,)$.

It is important that operations of this type don't induce entanglement in non-entangled states. To this end, let completely positive maps

$$\tau_k : \mathcal{A}_{\mathrm{aux}} \to \mathcal{A}\,, \quad \sigma_k : \mathcal{B}_{\mathrm{aux}} \to \mathcal{B}\,, \quad k = 1, \ldots, K\,,$$

be given.

**3.vii Prop** Let $\omega$ be a state on the bipartite quantum system $(\mathcal{A}, \mathcal{B})$. If $\omega$ is ppt, then the positive functional on the bipartite system $(\mathcal{A}_{\mathrm{aux}}, \mathcal{B}_{\mathrm{aux}})$,

$$\omega_{\mathrm{aux}}(A_{\mathrm{aux}} B_{\mathrm{aux}}) = \omega\Big( \sum_k \tau_k(A_{\mathrm{aux}})\sigma_k(B_{\mathrm{aux}}) \Big)$$

is also ppt.

In particular, a ppt state $\omega$ is not distillable by this kind of a LOCC process.

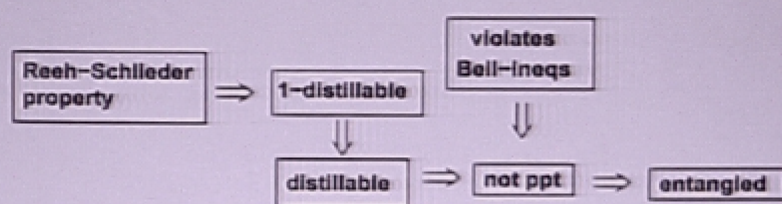
## 4. REEH-SCHLIEDER and DISTILLABILITY

**Standing assumptions:** $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras on a common Hilbert space $\mathcal{H}$. The states considered are normal states.

**4.i Def** Let $(\mathcal{A}, \mathcal{B})$ be a bipartite quantum system. A vector $\psi \in \mathcal{H}$ is said to have the **Reeh-Schlieder property** with respect to $\mathcal{A}$ if $\psi$ is **cyclic** for $\mathcal{A}$, ie.

$$\mathcal{A}\psi = \{A\psi | A \in \mathcal{A}\} \text{ is dense in } \mathcal{H}.$$

**4.ii Thm Let $(\mathcal{A}, \mathcal{B})$ be a bipartite quantum system with both $\mathcal{A}$ and $\mathcal{B}$ non-abelian and let $\psi \in \mathcal{H}$ be a unit vector. Suppose that $\psi$ is Reeh-Schlieder for $\mathcal{A}$. Then the state $\omega_\psi(\,.\,) = \langle\psi|\,.\,|\psi\rangle$ is 1-distillable.**



## 5. DISTILLABILITY in QFT

Let $(\{\mathcal{R}(O)\}_{O\subset\mathbb{R}^n}, \{U_a\}_{a\in\mathbb{R}^n}, \Omega)$ be a QFT on $n$-dimensional Minkowski spacetime in the operator-algebraic setting, fulling the usual assumptions:

**isotony, locality, covariance, weak additivity**

in a Hilbert space (separable) representation which is either:

- a **vacuum representation** with vacuum vector $\Omega$, ie. $U_a\Omega = \Omega$, and $\{U_a\}_{a\in\mathbb{R}^n}$ fulfills the **relativistic spectrum condition**.
- a **relativistic KMS representation** where $\langle\Omega|\,.\,|\Omega\rangle$ is an equilibrium state fulfilling the **relativistic KMS condition** [Bros, Buchholz (1994)] at inverse temperature $\beta > 0$.

**5.i Thm** **If the open regions $O_{\mathbf{A}}$ and $O_{\mathbf{B}}$ in Minkowski spacetime are spacelike separated by a non-zero spacelike distance, then the state $\omega = \langle\Omega|\,.\,|\Omega\rangle$ is 1-distillable on the bipartite system $(\mathcal{A} = \mathcal{R}(O_{\mathbf{A}}), \mathcal{B} = \mathcal{R}(O_{\mathbf{B}}))$.**

**The set of vector states $\langle\chi|\,.\,|\chi\rangle$ on $\mathcal{R} = \mathcal{A} \vee \mathcal{B}$ which are 1-distillable on $(\mathcal{A}, \mathcal{B})$ is strongly dense in the set of all normal states on $\mathcal{R}$.**

**Remark** There are related results:

*Bell-inequalities*: Summers and Werner (1985...1995), Reznik, Retzker and Silman (2003)
*Entanglement*: Clifton and Halvorson (2000), Jäkel (2001)

**5.ii Thm** If the open regions $O_A$ and $O_B$ in $M$ are causally separated by a non-zero spacelike distance, then the state $\omega = \langle\Omega|\,.\,|\Omega\rangle$ is 1-distillable on the bipartite system $(\mathcal{A} = \mathcal{R}(O_A), \mathcal{B} = \mathcal{R}(O_B))$.

The set of vector states $\langle\chi|\,.\,|\chi\rangle$ on $\mathcal{R} = \mathcal{A} \vee \mathcal{B}$ which are 1-distillable on $(\mathcal{A}, \mathcal{B})$ is strongly dense in the set of all normal states on $\mathcal{R}$.

**Remark: "Distillability beyond spacetime horizons"**

The regions $O_A$ and $O_B$ can be separated by a spacetime horizon (event horizon or cosmological horizon) — then one finds an abundance of distillable states on $(\mathcal{R}(O_A), \mathcal{R}(O_B))$ (but the actual distillation process requiring two-way classical communication between "Alice" localized in $O_A$ and "Bob" localized in $O_B$ cannot be carried out)

**Example:** Klein-Gordon field in representation of the Hartle-Hawking state on a static black hole spacetime.