Title: A unified and generalized approach to quantum error correction

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Abstract: In this talk I'll discuss recent joint work with Raymond Laflamme, David Poulin and Maia Lesosky in which a unified approach to quantum error correction is presented, called "operator quantum error correction". This scheme relies on a generalized notion of noiseless subsystems and includes the known techniques for the error correction of quantum operations --i.e., the standard model, the method of decoherence-free subspaces, and the noiseless subsystem method--as special cases. Correctable codes in this approach take the form of operator algebras and operator semigroups. The condition from the standard model is shown to be necessary for all of the known methods of error correction, and we'll see this as part of a discussion on conditions that characterize correctability for the general case.

Operator Quantum Error Correction

David Kribs

Joint work with R. Laflamme, D. Poulin, M. Lesosky

Department of Mathematics and Statistics

University of Guelph

Institute for Quantum Computing

University of Waterloo

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Outline

I. Introduction

II. Evolution of Quantum Systems

III. Error Correction of Quantum Operations: Standard Model, Noiseless Subsystems and Decoherence-Free Subspaces

IV. Generalized Noiseless Subsystems

V. Unified Approach – "Operator Quantum Error Correction"

VI. Conclusions for Correction

VII. Conclusion

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Introduction

- For every quantum system, there is an associated Hilbert space $\mathcal{H}$.
  $\Rightarrow \mathcal{H} = \mathbb{C}^n$ for some $n \geq 1$.

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Evolution of Quantum Systems

(A) Closed Quantum Systems:

- $U : \mathcal{H} \to \mathcal{H}$ unitary, $|\psi\rangle \in \mathcal{H}$ state;

$$|\psi\rangle \mapsto U|\psi\rangle$$

Evolution of Quantum Systems

(A) Closed Quantum Systems:

- $U : \mathcal{H} \to \mathcal{H}$ unitary, $|\psi\rangle \in \mathcal{H}$ state;

$$|\psi\rangle \mapsto U|\psi\rangle$$

- At the level of operators this can be written as

$$\rho \mapsto U\rho U^\dagger,$$

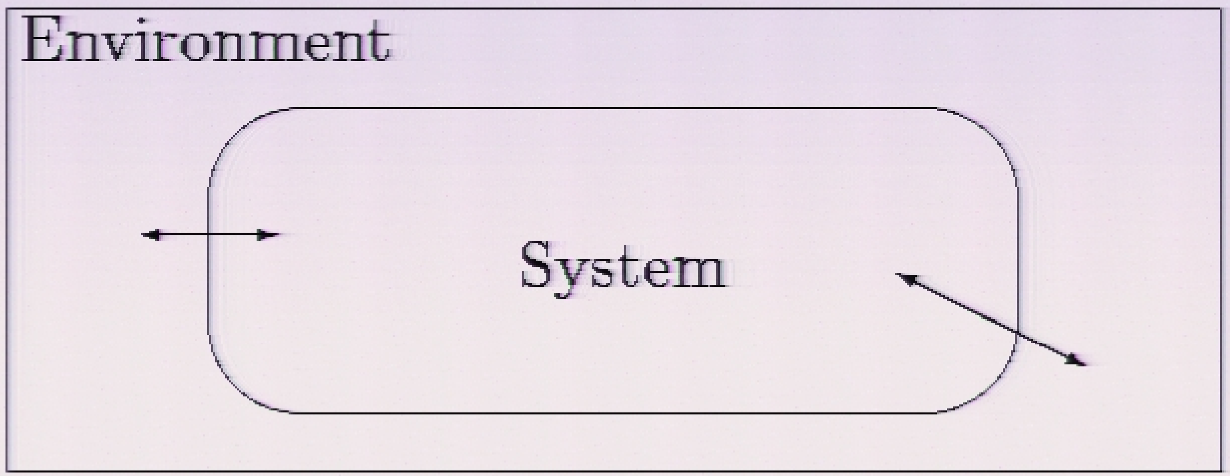
where ρ is a *density operator* ($\rho \geq 0$ and trace$(\rho) = 1$).

- As a "superoperator" $\mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \equiv$ operators on $\mathcal{H}$,

$$\mathcal{E}(\rho) = U\rho U^\dagger.$$

(B) Open Quantum Systems:

Figure 1:



$$\mathcal{H} = \mathcal{H}_E \otimes \mathcal{H}_S$$

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- A given evolution map

$$\mathcal{E} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$$

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must be **positive and trace preserving.**

- We also need the same for $id \otimes \mathcal{E}$ on $\mathcal{B}(\mathcal{H}_E \otimes \mathcal{H}_S)$ for any choice of E. i.e. $\mathcal{E}$ is *completely positive* and trace preserving.

- **Theorem.** (Choi-Kraus) Let $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ be a quantum operation. Then there are operators $\{A_i\}$ inside $\mathcal{B}(\mathcal{H})$ such that

$$\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger \quad \forall \rho \quad -(CP)$$

$$\sum_i A_i^\dagger A_i = I. \quad -(TP)$$

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- We write $\mathcal{E} = \{A_i\}$. The A_i are the *errors* or *noise operators* associated with $\mathcal{E}$.

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Error Correction of Quantum Operations

Basic Problem: Correct for the errors $\mathcal{E} = \{A_i\}$ associated with a given quantum operation $\mathcal{E}$.

e.g. $\mathcal{E} = \{U\}$

$$\begin{array}{lll} \rho & \mapsto & U\rho U^\dagger \quad \text{error} \\ & \mapsto & U^\dagger(U\rho U^\dagger)U = \rho \quad \text{recovery} \end{array}$$

Standard Model for Quantum Error Correction:

(Shor, Steane, Bennett-DiVincenzo-Smolin-Wootters, Knill-Laflamme)

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$$(\mathcal{R} \circ \mathcal{E})(\rho) = \rho \quad \forall \rho \in \mathcal{B}(\mathcal{C}) \subseteq \mathcal{B}(\mathcal{H}).$$

- The Standard Model consists of triples $(\mathcal{R}, \mathcal{E}, \mathcal{C})$. Note that we could focus on $\mathcal{B}(\mathcal{C})$ instead of $\mathcal{C}$.

Example

- Let $\mathcal{C} = \text{span}\{|00\rangle, |11\rangle\} \subseteq \mathbb{C}^4$ and

$$\mathcal{E} = \left\{ \frac{1}{\sqrt{2}} \mathbb{1}_4, \frac{1}{\sqrt{2}} X_1 \right\} \quad \text{where} \quad X_1 = X \otimes \mathbb{1}_2.$$

- **Theorem.** (K-L, B-D-S-W) The following are equivalent:

  (1) $\mathcal{C}$ is correctable for $\mathcal{E} = \{A_i\}$.

  (2) There are scalars $\Lambda = (\lambda_{ij})$ such that

$$P_{\mathcal{C}} A_i^\dagger A_j P_{\mathcal{C}} = \lambda_{ij} P_{\mathcal{C}} \quad \forall i, j.$$

- In the previous example, $\Lambda = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$.

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Noiseless Subsystems and Decoherence-Free Subspaces:

(Palma-Suominen-Ekert, Duan-Guo, Zanardi-Rasetti, Lidar-Chuang-Whaley, Knill-Laflamme-Viola, Zanardi, Kempe-Bacon-Lidar-Whaley)

- Given $\mathcal{E} = \{A_i\}$, let $\mathcal{A} = \mathrm{C}^*(\{A_i\}) = \mathrm{Alg}\{A_i, A_i^\dagger\}$ be the *interaction algebra* associated with $\mathcal{E}$.

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- The corresponding *noise commutant* is the algebra

$$\mathcal{A}' = \{\rho \in \mathcal{B}(\mathcal{H}) : \rho A_i = A_i \rho \ \ \forall i\}.$$

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- Assume $\mathcal{E}$ is unital (so $E(\mathbb{1}) = \mathbb{1}$). Then

$$\rho \in \mathcal{A}' \Rightarrow \mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger = \rho \mathcal{E}(\mathbb{1}) = \rho.$$

In fact...

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- As a finite dimensional C*-algebra, there are positive integers $m_k, n_k \geq 1$ such that

$$\text{Fix}(\mathcal{E}) = \mathcal{A}' \cong \oplus_k (\mathbb{1}_{m_k} \otimes \mathcal{M}_{n_k}).$$

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$$\mathrm{Fix}(\mathcal{E}) = \mathcal{A}' \cong \oplus_k (\mathbb{1}_{m_k} \otimes \mathcal{M}_{n_k}).$$

- Hence, $\rho \in \mathbb{1}_{m_k} \otimes \mathcal{M}_{n_k} \Rightarrow \mathcal{E}(\rho) = \rho$.

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- i.e. $\rho_0 \in \mathcal{M}_n \Rightarrow (id \circ \mathcal{E})(\mathbb{1}_m \otimes \rho_0) = \mathbb{1}_m \otimes \rho_0$.

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$\Rightarrow$ $\mathbb{1}_m \otimes \mathcal{M}_n$ is a *noiseless subsystem* (*decoherence-free subspace* when $m = 1$) for $\mathcal{E}$.

- **Note.** The commutant $(\mathbb{1}_m \otimes \mathcal{M}_n)' \cong \mathcal{M}_m \otimes \mathbb{1}_n$ contains "matrix units" $\{P_{kl} : 1 \leq k, l \leq m\}$.
- e.g. For $m = 2 = n$,

$$P_{11} = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & 0 \end{pmatrix} \quad P_{12} = \begin{pmatrix} 0 & \mathbb{1}_2 \\ 0 & 0 \end{pmatrix}$$

$$P_{21} = \begin{pmatrix} 0 & 0 \\ \mathbb{1}_2 & 0 \end{pmatrix} \quad P_{22} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}$$

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Generalized Noiseless Subsystems

- Assume we have a decomposition

$$\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}.$$

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Generalized Noiseless Subsystems

- Assume we have a decomposition

$$\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}.$$

- Standard NS (for $\mathcal{E}$ unital) we have

$$\mathcal{E}(\mathbb{1}^A \otimes \sigma^B) = \mathbb{1}^A \otimes \sigma^B \quad \forall \sigma^B \in \mathcal{B}(\mathcal{H}^B).$$

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- **Lemma.** The following conditions are equivalent, and are the defining properties for $\mathcal{H}^B$ to be a *(generalized) noiseless subsystem* for $\mathcal{E}$:
  (1) $\forall \sigma^A \, \forall \sigma^B, \, \exists \tau^A \; : \; \mathcal{E}(\sigma^A \otimes \sigma^B) = \tau^A \otimes \sigma^B$
  (2) $\forall \sigma^B, \, \exists \tau^A \; : \; \mathcal{E}(\mathbb{1}^A \otimes \sigma^B) = \tau^A \otimes \sigma^B$
  (3) $\forall \sigma \in \mathfrak{A} \; : \; \left(\mathrm{Tr}_A \circ \mathcal{P}_{\mathfrak{A}} \circ \mathcal{E} \right)(\sigma) = \mathrm{Tr}_A(\sigma).$

Example.

- Let $\Psi = \{E_0, E_1\}$ on $\mathcal{B}(\mathbb{C}^2)$ – "spontaneous emission" – where $E_0 = |0\rangle\langle 0|$, $E_1 = |0\rangle\langle 1|$ and let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a quantum operation with a NS $\mathcal{H}^B \subseteq \mathcal{H}$, so that $\Phi(\sigma) = \sigma$ for all $\sigma \in \mathcal{B}(\mathcal{H}^B)$.

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- Let $\mathcal{E} = \Psi \otimes \Phi$ on $\mathcal{B}(\mathbb{C}^2 \otimes \mathcal{H})$. Then $\mathcal{H}^B$ is noiseless for $\mathcal{E}$ and $\forall \rho \in \mathcal{B}(\mathbb{C}^2)$, $\forall \sigma \in \mathcal{B}(\mathcal{H}^B)$,

$$\mathcal{E}(\rho \otimes \sigma) = \Psi(\rho) \otimes \Phi(\sigma) = \Big(\sum_i E_i \rho E_i^\dagger\Big) \otimes \sigma.$$

- Given a decomposition $\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}$, the previous Lemma suggests consideration of the operator semigroup inside $\mathcal{B}(\mathcal{H})$,

$$\mathfrak{A} = \{\sigma^A \otimes \sigma^B : \sigma^A \in \mathcal{B}(\mathcal{H}^A), \sigma^B \in \mathcal{B}(\mathcal{H}^B)\}.$$

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- Suppose $\dim \mathcal{H}^A = m$, $\dim \mathcal{H}^B = n$. Let

$$\{P_{kl} : 1 \leq k, l \leq m\}$$

be a family of matrix units for $\mathcal{B}(\mathcal{H}^A) \otimes \mathbb{1}^B$. Further let $P_{\mathfrak{A}}$ be the projection of $\mathcal{H}$ onto $\mathcal{H}^A \otimes \mathcal{H}^B$ and let $\mathcal{P}_{\mathfrak{A}}$ be the map $\mathcal{P}_{\mathfrak{A}}(\cdot) = P_{\mathfrak{A}}(\cdot)P_{\mathfrak{A}}$.

- **Theorem.** Let $\mathcal{E} = \{E_i\}$ be a quantum operation on $\mathcal{B}(\mathcal{H})$ and let $\mathfrak{A}$ be a semigroup in $\mathcal{B}(\mathcal{H})$ as above. Then the following statements are equivalent:
(1) $\mathfrak{A}$ encodes a noiseless subsystem for $\mathcal{E}$ (decoherence-free subspace in the case m=1), as in the previous Lemma.
(2) The following two conditions hold:

$$P_{kk}E_iP_{ll} = \lambda_{ikl}P_{kl} \quad \forall\, i,k,l$$

for some set of scalars $\{\lambda_{ikl}\}$ and

$$E_iP_{\mathfrak{A}} = P_{\mathfrak{A}}E_iP_{\mathfrak{A}} \quad \forall\, i.$$

- In particular, each E_i must have a matrix representation with respect to $\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \otimes K$ of the form

$$E_i = \begin{pmatrix} (\lambda_{ikl} \mathbb{1}^B) & * \\ 0 & * \end{pmatrix}.$$

Unified Approach: "Operator Quantum Error Correction"

- The unified scheme consists of triples $(\mathcal{R}, \mathcal{E}, \mathfrak{A})$ where $\mathcal{R}$ and $\mathcal{E}$ are quantum operations on some $\mathcal{B}(\mathcal{H})$, and $\mathfrak{A}$ is a semigroup in $\mathcal{B}(\mathcal{H})$ defined as above with respect to a fixed decomposition $\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}$.

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- **Definition.** Given such a triple $(\mathcal{R}, \mathcal{E}, \mathfrak{A})$ we say that $\mathfrak{A}$ is *correctable for* $\mathcal{E}$ if

$$\left(\mathrm{Tr}_A \circ \mathcal{P}_{\mathfrak{A}} \circ \mathcal{R} \circ \mathcal{E}\right)(\sigma) = \mathrm{Tr}_A(\sigma) \quad \text{for all} \quad \sigma \in \mathfrak{A}.$$

- i.e. $(\mathcal{R}, \mathcal{E}, \mathfrak{A})$ is a correctable triple if the $\mathcal{H}^B$ sector of the semigroup $\mathfrak{A}$ encodes a (generalized) noiseless subsystem for the error map $\mathcal{R} \circ \mathcal{E}$.

- **Theorem.** The following conditions are equivalent:

(1) $\mathfrak{A}$ is correctable for $\mathcal{E}$.

(2) $\forall \sigma^A \, \forall \sigma^B, \, \exists \tau^A \, : \, (\mathcal{R} \circ \mathcal{E})(\sigma^A \otimes \sigma^B) = \tau^A \otimes \sigma^B$.

(3) $\forall \sigma^B, \, \exists \tau^A \, : \, (\mathcal{R} \circ \mathcal{E})(\mathbb{1}^A \otimes \sigma^B) = \tau^A \otimes \sigma^B$.

(4) $\forall \sigma \in \mathfrak{A} \, : \, \big(\mathrm{Tr}_A \circ \mathcal{P}_{\mathfrak{A}} \circ \mathcal{R} \circ \mathcal{E}\big)(\sigma) = \mathrm{Tr}_A(\sigma)$.

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- Table: Special Cases of Operator QEC

Given an error triple $(\mathcal{R}, \mathcal{E}, \mathfrak{A})$:

$\mathfrak{A} =$ subspace	Standard QEC
$\mathcal{R} = id$	Generalized NS
$\mathcal{R} = id \ + \ \mathfrak{A} =$ algebra	Standard NS
$\mathcal{R} = id \ + \ \mathfrak{A} =$ subspace	DFS

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Conditions for Correction

- Let $\mathfrak{A}_0$ be the C*-algebra $\mathfrak{A}_0 = \mathbb{1}^A \otimes \mathcal{B}(\mathcal{H}^B) \subseteq \mathfrak{A}$.

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Conditions for Correction

- Let $\mathfrak{A}_0$ be the C*-algebra $\mathfrak{A}_0 = \mathbb{1}^A \otimes \mathcal{B}(\mathcal{H}^B) \subseteq \mathfrak{A}$.
- Correctability of a given semigroup $\mathfrak{A}$ is equivalent to the *precise* correction of the algebra $\mathfrak{A}_0$:
- **Theorem.** Let $\mathcal{E} = \{E_i\}$ be a quantum operation on $\mathcal{B}(\mathcal{H})$ and let $\mathfrak{A}$ be a semigroup in $\mathcal{B}(\mathcal{H})$ as above. Then $\mathfrak{A}$ is correctable for $\mathcal{E}$ if and only if there is a quantum operation $\mathcal{R}$ on $\mathcal{B}(\mathcal{H})$ such that

$$(\mathcal{R} \circ \mathcal{E})(\sigma) = \sigma \quad \forall\, \sigma \in \mathfrak{A}_0.$$

- **Theorem.** Let $\mathcal{E} = \{E_i\}$ be a quantum operation on $\mathcal{B}(\mathcal{H})$ and let $\mathfrak{A}$ be a semigroup in $\mathcal{B}(\mathcal{H})$ as above. If $\mathfrak{A}$ is correctable for $\mathcal{E}$, then there are scalars $\Lambda = \{\lambda_{ijkl}\}$ such that

$$P_{kk} E_i^\dagger E_j P_{ll} = \lambda_{ijkl} P_{kl} \quad \text{for all} \quad i, j, k, l.$$

- **Theorem.** Let $\mathcal{E} = \{E_i\}$ be a quantum operation on $\mathcal{B}(\mathcal{H})$ and let $\mathfrak{A}$ be a semigroup in $\mathcal{B}(\mathcal{H})$ as above. If $\mathfrak{A}$ is correctable for $\mathcal{E}$, then there are scalars $\Lambda = \{\lambda_{ijkl}\}$ such that

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- The converse implication holds under the additional hypothesis:

$$\lambda_{ijkl} = 0 \;\forall\, i \neq j \quad \text{and} \quad |\lambda_{iikl}|^2 = \lambda_{iikk} \lambda_{iill} \;\forall\, i, k, l.$$

- **Remark.** Observe that setting $k = l$ in the equation above gives the condition from the Standard Model with $P_C = P_{kk}$.

- **Remark.** Observe that setting $k = l$ in the equation above gives the condition from the Standard Model with $P_C = P_{kk}$.
- It follows from this that the standard error correction condition is necessary for the existence of correctable codes in *all* of the known schemes for the error correction of quantum operations (Standard NS, Generalized NS, Operator QEC, etc).

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- **Theorem.** Let $\mathcal{E} = \{E_i\}$ be a quantum operation on $\mathcal{B}(\mathcal{H})$ and let $\mathfrak{A}$ be a semigroup in $\mathcal{B}(\mathcal{H})$ as above. If $\mathfrak{A}$ is correctable for $\mathcal{E}$, then there are scalars $\Lambda = \{\lambda_{ijkl}\}$ such that

$$P_{kk}E_i^\dagger E_j P_{ll} = \lambda_{ijkl} P_{kl} \quad \text{for all} \quad i,j,k,l.$$

- The converse implication holds under the additional hypothesis:

$$\lambda_{ijkl} = 0 \ \forall\, i \neq j \quad \text{and} \quad |\lambda_{iikl}|^2 = \lambda_{iikk}\lambda_{iill} \ \forall\, i,k,l.$$

- **Remark.** Observe that setting $k = l$ in the equation above gives the condition from the Standard Model with $P_C = P_{kk}$.

- It follows from this that the standard error correction condition is necessary for the existence of correctable codes in *all* of the known schemes for the error correction of quantum operations (Standard NS, Generalized NS, Operator QEC, etc).

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Conclusion

- *Operator quantum error correction* is a scheme for the error correction of quantum operations that includes the Standard Model, and the NS and DFS methods as special cases.

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Conclusion

- *Operator quantum error correction* is a scheme for the error correction of quantum operations that includes the Standard Model, and the NS and DFS methods as special cases.
- In addition, Operator QEC includes a generalized version of NS that applies to arbitrary (not necessarily unital) quantum operations.
- A consequence of the error correction conditions derived is that the condition from the Standard Model is a requisite for any of these schemes to be feasible.

References

1. D.W. Kribs, R. Laflamme, D. Poulin, *A unified and generalized approach to quantum error correction*, Phys. Rev. Lett., **94** (2005), 180501.
2. D.W. Kribs, R. Laflamme, D. Poulin, M. Lesosky, *Operator quantum error correction*, arxiv.org/quant-ph/0504189.

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