Title: Self-organized criticality

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Abstract:

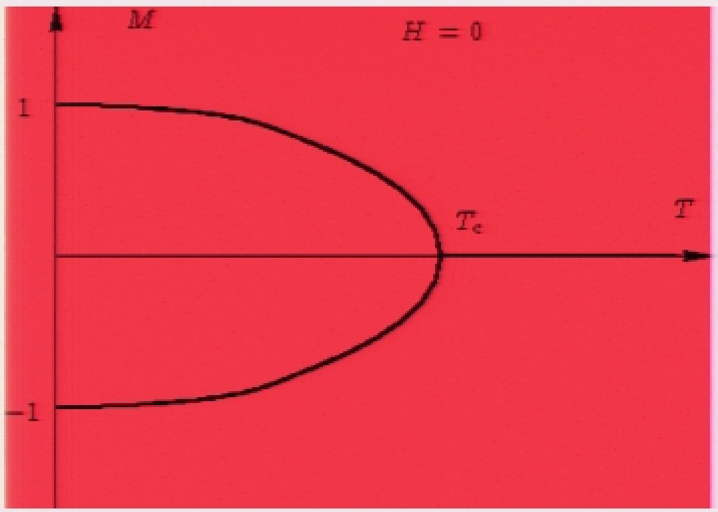
Self-organized Criticality

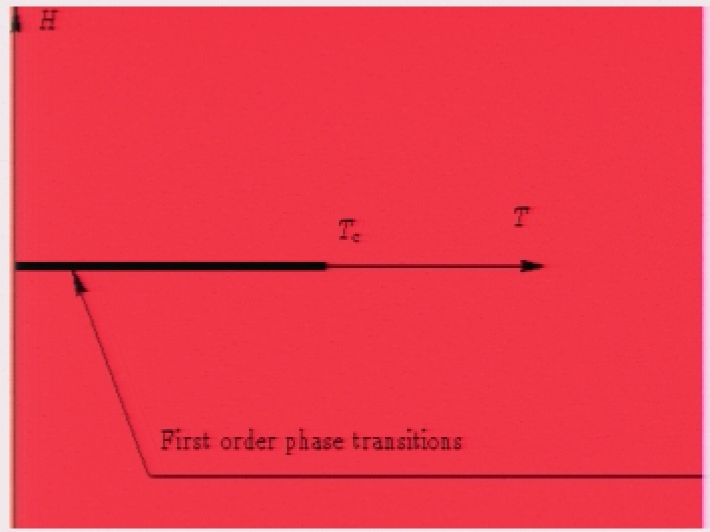
Maya Paczuski

What is complexity?

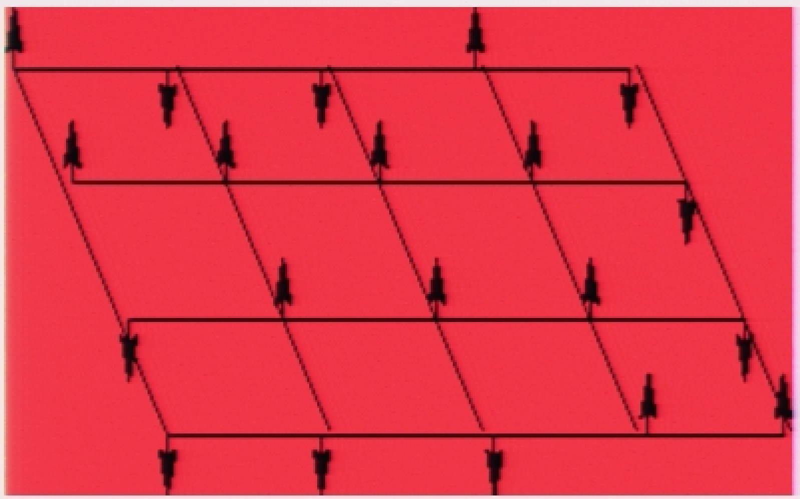
- No formal definition
- A pedestrian approach
  - Out of equilibrium
  - Variations across many different space and time scales – “emergence”.
  - Same patterns observed in many different physical, biological, and social phenomena
  - Dynamical models (SOC)

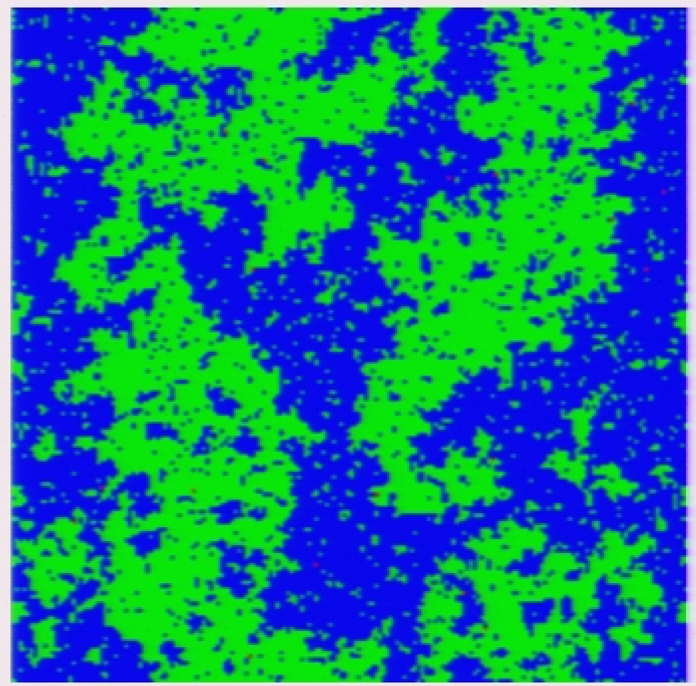
Critical points in equilibrium (magnets)





What is criticality?
(2d Ising Model)



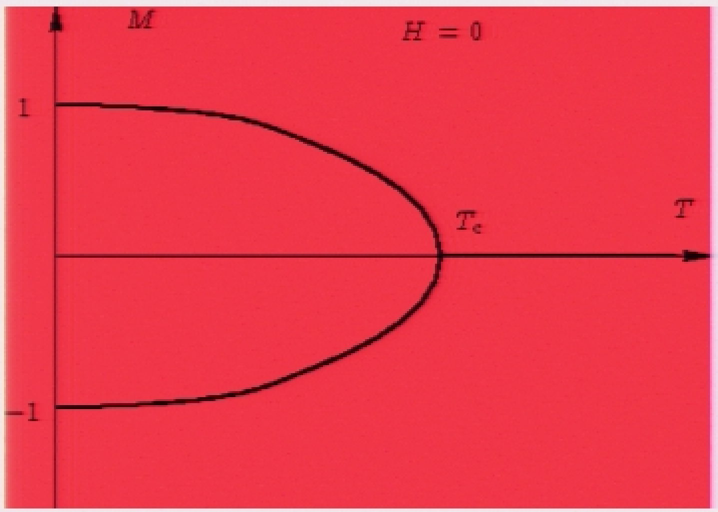


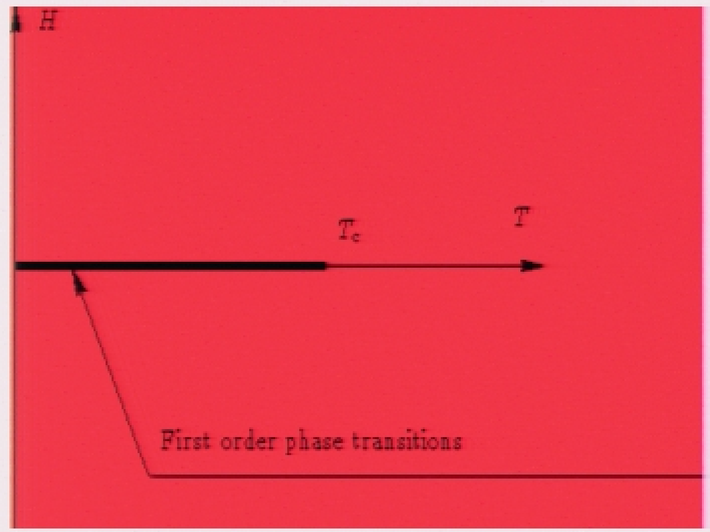
ξ diverges as T approaches Tc

Fractal structure, long time correlations

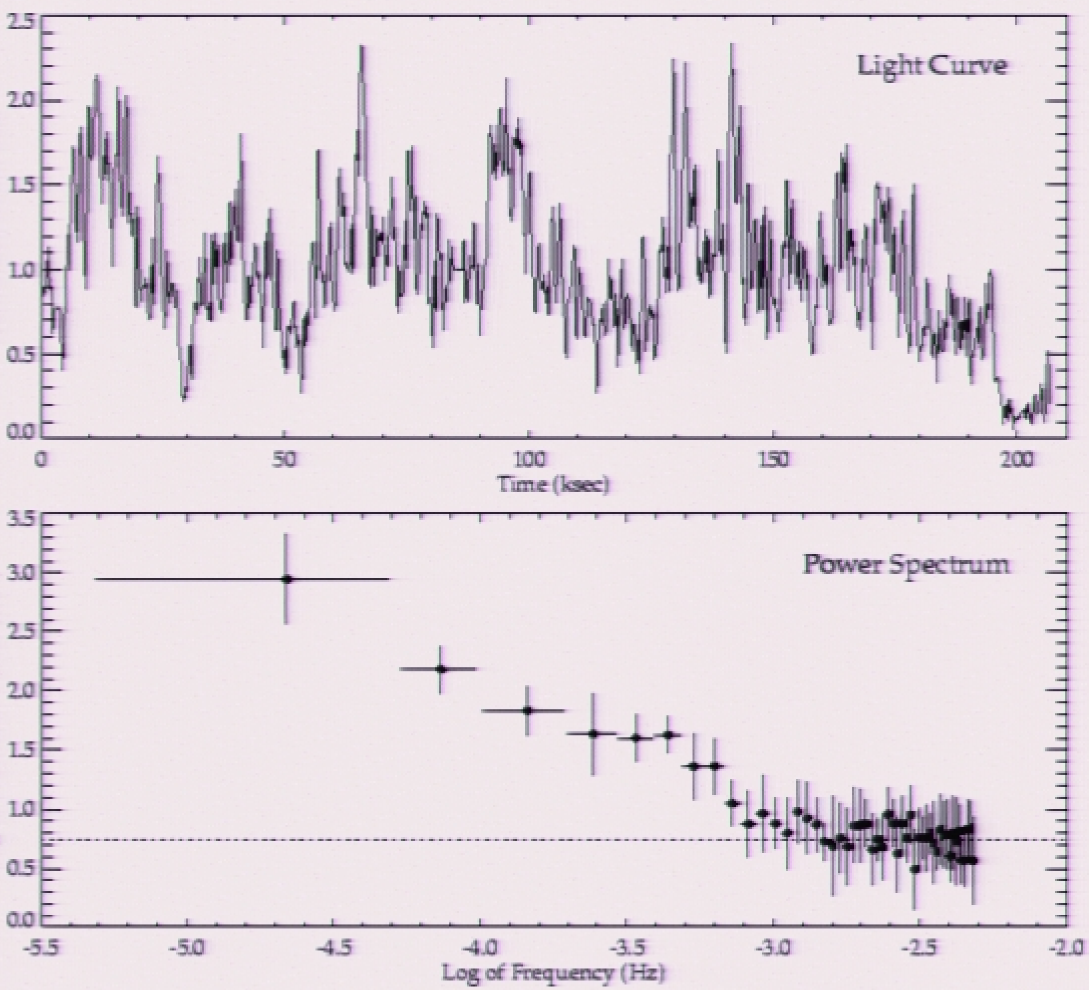
Requires fine tuning temperature

Critical points in equilibrium (magnets)



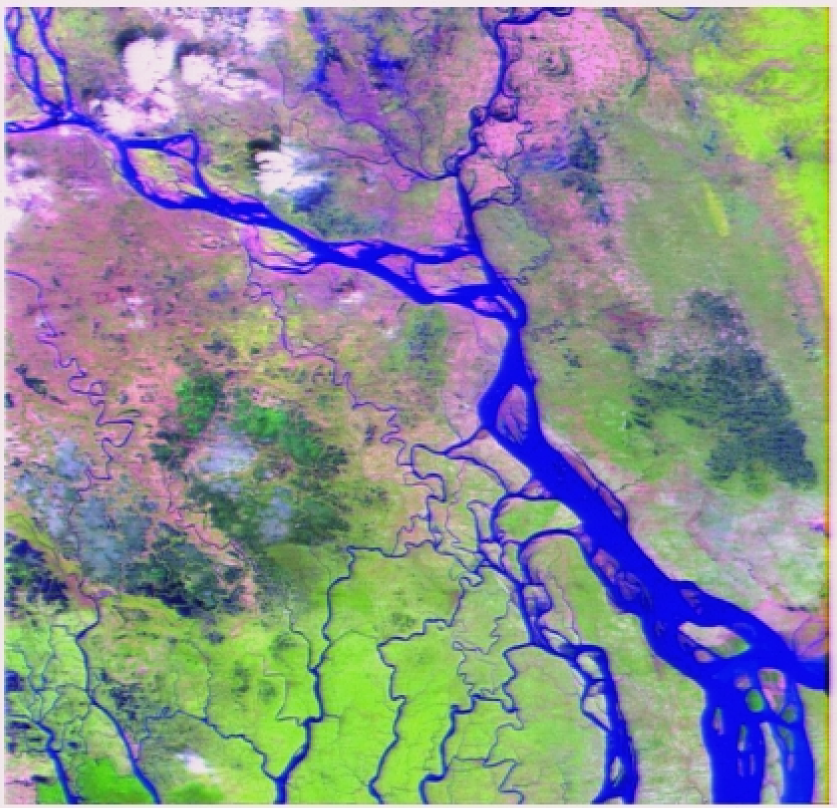


Out of equilibrium: 1/f noise



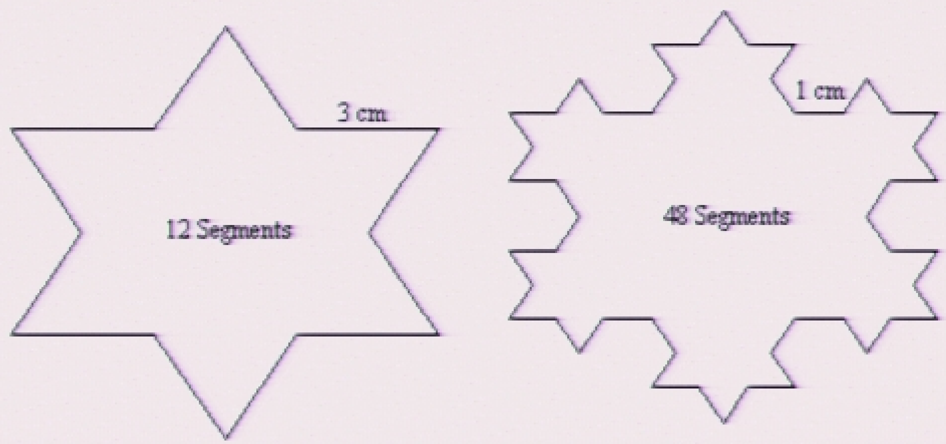
- *X-ray variability of the Seyfert galaxy NGC 4051. Top: Low Energy (0.05 - 2 keV) EXOSAT light curve binned on a timescale of 300 s. Bottom: Temporal power spectrum displaying characteristic `red noise' and a possible quasi-periodic oscillation at ~ 4×10-4 Hz. Data provided by A. Lawrence*

Out of Equilibrium: Fractals



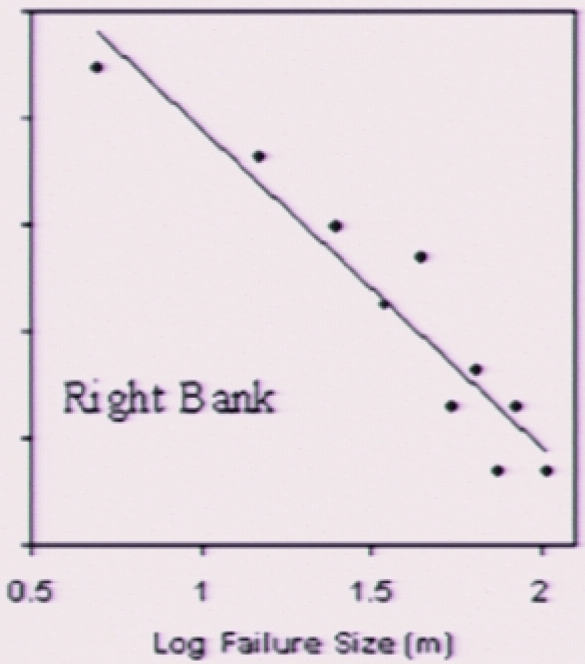
$$N = \left(\frac{P}{p}\right)^d \qquad d = \frac{\log N}{\log (P/p)}$$

N = 4, P/p = 3, so d = log 4 / log 3



Self-organization in fluvial landscapes: sediment dynamics as an emergent property

(B)



Avalanches have a (sort of) a power law distribution of sizes

Common features for many different systems

- They are far from equilibrium
- Thresholds (nonlinearities) are crucial
- Avalanches are broadly distributed
- They change the fractal structure of the system, which in turn changes the pattern of avalanches.
- Dynamics has correlations over long time scales

Evolution as a self-organized critical phenomenon

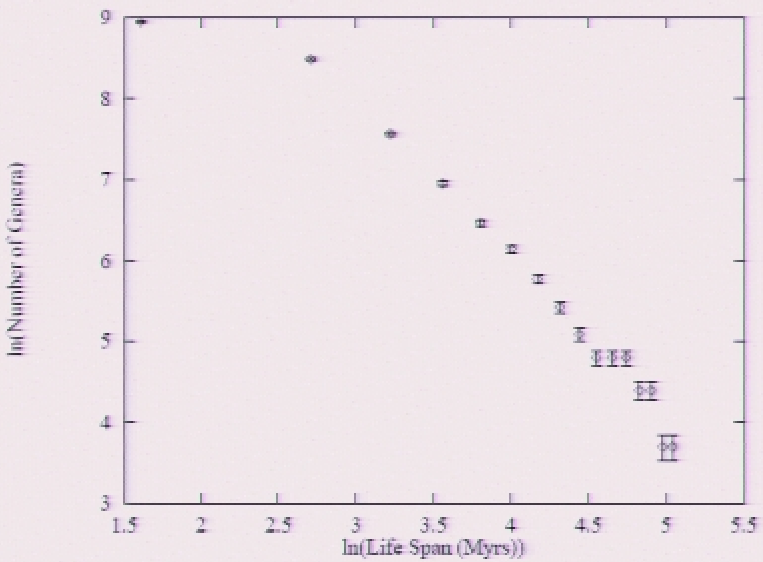


Figure 4: Lifetime distribution for genera as recorded by Sepkoski and Raup. The distribution can be well fitted by a power law $N(T) \propto 1/T^{-2}$ except at its lowest T-values (Sneppen, Bak, Flyvbjerg, Jensen [51]).

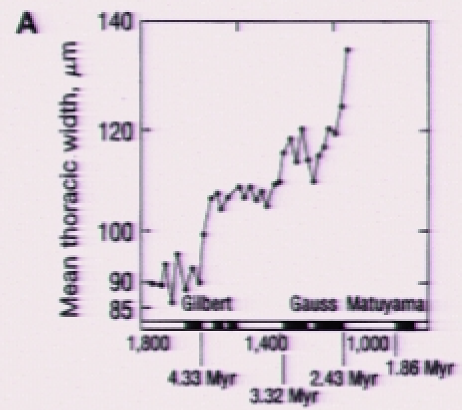
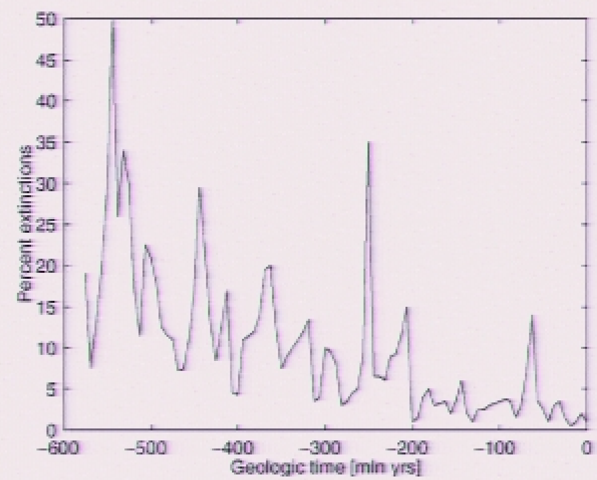
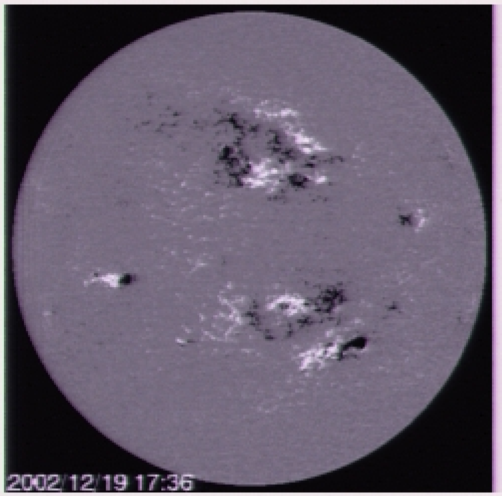
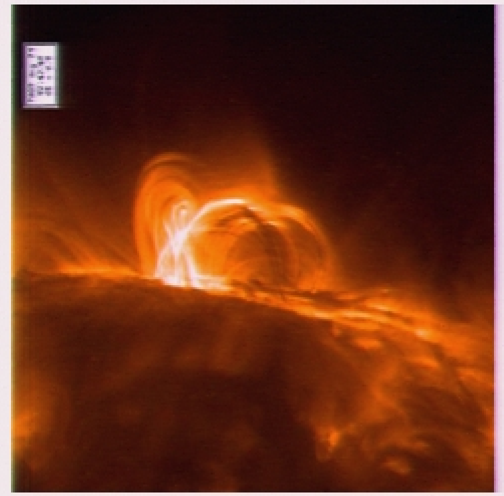


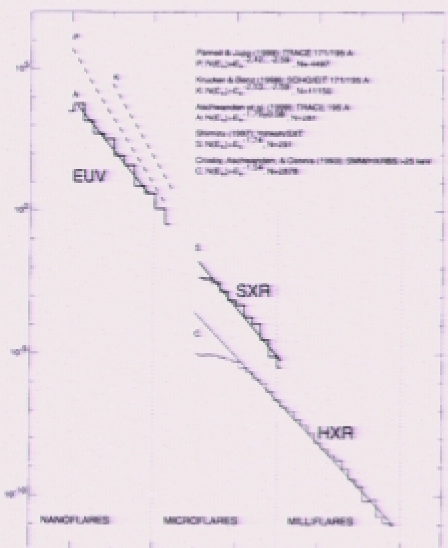
FIG. 1. (*A*) Time series for the variation of the morphology of a single species. The figure shows the increase in thoracic width of the Antarctic radiolarian *Pseudocubus vema* over 2.5 million years (Myr) according to Kellogg (1). (*B*) Model prediction for time series for

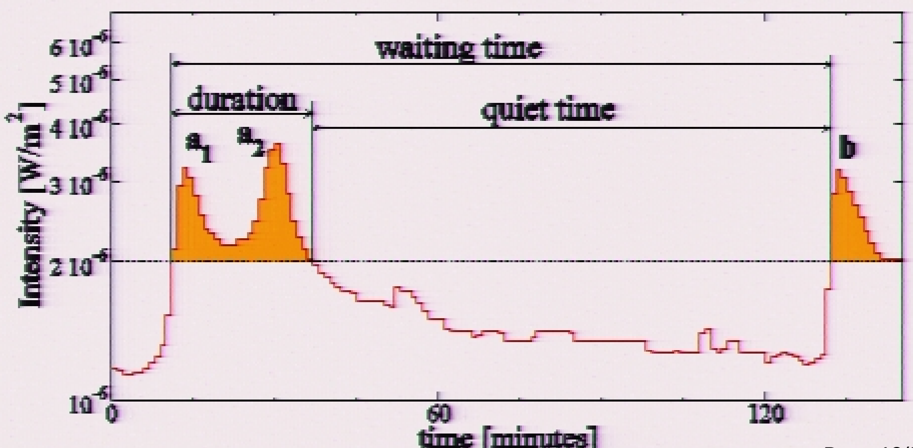


Solar flares

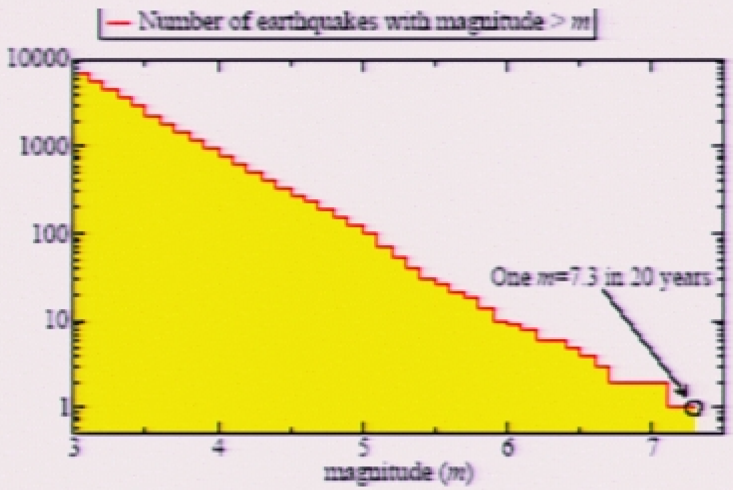


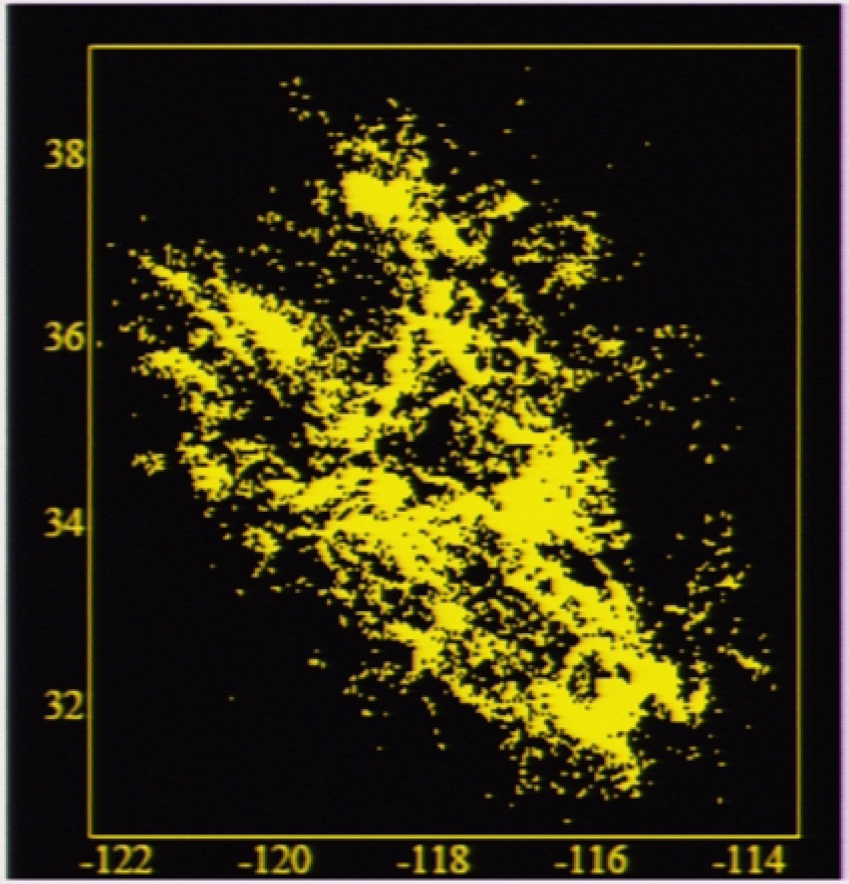


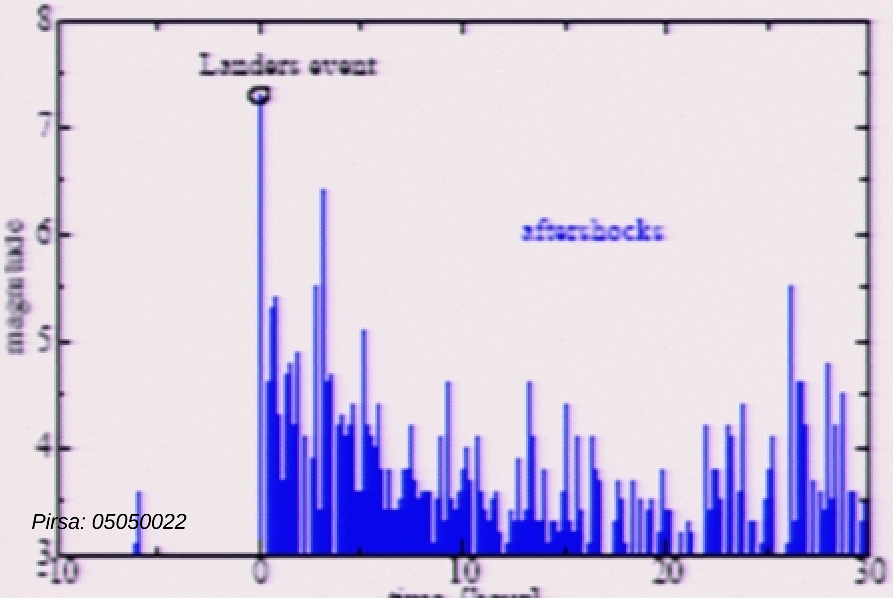




Earthquakes in California

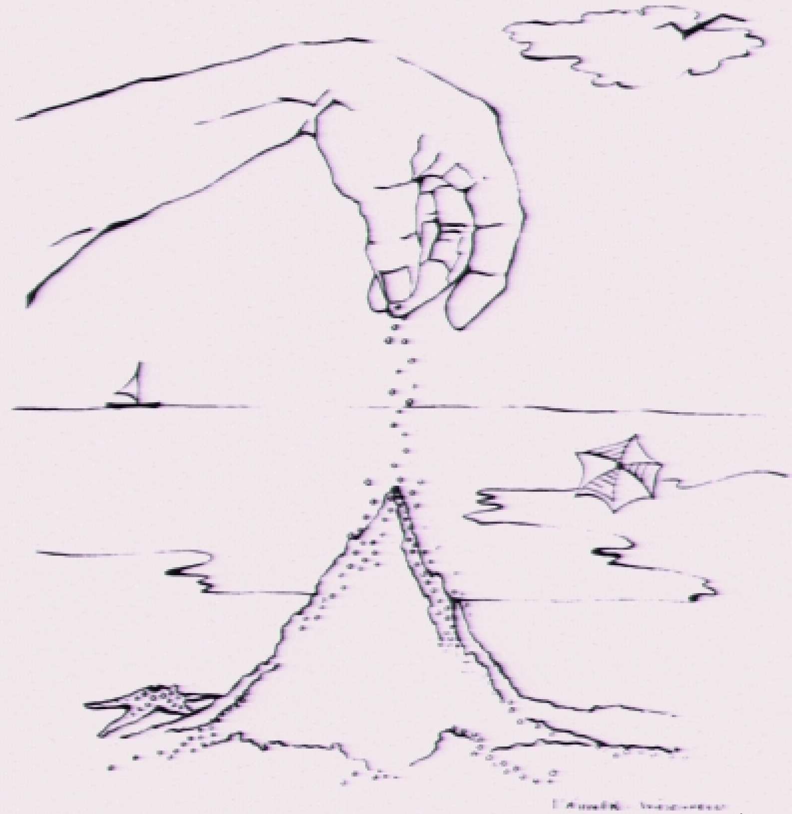






Self Organized Criticality

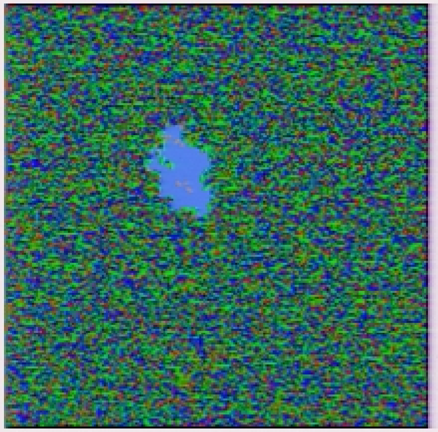
- Avalanches with a power law distribution
- Correlations over many space and time scales
- Solves 'fine-tuning' problems
- Fundamental parameters are emergent



Robust & universal mechanism ⇨ simple models

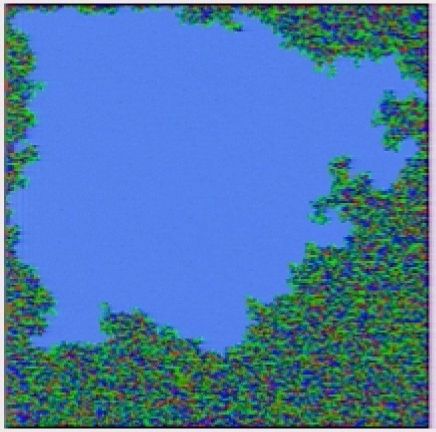
P. Bak, C. Tang and K. Wiesenfeld ('87)

The BTW (Abelian) Sandpile



Add sand randomly such that $z_i \longrightarrow z_i + 1$

Instability criterion:
If $z_i > 3$, site i topples, distributing one grain to each of its 4 neighbors on the lattice.



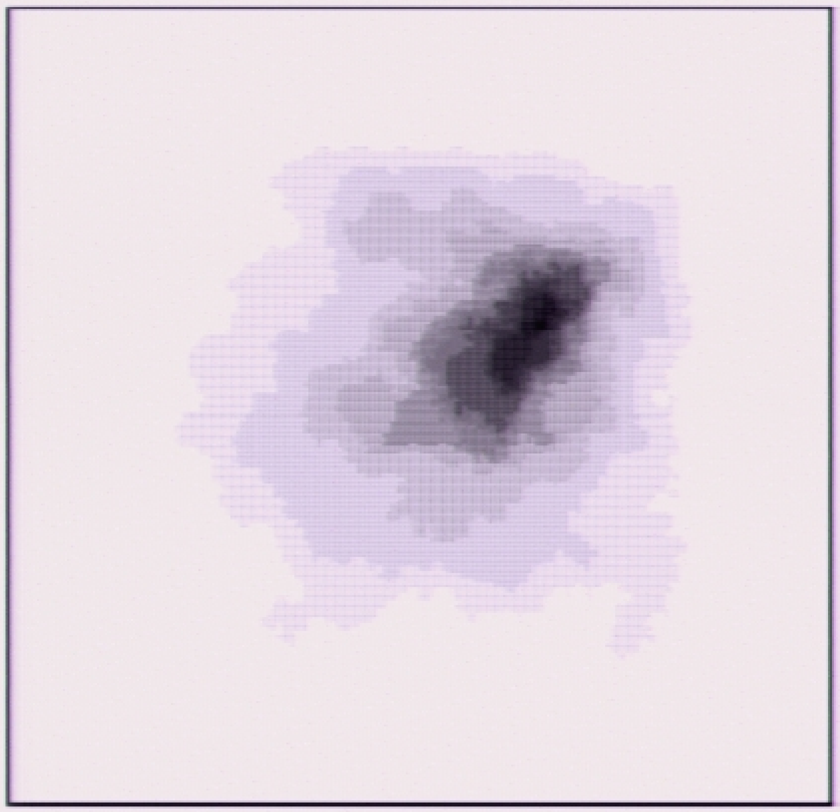
Repeat for all unstable sites that may be created.

Avalanche stops when the entire pile is stable.

Add another grain…

Order of topplings doesn't change final state (Abelian property)

Waves in BTW



Exact results (Priezzev et al.)

Mathematical formulation (Dhar)

$$\text{If } z_i > z_{i,c}, \text{ then } z_j \to z_j - \Delta_{ij}, \text{ for every } j. \tag{3}$$

Without loss of generality we may choose $z_{i,c} = \Delta_{ii}$ (this amounts to a particular choice of the origin of the z_i variables). In this case, the allowed values of z_i in a stable configuration are $1, 2, \ldots, \Delta_{ii}$.

Evidently the matrix Δ has to satisfy some conditions to ensure that the model is well behaved.

1. $\Delta_{ii} > 0$, for every i. (Otherwise topplings never terminate.)
2. For every pair $i \neq j$, $\Delta_{ij} \leq 0$. (This condition is required to establish the Abelian property, as will be shown shortly.)
3. $\sum_j \Delta_{ij} \geq 0$ for every i. (This condition states that sand is not generated in the toppling process.)

3.2 The Abelian Property

Let $\mathcal{C}$ be a stable configuration, and define the operator a_i such that the stable configuration $\mathcal{C}' = a_i\mathcal{C}$ is the one achieved after addition of sand at site i and relaxing. The mathematical treatment of the sandpile models relies on one simple property they possess [9]: The order in which the operations of particle addition and site toppling are performed does not matter. Thus the operators a_i commute, *i.e.*,

$$a_i a_j = a_j a_i \ , \qquad \text{for every } i, \ j. \tag{4}$$

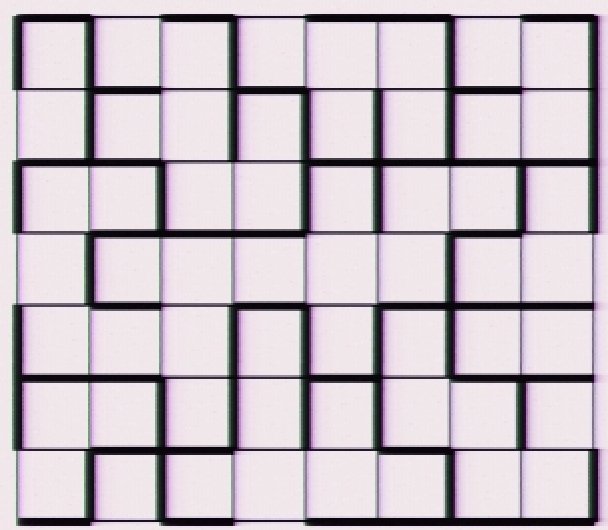
To prove this we start by noting that if we have a configuration with two or more unstable sites, then these sites can be relaxed in any order, and the resulting configuration is independent of the order of toppling. Consider two unstable sites i and j. If we topple at i first, this can only increase the value of z_j (by condition 2), and site j remains unstable. After toppling at j also, the height at any site k, as a result of these two topplings, undergoes a net change of $-\Delta_{ik} - \Delta_{jk}$. Clearly toppling first at j, then at i gives the same result. By a repeated use of this property, any number of unstable sites in a configuration can be relaxed in any order, always giving the same result.

Exact results using operator algebra (Dhar)

$$
\begin{aligned}
G_{ik} &\sim r_{ik}^{2-d}, \qquad \text{for } d > 2; \\
&\sim \log(L/r_{ik}), \qquad \text{for } d = 2; \\
&\sim (L - r_{ik}), \qquad \text{for d} = 1.
\end{aligned}
$$

$$|\mathbf{R}| = \det \Delta \ .$$

Mapping of configurations on the attractor to spanning trees



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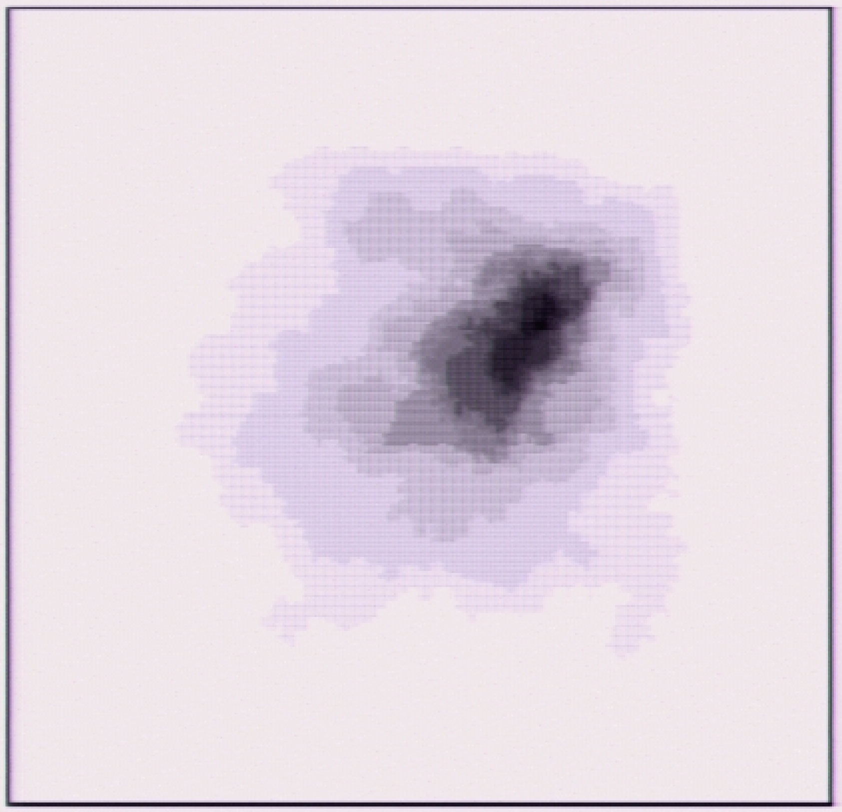

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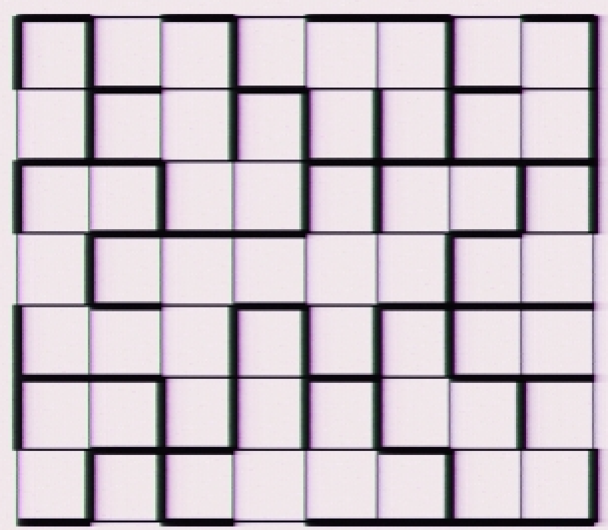
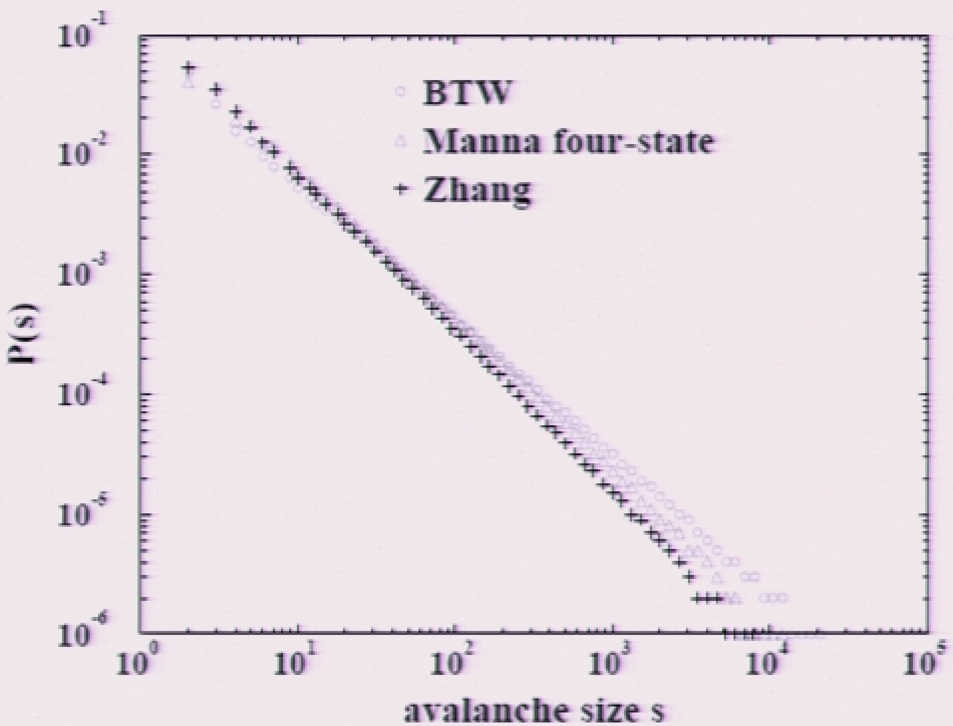


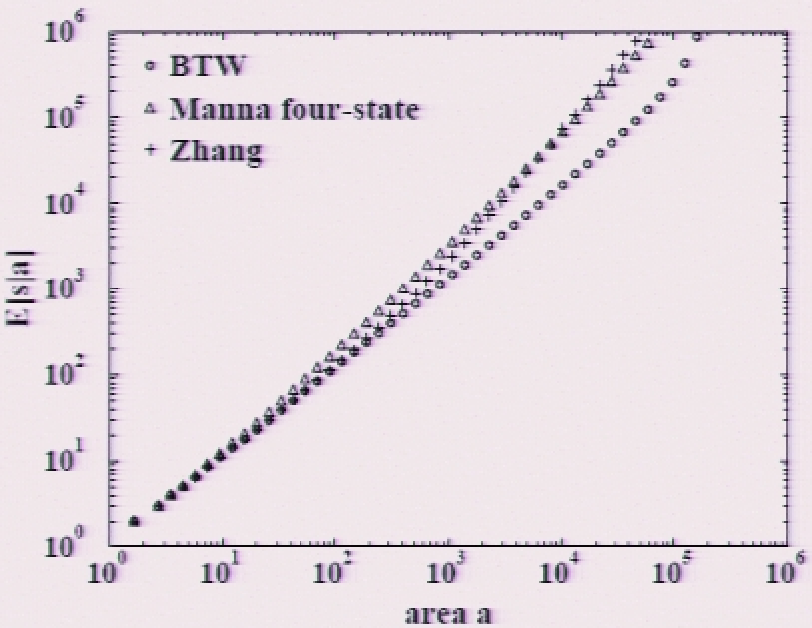
Distribution of avalanche sizes (number of topplings)

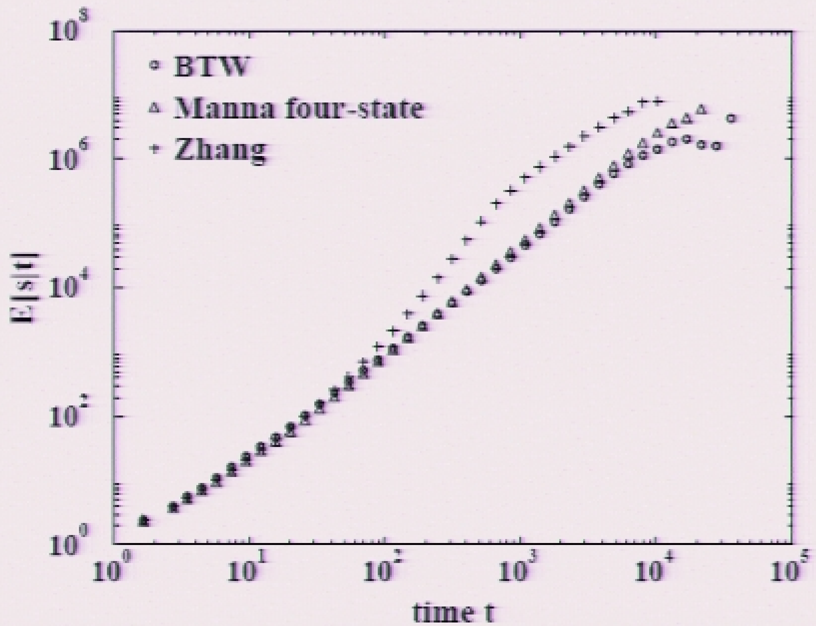


$P(s) \sim s^{-\tau}$

Only numerics – no theory yet

Theory should also obtain scaling relations for different variables





$a \sim r^2$ $s \sim a^{D/2}$ $t \sim a^{z/2}$

Flares (or bursts in magnetically confined plasmas) described by shell models not sandpiles

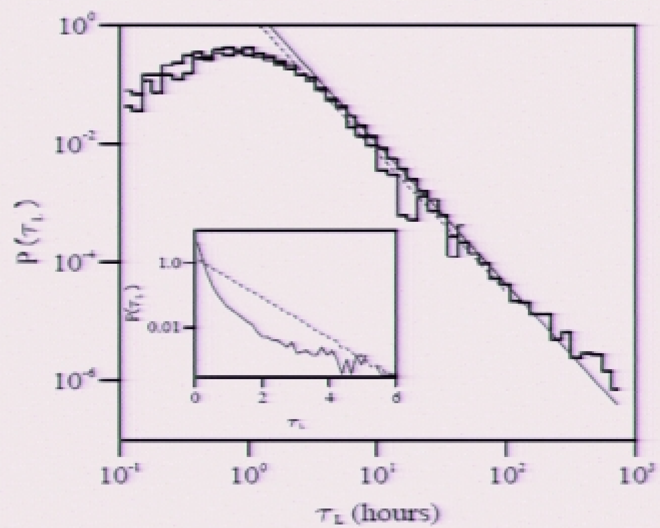


FIG. 1. Probability distribution of the laminar time $P(\tau_L)$ between two X–ray flares for dataset A (dashed line) and dataset B (full line). The straight lines are the respective power law fits. In the inset we show, in lin–log scale, the distribution for dataset B (full line) and the distribution obtained through the SOC model (dashed line) which displays a

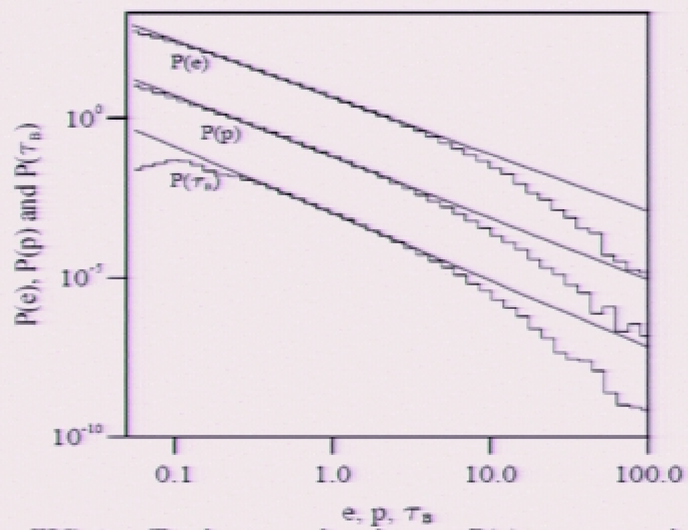
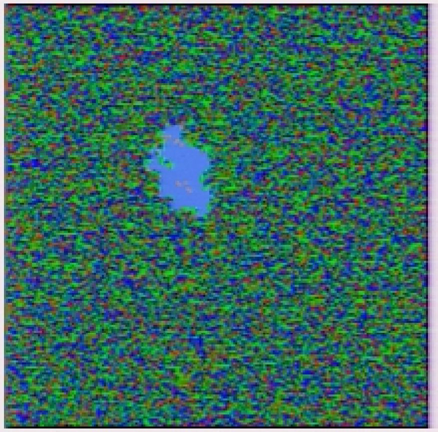
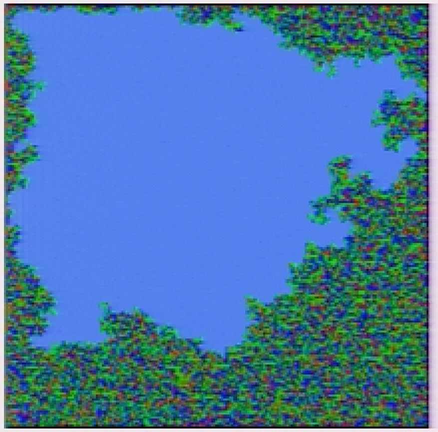


FIG. 3. Total energy distribution $P(e)$, energy peak distribution $P(p)$ and bursts duration distribution $P(\tau_B)$ for the shell model. The variables have been normalized to the respective root–mean–square values. The straight lines are the fits with power laws. The values of $P(e)$ and $P(\tau_B)$ are offset by a factor 100 and 10^{-2} respectively.

The BTW (Abelian) Sandpile





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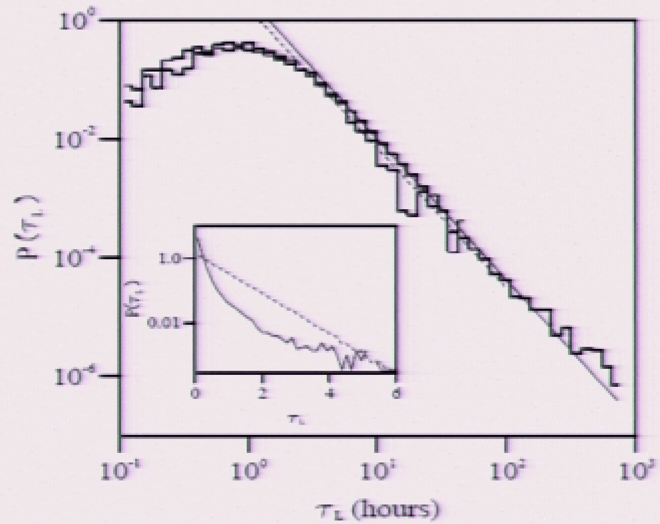

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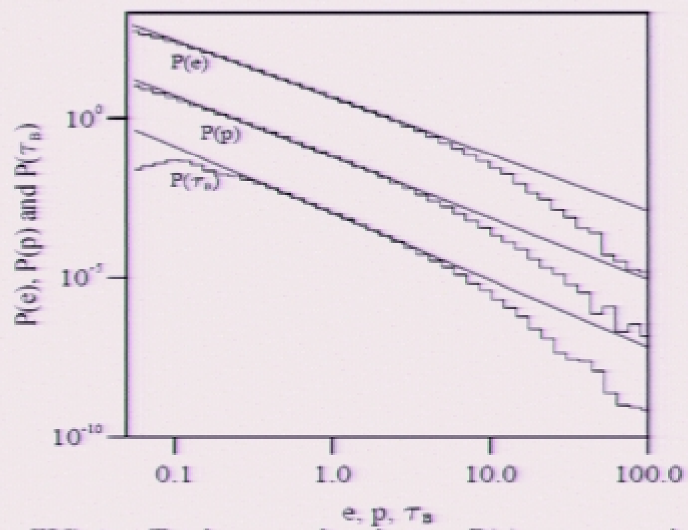

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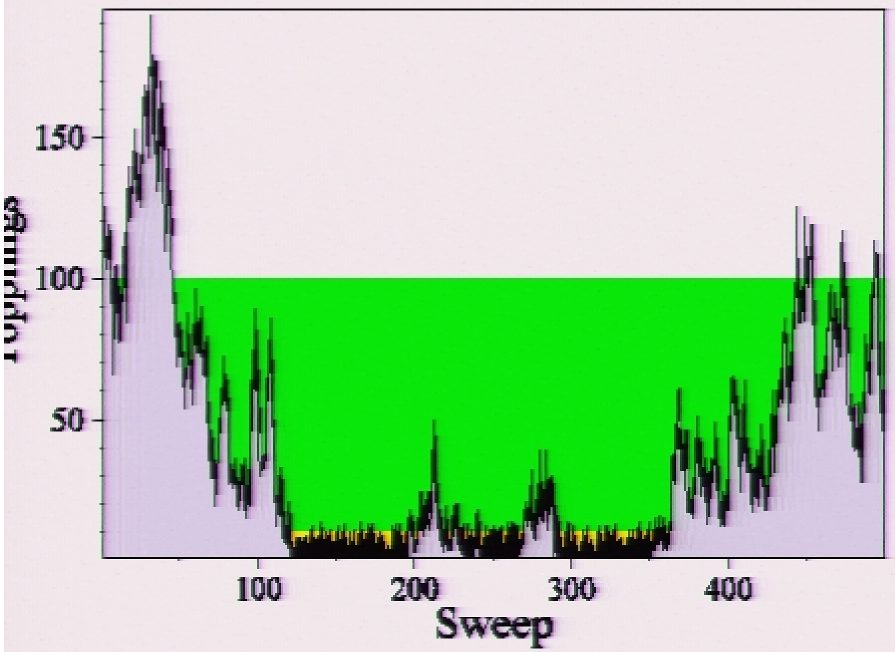
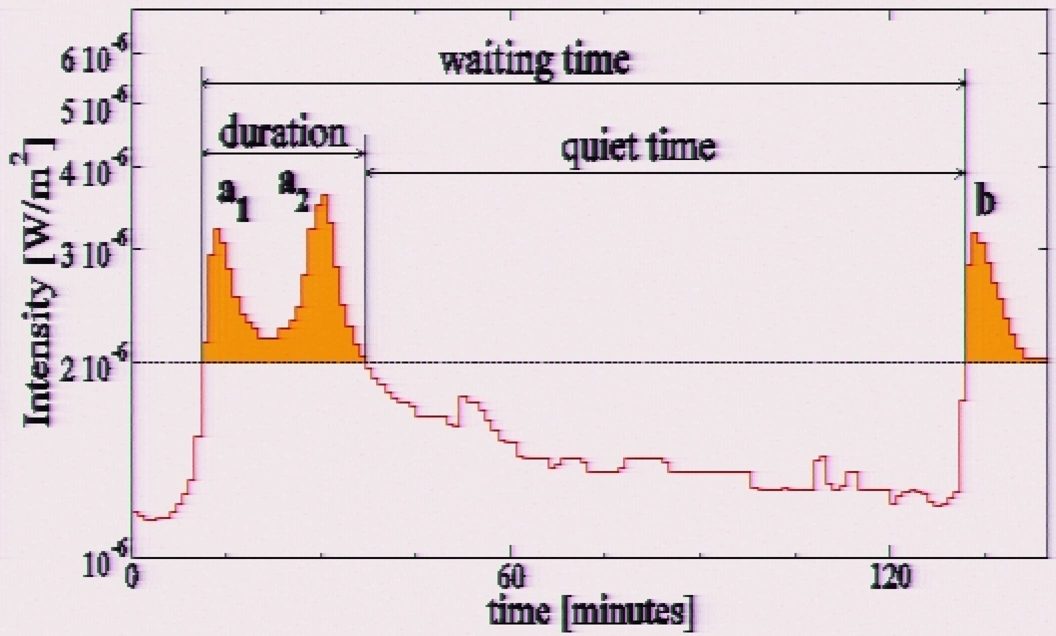
Flares may be described by SOC

- Use the same analysis technique for all models and for data
  - Bursts not resolvable down to smallest energy scales. Must impose a threshold
  - Use the same clock for both durations and quiet times
  - Look for universal properties that are independent of threshold over a wide range.

A common definition of events using thresholds one BTW sweep = one minute (resolution)

Duration and quiet times measured on a single clock – no "infinite" time scale separation

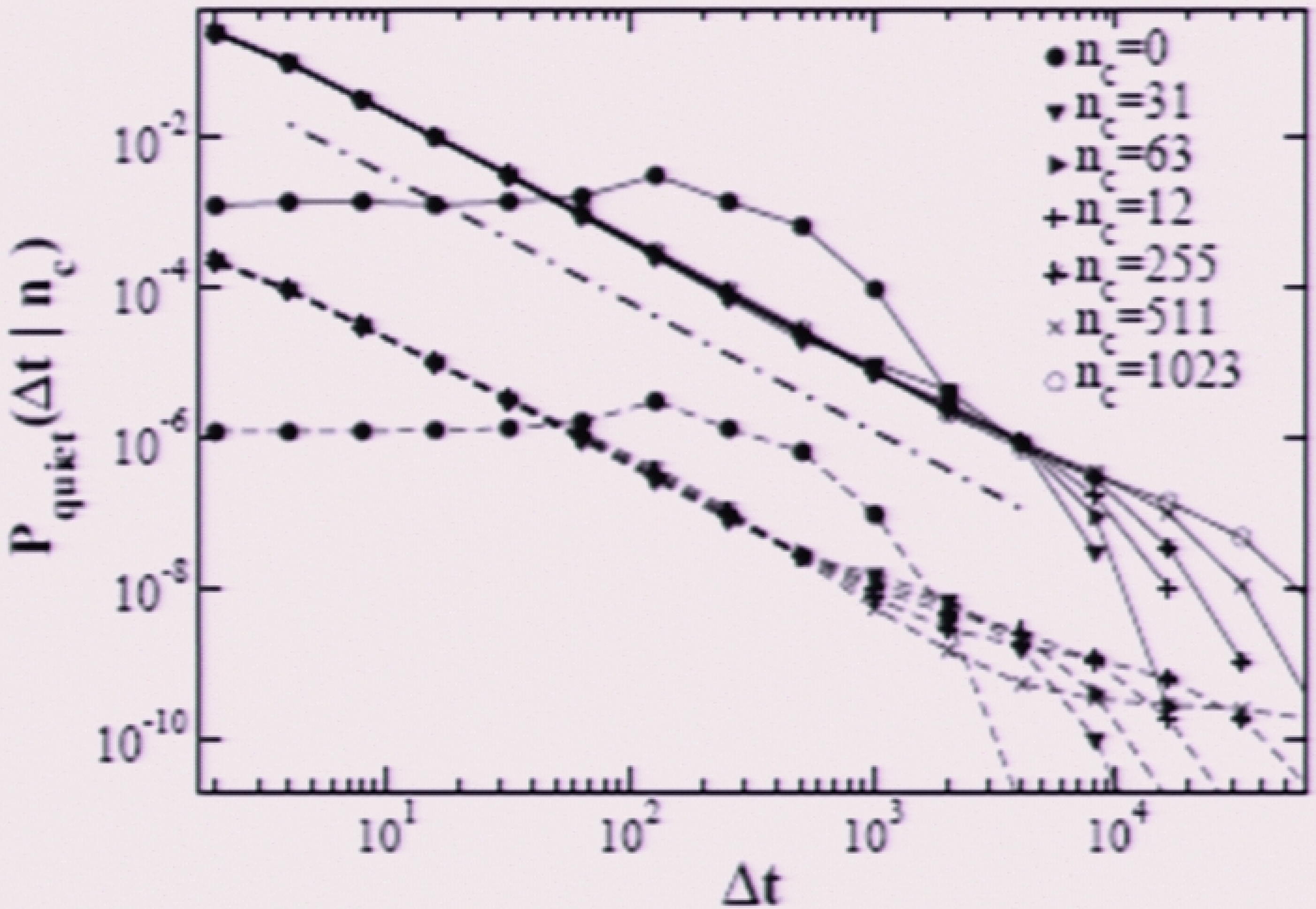




BTW
(Boettcher & MP)

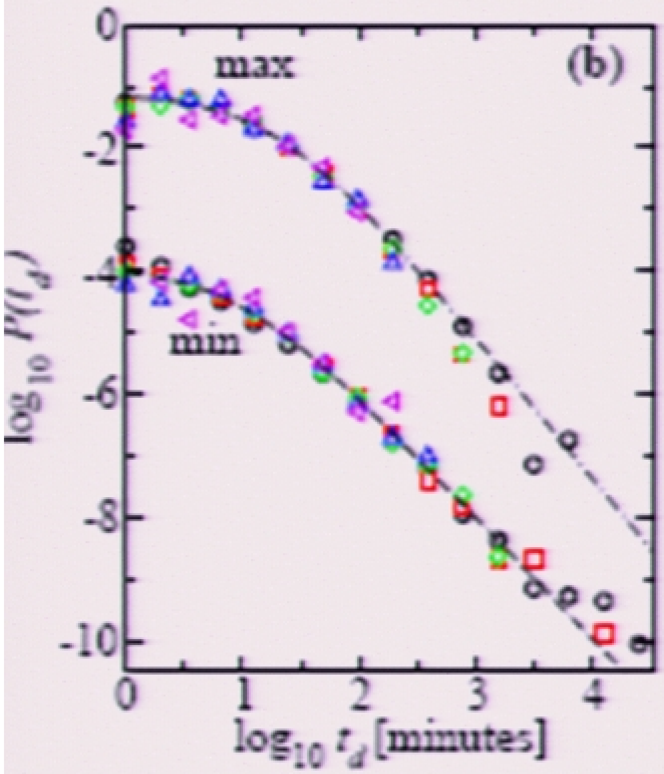
GOES emission
(Baiesi, MP & Stella)

A common definition of events using thresholds to power law on increasing the threshold n_c



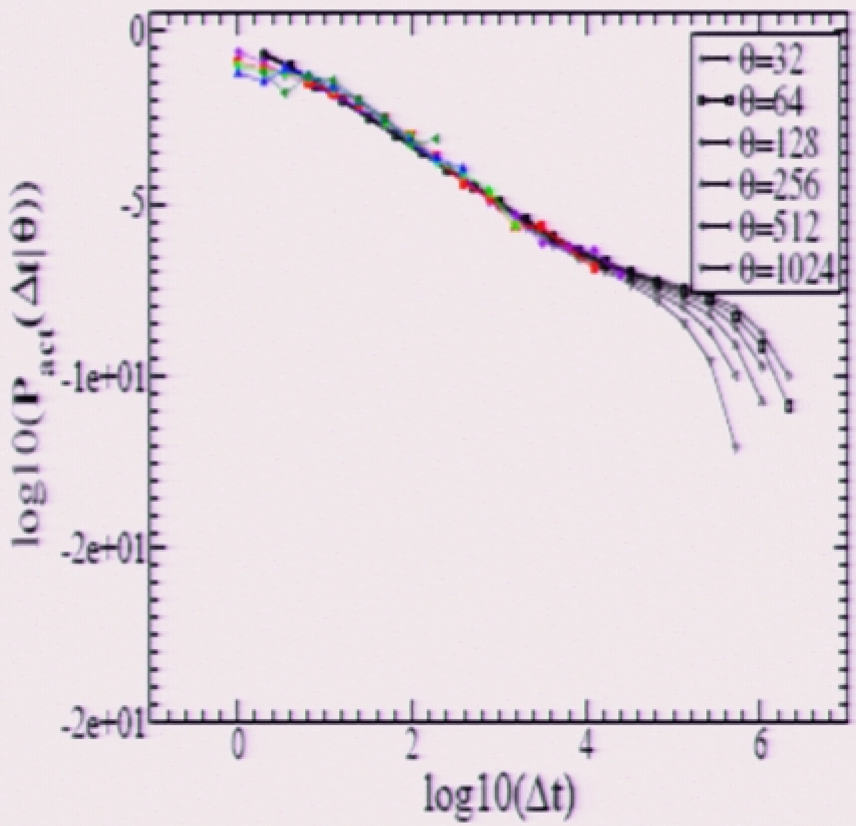
(Boettcher and MP (2005))

Invariant solar flare and BTW event durations at different thresholds



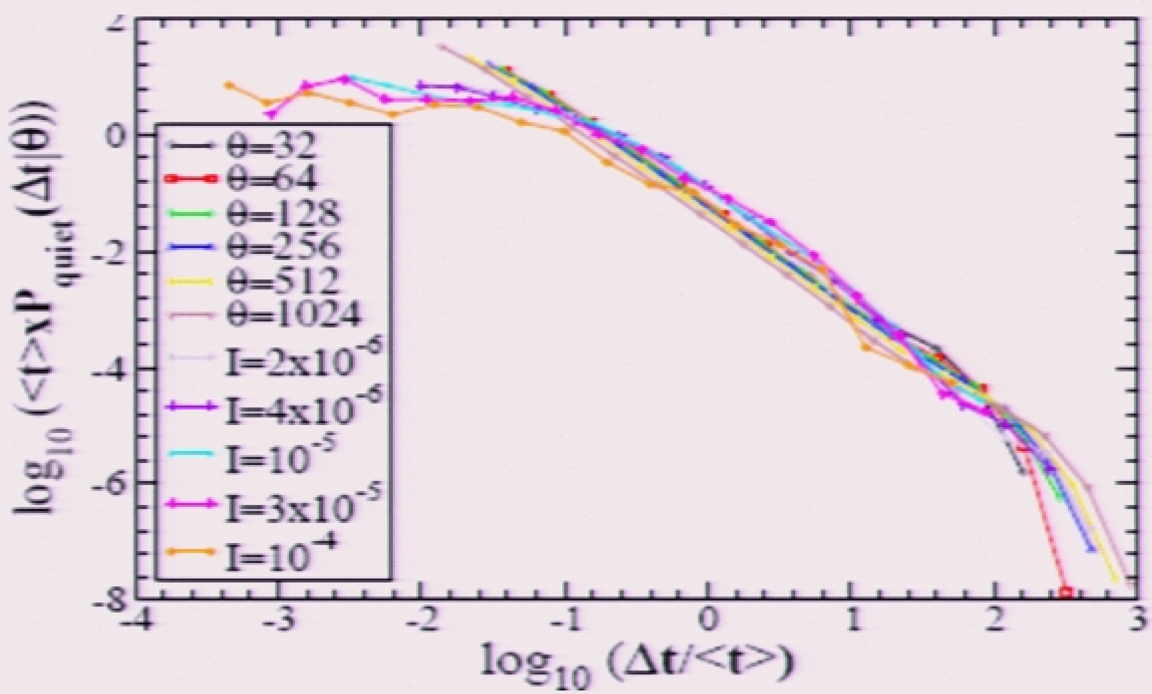
$$P(t_d) \sim (1 + t_d/t_d^*)^{-\gamma_{\rm dur}}$$

Superstatistics "q-exponential"



Comparing model & flare data

Solar Flare and BTW quiet times at different thresholds



Rescaling to get dimensionless quantities, BTW and flare quiet times are comparable

Forest Fire Model: An "Ising" Model of Turbulence (Bak, Chen, Tang (1990))

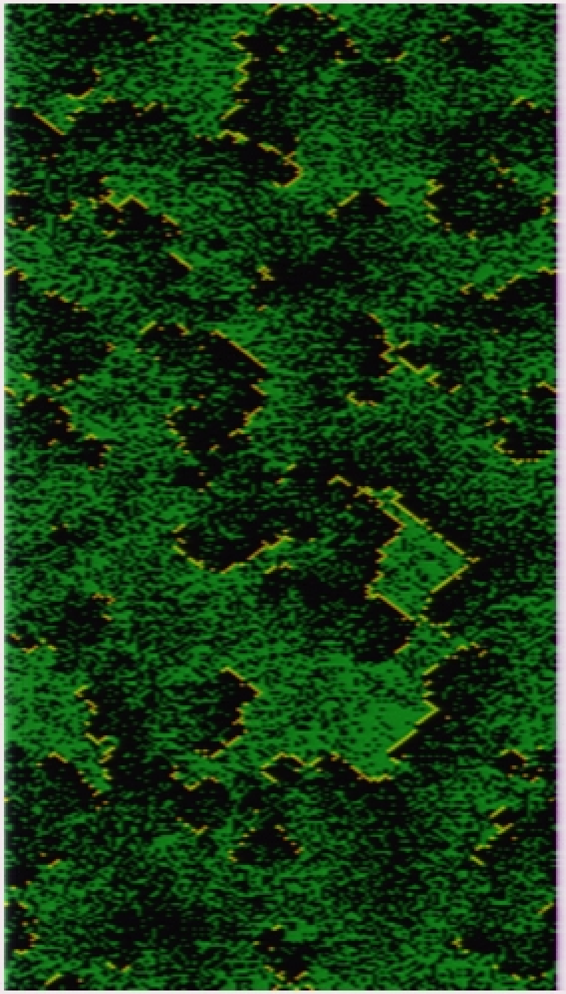
AT EACH TIME STEP

- Fires ignite neighboring trees and burn down, leaving an empty site
- Trees grow randomly on empty sites at rate p

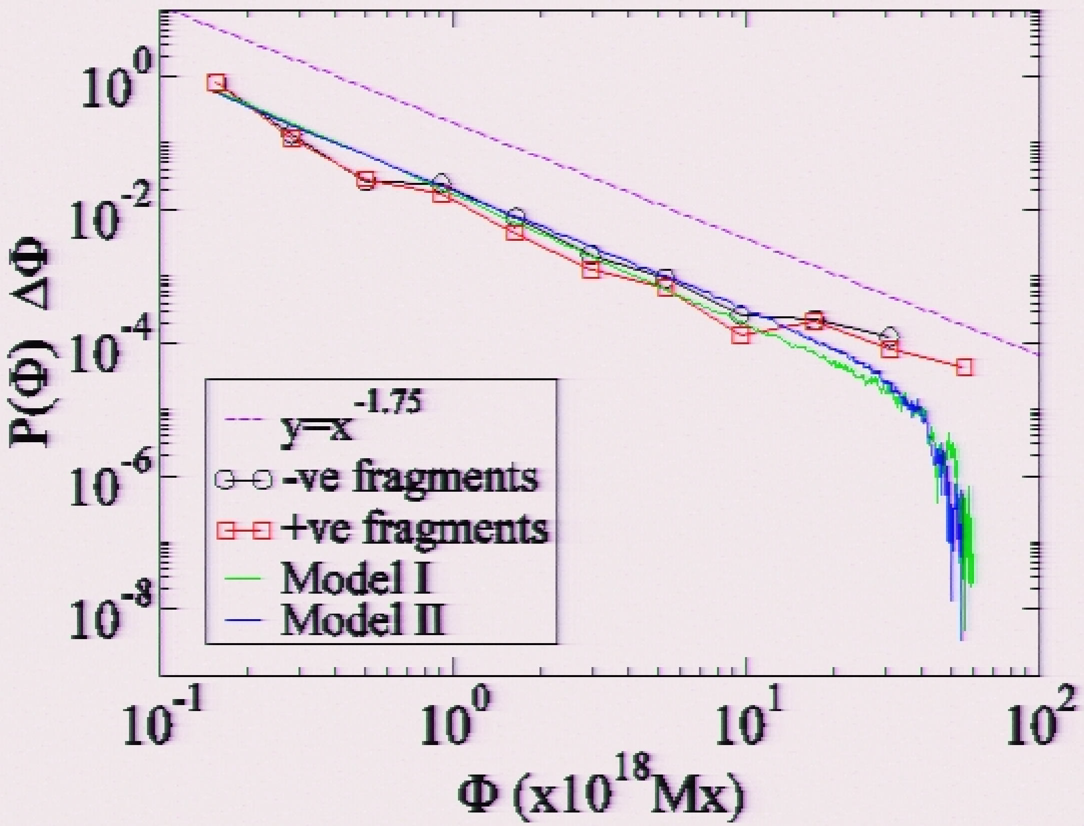
CORRELATED STEADY STATE WITH FIRES WHEN

$$1 < l < \xi \ll L$$

$$\xi = (0.77p)^{-2/3}$$

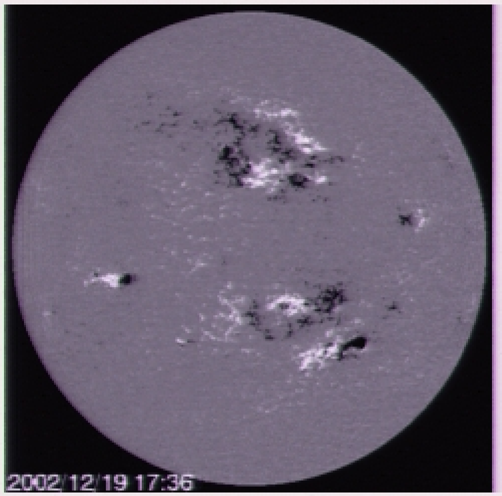


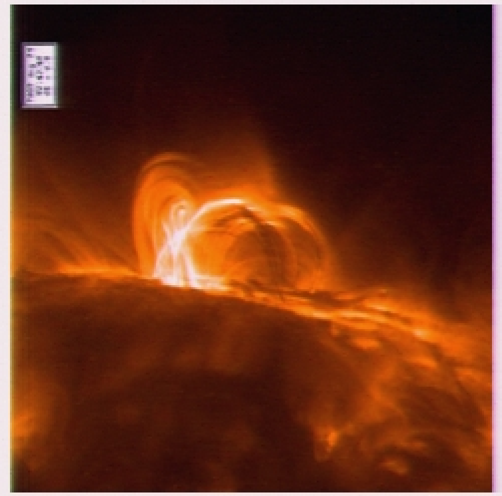
The scale free magnetic network

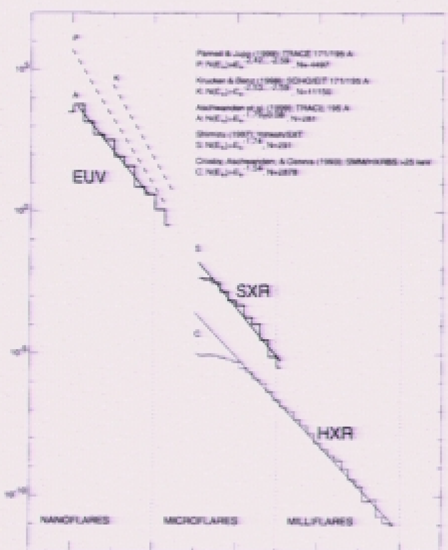


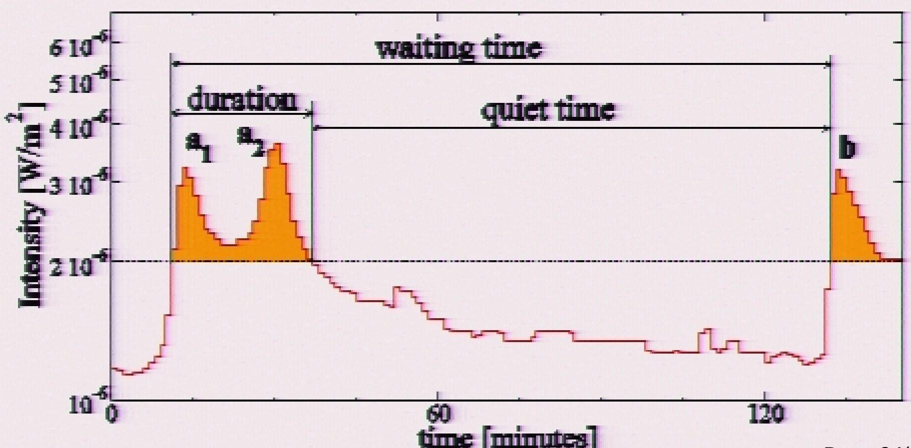
Hughes and MP (2004)

Solar flares









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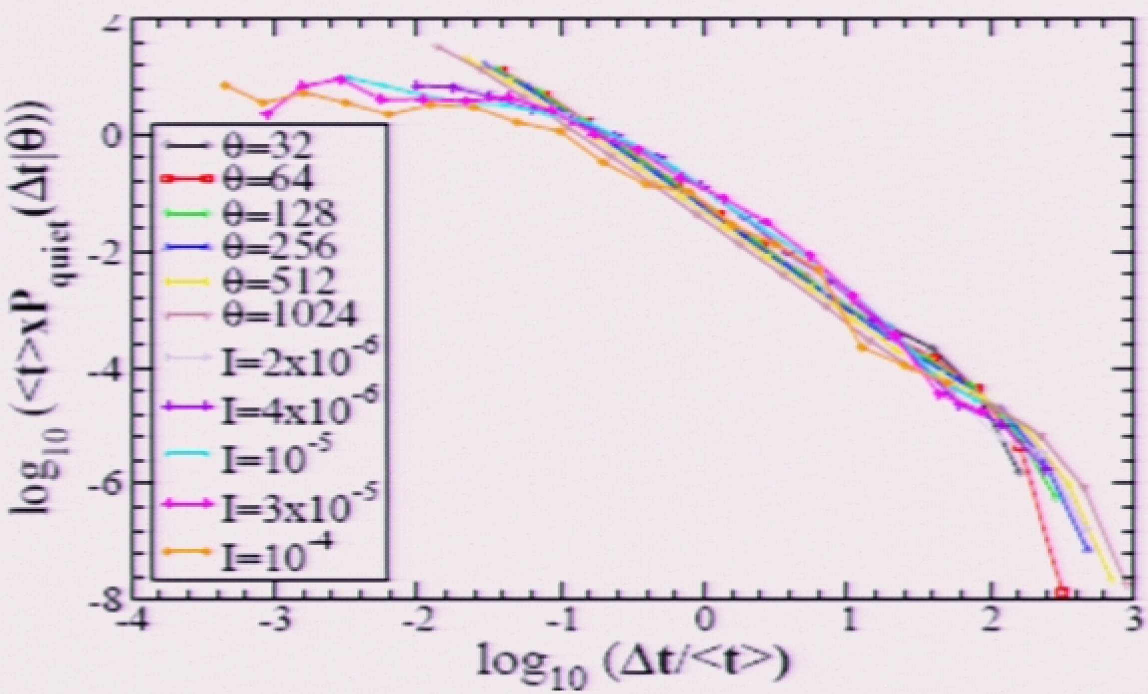

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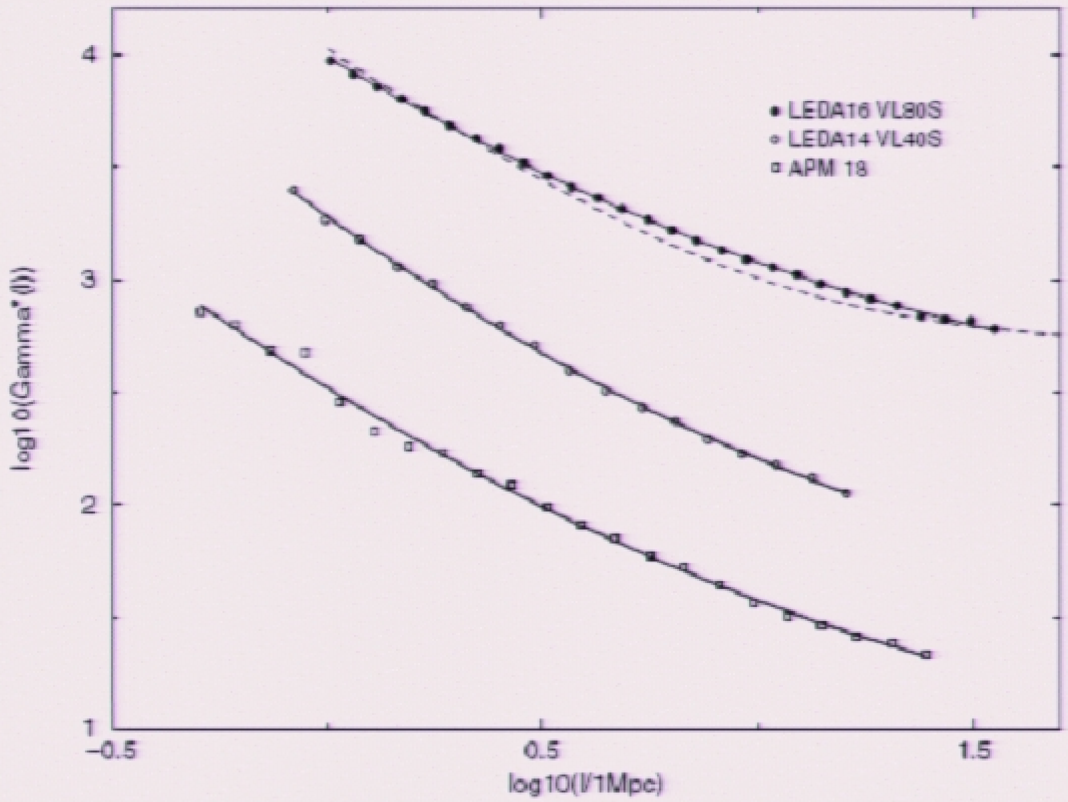
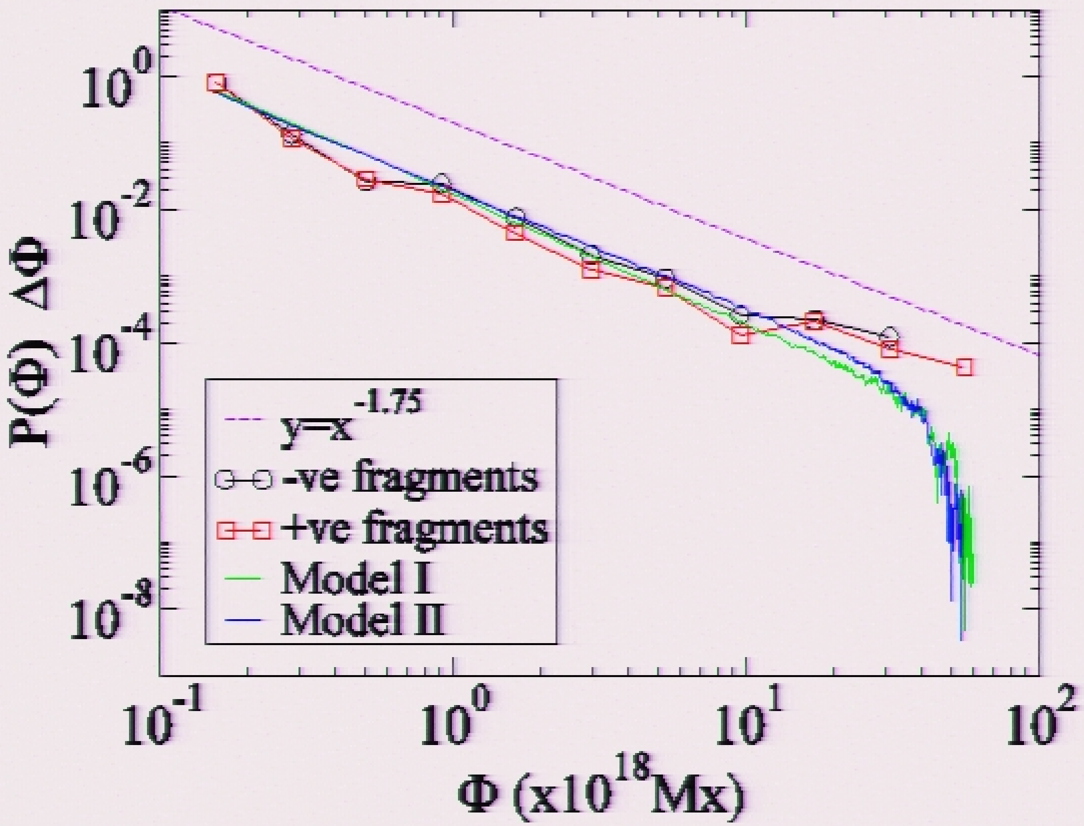
Analysis of Galaxy Maps



$$\log[\Gamma^*(l)] \sim \left[\frac{3}{2} \left(\frac{\log(l/l_0)}{\log(\xi/l_0)} \right) - 3 \right] \log(l/l_0).$$ **scale covariant**

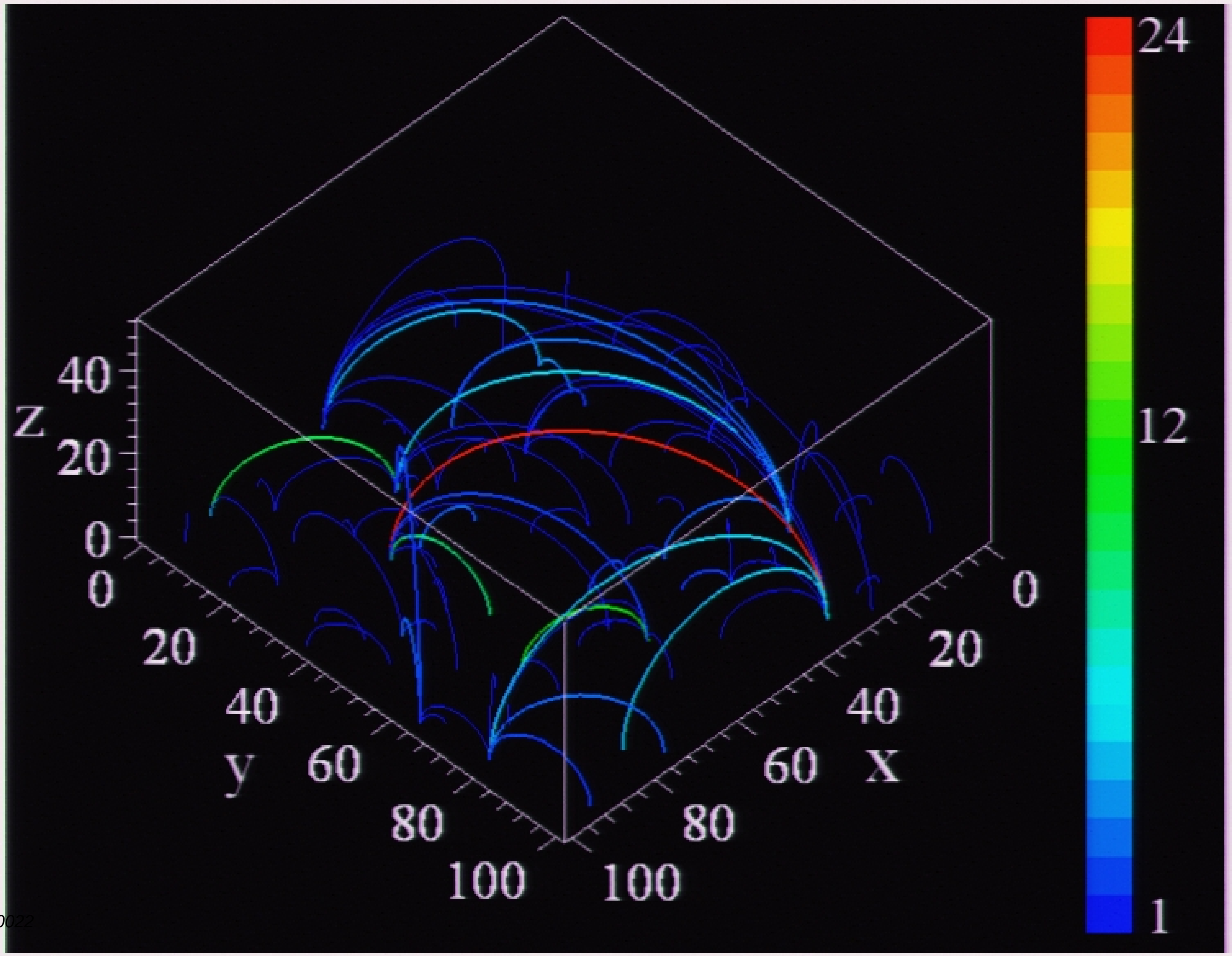
$$\Gamma^*(l) \sim \langle n \rangle [1 + g(l)], \quad g(l) = (r_0/l)^\gamma$$ **scale invariant**

The scale free magnetic network

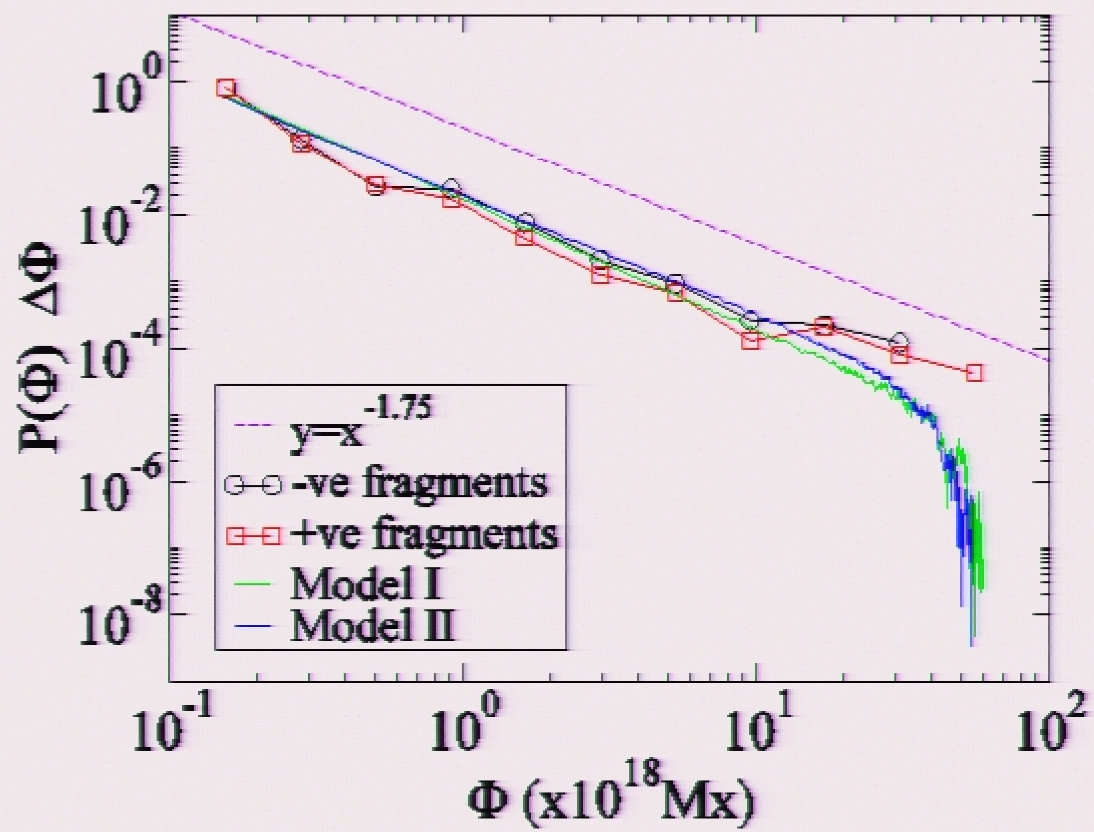


Hughes and MP (2004)

SOC model of reconnecting flux tubes
(PRL 2003- Hughes, MP, and the Culham group)



The scale free magnetic network



Hughes and MP (2004)