Title: Quantizing the spectral curve(s) of AdS/CFT

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Abstract:

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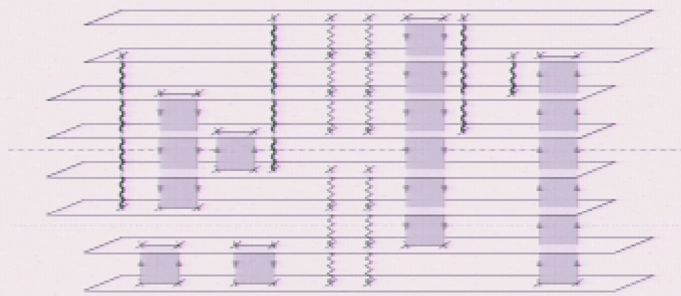

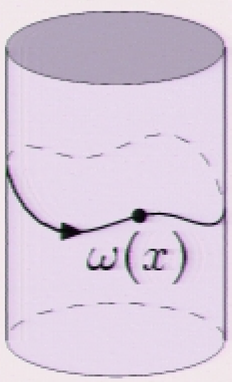

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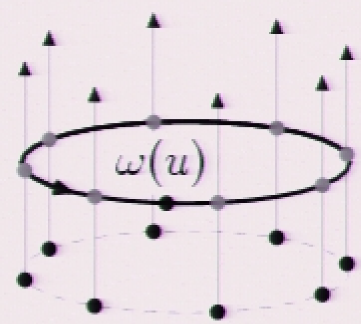
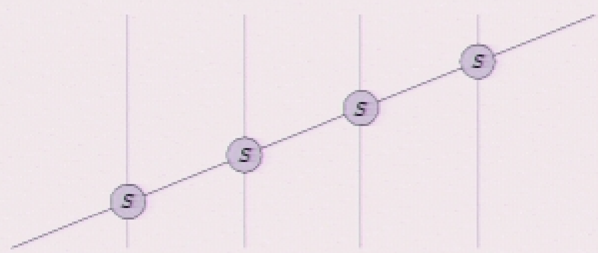
May 2005





Based on work with V. Kazakov, K. Sakai, M. Staudacher, K. Zarembo:
`hep-th/0410253,0502226,0503200,0504190.`

Large Spin Limits of AdS/CFT

AdS/CFT: String energies & gauge dimensions match: $\{E\} = \{D\}$

Proposal: Consider states with large spin J on S^5/of flavour $\mathfrak{so}(6)$

- BMN limit; non-planar and near $\mathcal{O}(1/J)$ extensions. [Berenstein Maldacena Nastase]. . .
- Semiclassical spinning strings: [Frolov Tseytlin]

$$\mathcal{O} \sim \operatorname{Tr} \phi_1 \ldots \phi_1 \phi_2 \ldots \phi_2 \phi_1 \ldots \phi_1 \phi_2 \ldots \phi_2 \longleftrightarrow \text{long classical strings.}$$

Effective coupling constant $$\lambda' = \frac{\lambda}{J^2}.$$

- String theory: Expansion in λ' and $1/J \sim 1/\sqrt{\lambda}$,
- Gauge theory: ℓ-loop contribution suppressed by (at least) $1/J^{2\ell}$.

Expansion in λ' apparently equivalent to expansion in λ. Compare!

Three-loop mismatch in near BMN limit. [Callan, Lee, McLoughlin Schwarz, Swanson, Wu] [Callan McLoughlin Swanson]

Similar disagreement for spinning strings. [Serban Staudacher]

Outline

- String Solutions:
  What has been done and what cannot be done?
- Classical superstrings on $AdS_5 \times S^5$:
  Construction of the spectral curve for any solution.
  Express curve through integral equations.
- Perturbative $\mathcal{N} = 4$ Gauge Theory:
  Derive the spin chain S-matrix in a subsector.
  Construct algebraic equations by nested Bethe ansatz.
- Combine results into algebraic equations for the complete model.

General Assumptions:

- Classical & non-interacting strings.
- Perturbative & planar gauge theory.

Strings in Flat Space

Equations of motion: $\partial_+\partial_-\vec{X} = 0$.

Solved by Fourier transformation. Mode decomposition:

$$\vec{X}(\tau,\sigma) = \vec{x}_0 + \tau\vec{p} + \mathrm{Re}\sum_{n\neq 0} \vec{a}_n \exp(i|n|\tau + in\sigma)$$

subject to Virasoro constraint $(\partial_\pm\vec{X})^2 = 0$.

⋆ Solutions classified by amplitudes $|\vec{a}_n|$ (as well as $|\vec{p}|$).

⋆ Quantize string modes $\vec{a}_n \to \vec{\alpha}_n$ (as well as $\vec{x}_0$, $\vec{p}$).

String Hamiltonian:

$$H = \frac{1}{\alpha'}\sum\nolimits_n |n|\vec{\alpha}_n^\dagger\vec{\alpha}_n.$$

Strings in (Near) Plane Waves

Consider $AdS_5 \times S^5$ as expansion around plane waves.

Solution for strings on plane waves [Metsaev hep-th/0112044] [Metsaev Tseytlin]

$$\vec{X}(\tau,\sigma) = \mathrm{Re} \sum_n \vec{a}_n \exp(i\omega_n \tau + in\sigma), \quad \omega_n = \sqrt{1/\lambda' + n^2}\,.$$

⋆ Solutions classified by amplitudes $|\vec{a}_n|$.

⋆ Quantize string modes $\vec{a}_n \to \vec{\alpha}_n$.

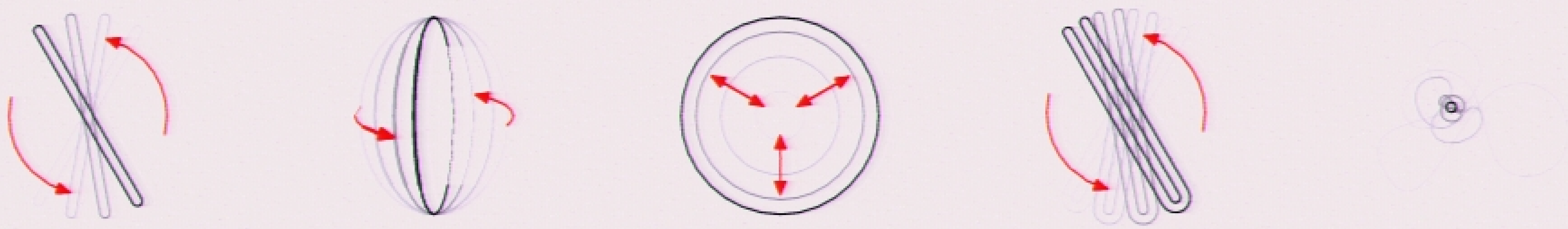
Recover $AdS_5 \times S^5$ Hamiltonian as expansion in $1/R$ [Callan, Lee, McLoughlin Schwarz, Swanson, Wu]

$$H = \sqrt{\lambda'} \sum_n \omega_n \vec{\alpha}_n^\dagger \vec{\alpha}_n + \mathcal{O}(\vec{\alpha}^4/R) + \mathcal{O}(\vec{\alpha}^6/R^2) + \ldots$$

- Coupling of arbitrarily many modes.
- $\mathcal{O}(1/R^2)$ already very complicated.

Spinning Strings

Many examples investigated: [Gubser Klebanov Polyakov][Frolov Tseytlin][Minahan hep-th/0209047][Frolov Tseytlin]. . .



folded | circular | pulsating | higher modes | plane waves

Ansatz, e.g. string on $\mathbb{R}_t \times S^2$: Energy $\mathcal{E} = E/\sqrt{\lambda}$, spin $\mathcal{J} = J/\sqrt{\lambda}$.

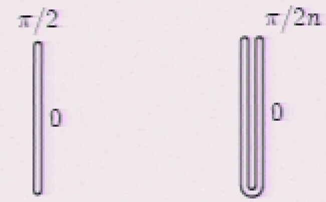
$$t(\tau,\sigma) = \mathcal{E}\,\tau, \quad \vec{X}(\tau,\sigma) = \begin{pmatrix} \sin\vartheta(\sigma)\,\cos\mathcal{J}\tau \\ \sin\vartheta(\sigma)\,\sin\mathcal{J}\tau \\ \cos\vartheta(\sigma) \end{pmatrix}.$$

Solve equations of motion and Virasoro constraint

$$\vartheta(\sigma) = \mathrm{am}(\mathcal{E}(\sigma-\sigma_0),\eta), \qquad \mathcal{J} = \eta\,\mathcal{E}.$$

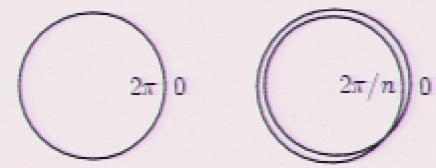
Global Charges

Folded string: $\vartheta(0) = 0$ and $\vartheta'(\pi/2n) = 0$



$$J = \sqrt{\lambda}\,\mathcal{J} = \sqrt{\lambda}\,\frac{2n}{\pi}\mathrm{K}(1/\eta), \qquad E = \sqrt{\lambda}\,\mathcal{E} = \sqrt{\lambda}\,\frac{2n}{\eta\pi}\mathrm{K}(1/\eta).$$

Circular string: $\vartheta(0) = 0$ and $\vartheta(2\pi/n) = 2\pi$



$$J = \sqrt{\lambda}\,\mathcal{J} = \sqrt{\lambda}\,\frac{2n\eta}{\pi}\mathrm{K}(\eta), \qquad E = \sqrt{\lambda}\,\mathcal{E} = \sqrt{\lambda}\,\frac{2n}{\pi}\mathrm{K}(\eta).$$

Global charges of generic solutions

$$J_k = \sqrt{\lambda}\,\mathcal{J}_k(\eta_a), \qquad S_k = \sqrt{\lambda}\,\mathcal{S}_k(\eta_a), \qquad E = \sqrt{\lambda}\,\mathcal{E}(\eta_a)$$

with algebraic, elliptic, hyperelliptic, ... functions of moduli $\{\eta_a\}$.

- Why elliptic functions? What is the meaning of moduli?

Strings on $AdS_5 \times S^5$

- Too difficult to solve the equations of motion in general.
  No direct way to quantization as in flat space or plane waves.
- Very difficult to expand around plane waves.
  Only expansion . . .

Now what?

- Give up on finding exact energy spectrum.
- Classify solutions to understand structure of spectrum.
- Try to quantize that.

How?!

- Extract all conserved charges: Lax pair, monodromy.
- Investigate their analyticity properties.
- Reconstruct the corresponding algebraic curve.
- Discretize the curve.

Supersymmetric Sigma Model

Field $g(\sigma,\tau) \in \mathrm{PSU}(2,2|4)$ with $\mathbb{Z}_4$-graded flat connection [Berkovits Bershadsky, Hauer Zhukov, Zwiebach]

$$J = -g^{-1}dg = H + Q_1 + P + Q_2, \qquad dJ - J \wedge J = 0.$$

Coset identification $g \simeq gh$ with $h(\sigma,\tau) \in \mathrm{Sp}(1,1) \times \mathrm{Sp}(2)$. Action [Roiban Siegel]

$$S_\sigma = \frac{\sqrt{\lambda}}{2\pi} \int \left(\tfrac{1}{2}\mathrm{str}\, P \wedge *P - \tfrac{1}{2}\mathrm{str}\, Q_1 \wedge Q_2 + \Lambda \wedge \mathrm{str}\, P\right).$$

$\mathfrak{psu}(2,2|4)$ Noether current and equation of motion

$$K = P + \tfrac{1}{2}{*Q_1} - \tfrac{1}{2}{*Q_2} - *\Lambda, \qquad d{*K} - J \wedge *K - *K \wedge J = 0,$$

Virasoro constraints

$$\mathrm{str}\, P_+^2 = \mathrm{str}\, P_-^2 = 0.$$

Start with (abstract) solution $g(\sigma,\tau)$ to the above equations.

Lax Pair

Lax pair: Family of flat connections ($\rightsquigarrow$ integrability) [Bena Polchinski Roiban]

$$A(z) = H + \tfrac{1}{2}(z^{-2} + z^2)P + \tfrac{1}{2}(z^{-2} - z^2)(*P - \Lambda) + z^{-1}Q_1 + z\,Q_2.$$

Flatness of $A(z)$ for all z due to flatness of J and conservation of K:

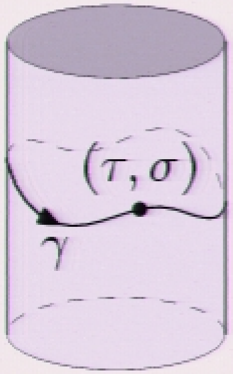
$$dA(z) - A(z) \wedge A(z) = 0.$$

Analytic for all $z \in \bar{\mathbb{C}}/\{0, \infty\}$. Poles at $z = 0$ and $z = \infty$.

Special point $z = 1$: $A(1) = H + Q_1 + P + Q_2 = J$.

Monodromy

Monodromy of Lax connection around closed string [NB, Kazakov Sakai, Zarembo]



$$\Omega(z) = \left(\mathrm{P}\exp\oint_{-\gamma} A(1)\right)\left(\mathrm{P}\exp\oint_{\gamma} A(z)\right).$$

Independent of path γ, but not of base point $\gamma(0) = (\tau, \sigma)$

$$d\Omega(z) - \left[A(z), \Omega(z)\right] = 0.$$

Shift generates similarity transformation. Eigenvalues preserved

$$\Omega(z) \simeq \mathrm{diag}\left(e^{i\hat{p}_1(z)}, \ldots, e^{i\hat{p}_4(z)} \middle|\middle| e^{i\tilde{p}_1(z)}, \ldots, e^{i\tilde{p}_4(z)}\right).$$

The quasi-momenta $p_k(z)$ are conserved, gauge-invariant quantities.
Complete (?) set of action variables in Hamilton-Jacobi formalism.

Global Charges

Expansion of Lax connection at $z = 1$:

$$A(1+\epsilon) = J - 2\epsilon * K + \mathcal{O}(\epsilon^2).$$

Global $\mathfrak{psu}(2,2|4)$ charges S can be read off from monodromy at $z = 1$

$$\Omega(1+\epsilon) = I - \epsilon \frac{4\pi S}{\sqrt{\lambda}} + \mathcal{O}(\epsilon^2).$$

Expansion of quasi-momenta (fix $\hat{p}_k(1) = \tilde{p}_k(1) = 0$)

$$\hat{p}_k(1+\epsilon) \sim \epsilon \frac{4\pi(E, S_1, S_2)}{\sqrt{\lambda}} + \ldots, \quad \tilde{p}_k(1+\epsilon) \sim \epsilon \frac{4\pi(J_1, J_2, J_3)}{\sqrt{\lambda}} + \ldots$$

Conjugation Symmetry

$\mathbb{Z}_4$ property of supertranspose: $X^{\mathrm{ST},\mathrm{ST}} = \eta X \eta$, $X^{\mathrm{ST},\mathrm{ST},\mathrm{ST},\mathrm{ST}} = X$.

Conjugation of connection $J = H + Q_1 + P + Q_2$

$$C\,(H, Q_1, P, Q_2)^{\mathrm{ST}}\,C^{-1} = (-H, -iQ_1, +P, +iQ_2).$$

Map $z \mapsto iz$ conjugates Lax connection and monodromy

$$A(iz) = -C\,A^{\mathrm{ST}}(z)\,C^{-1}, \qquad \Omega(iz) = C\,\Omega^{-\mathrm{ST}}(z)\,C^{-1}.$$

Transformation of quasi-momenta with $k' = (2, 1, 4, 3)$, $\varepsilon_k = (+, +, -, -)$

$$\hat{p}_k(iz) = -\hat{p}_{k'}(z), \qquad \tilde{p}_k(iz) = 2\pi m \varepsilon_k - \tilde{p}_{k'}(z).$$

$z \mapsto -z$ is a trivial symmetry of quasi-momenta. Reparametrize:

$$x = \frac{1+z^2}{1-z^2} \qquad z^2 = \frac{x-1}{x+1}.$$

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Analyticity

Monodromy $\Omega(z)$ is analytic in z except at $z = 0, \infty$. Consider $z = 0$:
Diagonalize Lax connection perturbatively with regular $T(z)$

$$\partial_\sigma - \bar{A}_\sigma(z) = T(z)\big(\partial_\sigma - A_\sigma(z)\big)T^{-1}(z).$$

Derivative $\partial_\sigma = \mathcal{O}(z^0)$ subleading w.r.t. $A_\sigma(z) = \mathcal{O}(1/z^2)$:

$$\begin{aligned}\bar{A}(z) &= \tfrac{1}{2}T(P_+ + \Lambda_\sigma)T^{-1}/z^2 + \mathcal{O}(1/z)\\ &= \operatorname{diag}(\alpha, \alpha, \beta, \beta || \alpha, \alpha, \beta, \beta)/z^2 + \mathcal{O}(1/z)\end{aligned}$$

Degeneracies due to conjugation $CP^{\mathrm{ST}}C^{-1} = P$, tracelessness $\operatorname{str} P = 0$ and Virasoro $\operatorname{str} P_+^2 = 0$.

$$\hat{p}_{1,2}(z) \sim \tilde{p}_{1,2}(z) \sim \alpha/z^2, \quad \hat{p}_{3,4}(z) \sim \tilde{p}_{3,4}(z) \sim \beta/z^2 \qquad \text{at } z = 0.$$

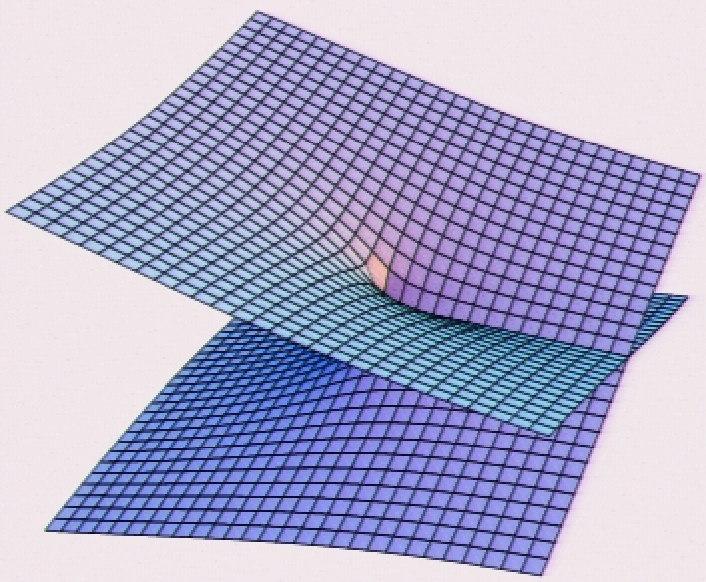
Diagonalization introduces new singularities $\{\hat{z}_a, \tilde{z}_a, z_a^*\}$ in $\hat{p}_k(z), \tilde{p}_k(z)$.

Bosonic Branch Points

Eigenvalue crossing: Consider 2×2 submatrix of AdS_5-part of $\Omega(z)$

$$\Gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \hat{\gamma}_{1,2} = \tfrac{1}{2}\left(a + d \pm \sqrt{(a-d)^2 + 4bc}\right) .$$

Generic behaviour at degenerate eigenvalues $e^{i\hat{p}_k(\hat{z}_a)} = e^{i\hat{p}_l(\hat{z}_a)}$:



$$e^{i\hat{p}_k(\hat{z}_a)}\left(1 \pm \hat{\alpha}_a\sqrt{z - \hat{z}_a} + \mathcal{O}(z - \hat{z}_a)\right) .$$

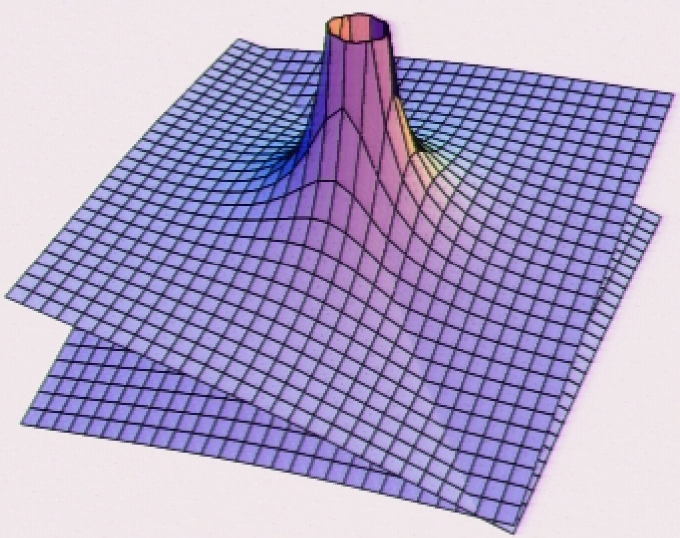
Full turn around $\hat{z}_a$ interchanges eigenvalues (labelling): Branch cut.

Fermionic Singularities

Mixed eigenvalue crossing: Consider $(1|1) \times (1|1)$ submatrix of $\Omega(z)$

$$\Gamma = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right), \qquad \hat{\gamma} = \frac{bc}{d-a} + a, \quad \tilde{\gamma} = \frac{bc}{d-a} + d\,.$$

Generic behaviour at degenerate eigenvalues $e^{i\hat{p}_k(z_a^*)} = e^{i\tilde{p}_l(z_a^*)}$:

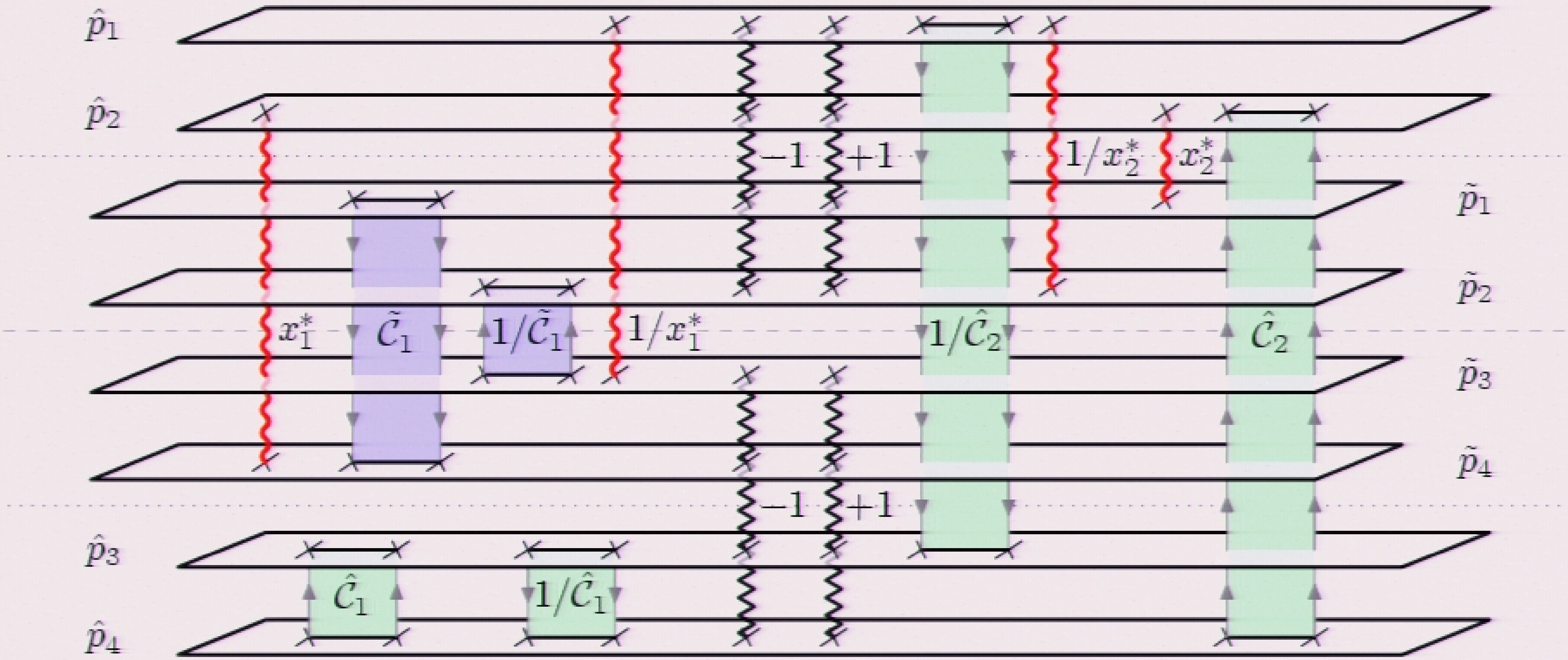


$$e^{i\hat{p}_k(z_a^*)}\left(\frac{\alpha_a^*}{z - z_a^*} + 1 + \mathcal{O}(z - z_a^*)\right).$$

Residue of fermionic singularity $\alpha_a^* \sim bc$ is nilpotent.

Spectral Curve

Using spectral parameter $x = (1+z^2)/(1-z^2)$



For finitely many cuts & singularities: $p'(x)$ is algebraic curve.

Simplest spinning strings have genus $0/1$: algebraic/elliptic functions.

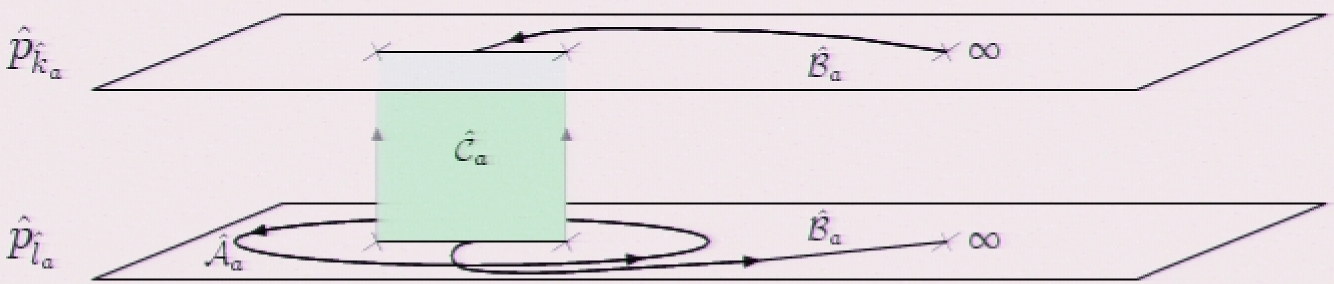
String Moduli

Single-valuedness of e^{ip}: All closed cycles must be integer

$$\oint dp \in 2\pi\mathbb{Z} \quad \text{as well as} \quad \int_{\infty_k}^{\infty_l} dp \in 2\pi\mathbb{Z} \quad \text{and} \quad \int_{\infty_k}^{0_k} dp \in 2\pi\mathbb{Z}.$$

Cuts/singularties: "mode number" $n_a \in \mathbb{Z}$ and "amplitude" $K_a \in \mathbb{R}$

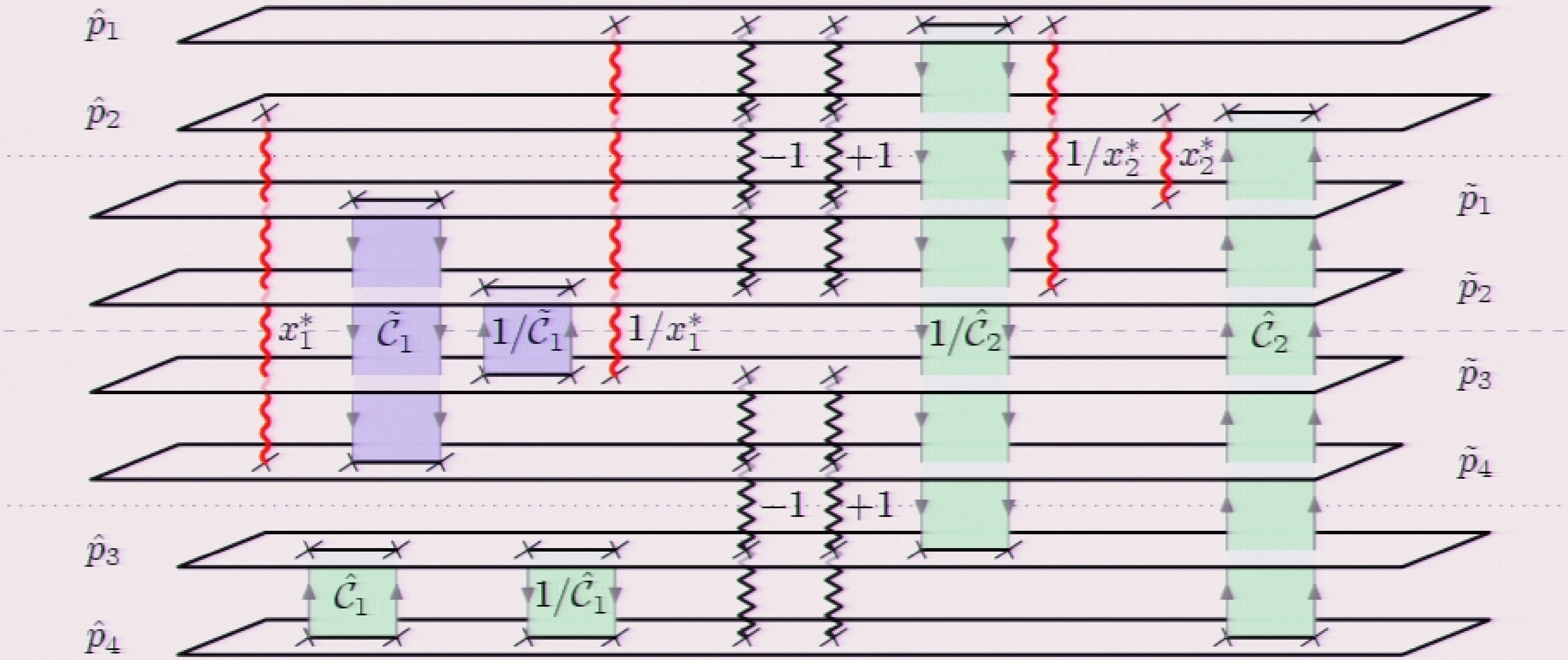
$$\int_{\mathcal{A}_a} dp = 0, \quad n_a = \frac{1}{2\pi}\int_{\mathcal{B}_a} dp, \quad K_a = -\frac{1}{2\pi i}\oint_{\mathcal{A}_a} p(x)\left(1 - \frac{1}{x^2}\right) dx.$$



Continuous moduli of curve: One K_a for each bosonic cut.

Spectral Curve

Using spectral parameter $x = (1 + z^2)/(1 - z^2)$



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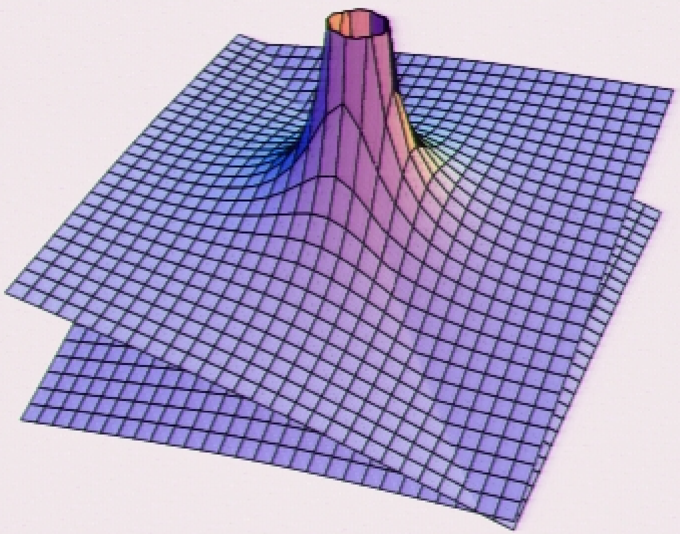


$$e^{i\hat{p}_k(z_a^*)}\left(\frac{\alpha_a^*}{z - z_a^*} + 1 + \mathcal{O}(z - z_a^*)\right).$$

Residue of fermionic singularity $\alpha_a^* \sim bc$ is nilpotent.

Integral Equations

Parametrize quasi-momenta $p(x)$ using 7 resolvents (cuts/poles)

$$G_j(x) = \int_{\mathcal{C}_{j,a}} \frac{dy\,\rho_j(y)}{1-1/y^2}\frac{1}{y-x} + \sum\nolimits_a \frac{\alpha_{j,a}}{1-1/x_{j,a}^2}\frac{1}{x_{j,a}-x}$$

Integral equations with $H_j(x) = G_j(x) + G_j(1/x) - G_j(0)$

$$-2\pi n_{j,a} = \sum_{j'=1}^{7} M_{j,j'} \not{H}_{j'}(x) + F_j(x), \qquad \text{for } x \in \mathcal{C}_{j,a}, x_{j,a}$$

$M_{j,j'}$: Cartan matrix of $\mathfrak{su}(2,2|4)$.

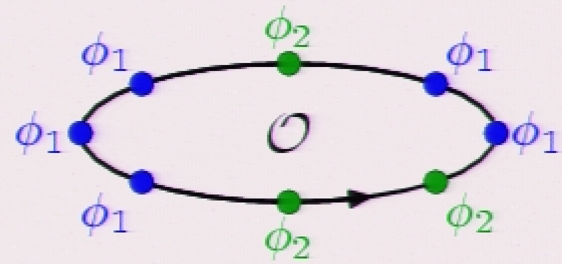
$F_j(x)$: Potential terms made from $G_{j'}(0)$, $G'_{j'}(0)$ and $G_{j'}(1/x)$.

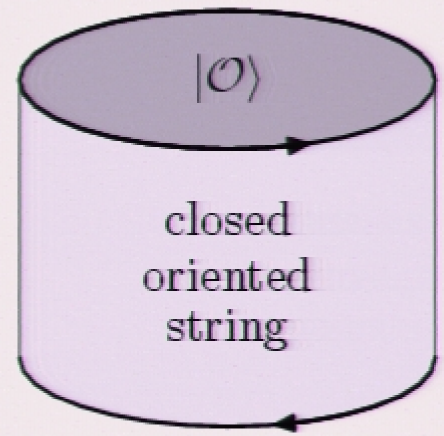
Gauge Theory and Spin Chains

Single trace operator, two complex scalars ϕ_1, ϕ_2 (a.k.a. $\mathcal{Z}, \phi$ or Z, X)

$$\mathcal{O} = \mathrm{Tr}\, \phi_1\phi_1\phi_2\phi_1\phi_1\phi_1\phi_2\phi_2$$

Length L: # of fields

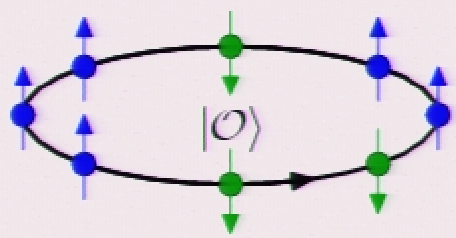




Identify $\phi_1 = |\uparrow\rangle$, $\phi_2 = |\downarrow\rangle$

$$|\mathcal{O}\rangle = |\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow\rangle$$

Length L: # of sites



Operator mixing, quantum superposition: $|\mathcal{O}\rangle = *|\ldots\rangle + *|\ldots\rangle + \ldots$

Dilatation Generator

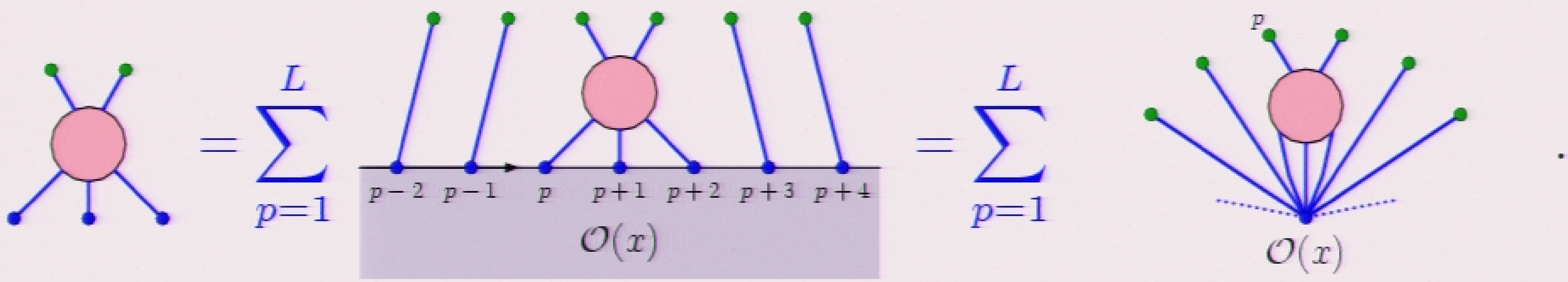
Scaling dimensions $D_{\mathcal{O}}(g)$ as eigenvalues of the dilatation generator $\mathfrak{D}(g)$

$$\mathfrak{D}(g)\,\mathcal{O} = D_{\mathcal{O}}(g)\,\mathcal{O}.$$

Quantum corrections in perturbation theory: $g \sim \sqrt{\lambda}$ [NB hep-th/0407277]

$$\mathfrak{D}(g) = \mathfrak{D}_0 + g^2\,\mathfrak{D}_2 + g^3\mathfrak{D}_3 + g^4\mathfrak{D}_4 + \dots .$$

Local action along spin chain (homogeneous, dynamic)



Bethe Ansatz

Consider $\mathfrak{su}(1|2)$ sector: Two scalars $\mathcal{Z}, \phi$ and fermion ψ. [NB Staudacher]

Hamiltonian for anomalous dimensions: $\mathcal{H} = (\mathfrak{D} - \mathfrak{D}_0)/g^2$.

Vacuum of an infinite chain:

$$|0\rangle = |\ldots \mathcal{Z}\mathcal{Z}\mathcal{Z} \ldots\rangle, \qquad \mathcal{H}|0\rangle = 0.$$

One-particle states with $\mathcal{A} = \phi, \psi$ at position a and momentum p:

$$|\mathcal{A}, p\rangle = \sum_a e^{ipa} |\ldots \mathcal{A}^a \ldots\rangle, \qquad \mathcal{H}|\mathcal{A}, p\rangle = e(p)|\mathcal{A}, p\rangle.$$

- Dispersion relation $e(p)$ derived at three loops,
- equal for ϕ and ψ and
- asympototic form conjectured (plane waves limit).

Elastic Scattering

Two-particle scattering state

$$|\mathcal{A}, p; \mathcal{B}, q\rangle = \sum_{a,b,\mathcal{C},\mathcal{D}} \Psi^{\mathcal{C}\mathcal{D}}_{\mathcal{A}\mathcal{B},ab}(p,q)|\ldots\mathcal{C}^a\ldots\mathcal{D}^b\ldots\rangle.$$

Demand eigenstate of elastic scattering process

$$\mathcal{H}|\mathcal{A}, p; \mathcal{B}, q\rangle = \big(e(p) + e(q)\big)|\mathcal{A}, p; \mathcal{B}, q\rangle$$

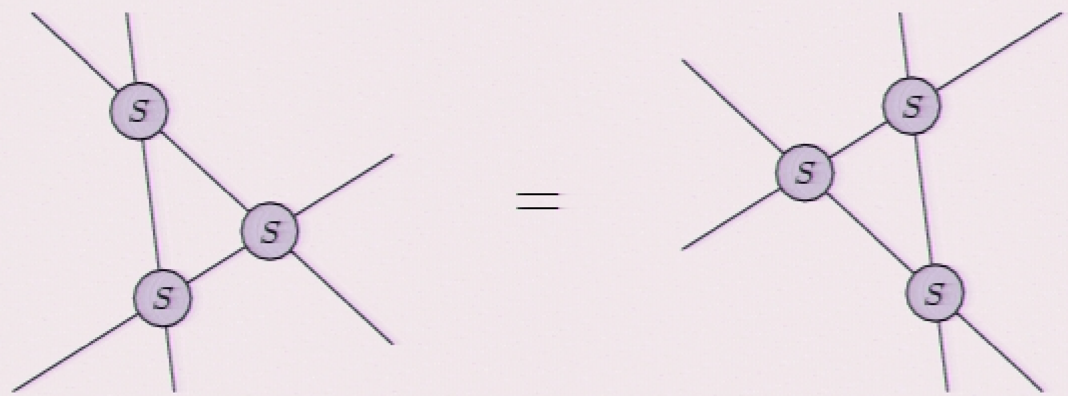
with asymptotics of the wavefunction Ψ

$$\Psi^{\mathcal{C}\mathcal{D}}_{\mathcal{A}\mathcal{B},a\ll b}(p,q) = e^{ipa+iqb}\,\delta^{\mathcal{C}}_{\mathcal{A}}\delta^{\mathcal{D}}_{\mathcal{B}},$$
$$\Psi^{\mathcal{C}\mathcal{D}}_{\mathcal{A}\mathcal{B},a\gg b}(p,q) = e^{ipa+iqb}\,S^{\mathcal{C}\mathcal{D}}_{\mathcal{A}\mathcal{B}}(p,q).$$

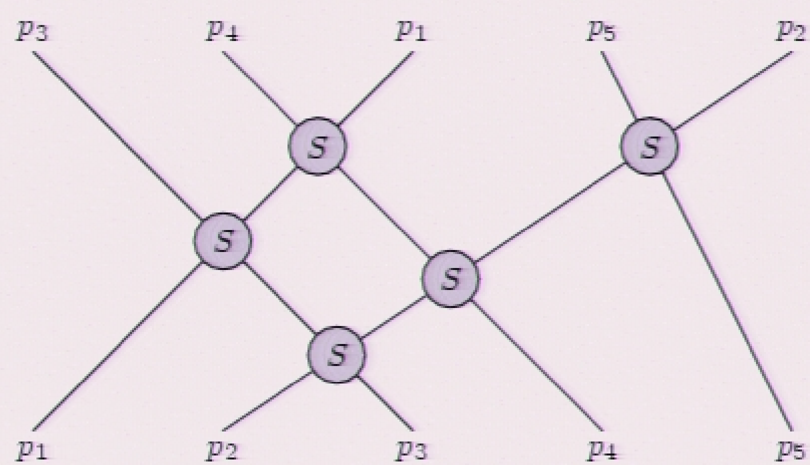
- S-matrix derived at three-loops and
- asympototic form conjectured (apparently new type of S-matrix).

Factorized Scattering

Yang-Baxter equation: Self-consistency condition for factorized scattering

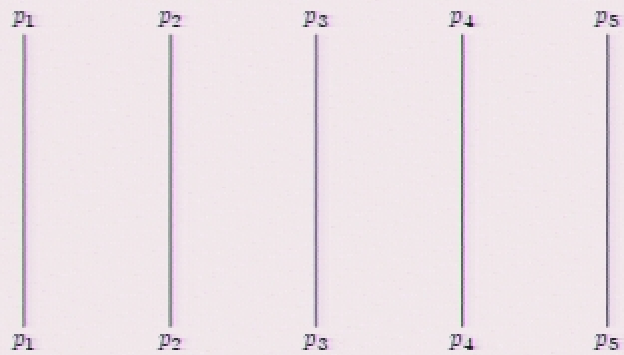


Phase between partial wave functions of an multi-excitation eigenstate

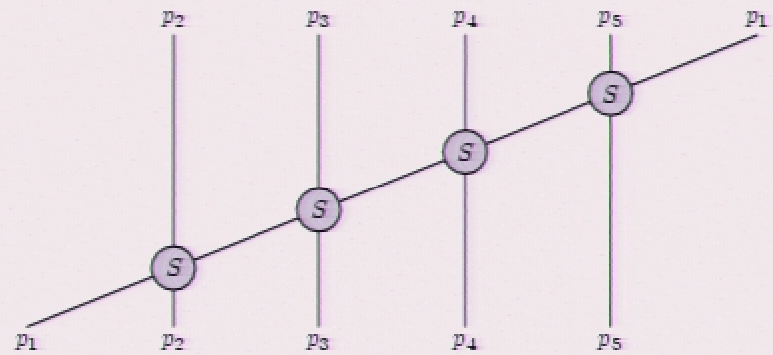


Periodicity

Periodicity of a state on the infinite chain: Bethe equation



$$= e^{ip_1 L}$$



Matrix Bethe equation for mixing between ϕ, ψ. Nested Bethe ansatz:

- Consider a lattice of K excitations ϕ, no mixing (vacuum).
- Replace one ϕ by ψ and diagonalize (dispersion relation).
- Replace two ϕ by ψ and diagonalize (S-matrix).

Two types of excitations with diagonal S-matrix:

- $\mathcal{Z} \to \phi$ from first level,
- $\phi \to \psi$ from second level.

Bethe Equations

$\mathfrak{su}(1|2)$: Two types of diagonal excitations. 4: $\mathcal{Z} \to \phi$, 5: $\phi \to \psi$.

Represent momentum by $x_k = x(p_k)$, $x_k^\pm = x^\pm(p_k)$, $u_k = u(p_k)$.

$$1 = \left(\frac{x_{4,k}^-}{x_{4,k}^+}\right)^L \prod_{\substack{k'=1 \\ k' \neq k}}^{K_4} \frac{u_{4,k} - u_{4,k'} + i}{u_{4,k} - u_{4,k'} - i} \prod_{k'=1}^{K_5} \frac{x_{4,k}^- - x_{5,k'}}{x_{4,k}^+ - x_{5,k'}},$$

$$1 = \prod_{k'=1}^{K_4} \frac{x_{5,k} - x_{4,k'}^+}{x_{5,k} - x_{4,k'}^-}.$$

Cyclicity of the trace & energy eigenvalue (anomalous dimension)

$$1 = \prod_{k'=1}^{K_4} \frac{x_{4,k'}^+}{x_{4,k'}^-}, \qquad E = (D - D_0)/g^2 = \sum_{k'=1}^{K_4} \left(\frac{i}{x_{4,k}^+} - \frac{i}{x_{4,k}^-}\right).$$

Complete Model

$\mathcal{N}=4$ SYM/$\mathfrak{psu}(2,2|4)$ has 7 types of excitations. Bethe equations:

$$1 = U_j(x_{j,k}) \prod_{j'=1}^{7} \prod_{\substack{k'=1 \\ (j',k')\neq(j,k)}}^{K_{j'}} \frac{u_{j,k} - u_{j',k'} + \frac{i}{2}M_{j,j'}}{u_{j,k} - u_{j',k'} - \frac{i}{2}M_{j,j'}}.$$

with Cartan matrix $M_{j,j'}$ and some interactions U_j.

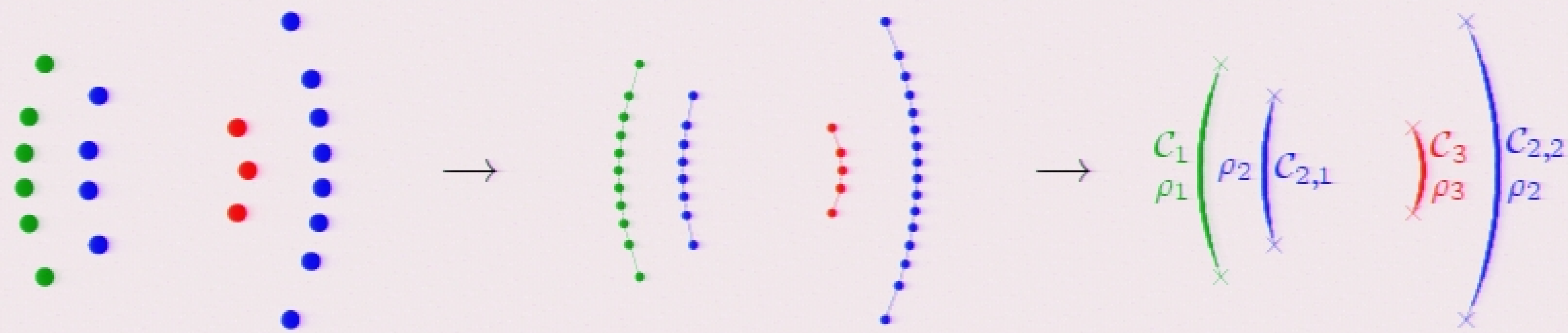
Properties of the Bethe equations: [NB Staudacher]

- Leads to one-loop equations and multiplet splitting at $g=0$. [NB Staudacher]
- Three-loop energies for several states in $\mathfrak{su}(2|3)$ sector agree. [NB hep-th/0310252]
- Aspects of dynamic spin chain can be observed in Bethe equations.
- Restriction to zero-momentum sector (cyclic states) crucial.
- Structure of the algebra $\mathfrak{psu}(2,2|4)$ is important.
- Model has several free parameters: $\mathcal{N}=4$ SYM & string chain.

Thermodynamic Limit

- Long spin chains, $L \to \infty$.
- Large number of Bethe roots $K_j \sim L$.
- Low energy, $E \sim 1/L$.

Roots $x_{j,k}$ condense on contours $\mathcal{C}_j$ (or remain single $x_{j,a}$):



Discrete sums turn into integrals ($\prod = \exp \sum \log$) with densities ρ_j

$$\sum_{k=1}^{K_j} f(x_{j,k}) \to \int_{\mathcal{C}_j} \frac{dx\, \rho_j(x)}{1 - g^2/2x^2}\, f(x).$$

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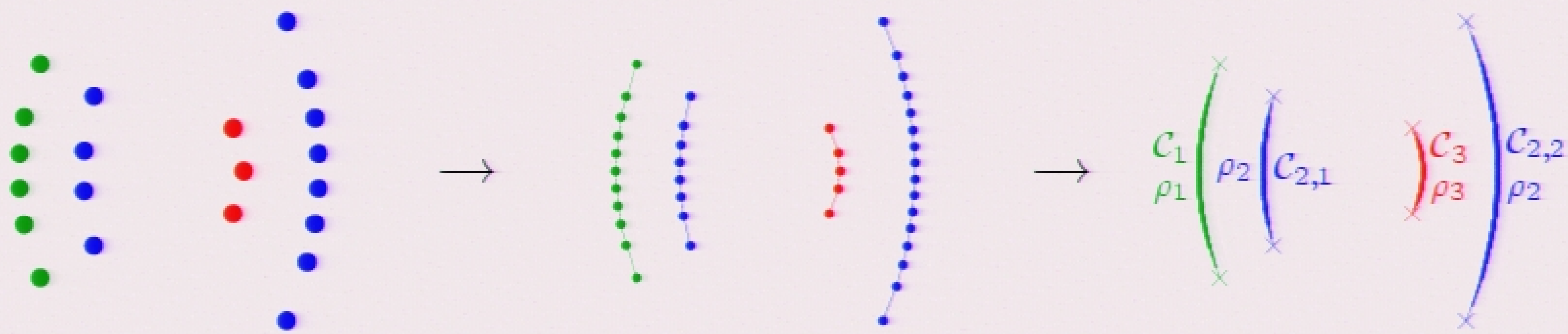

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Comparison

The thermodynamic limit of the Bethe equations yields integral equations

$$-2\pi n_{j,a} = \sum_{j'=1}^{7} M_{j,j'} \not{H}_{j'}(x) + F_j(x), \qquad \text{for } x \in \mathcal{C}_{j,a}, x_{j,a}$$

with some potentials $F_j(x)$ from the functions $U_j(x)$.

- Potentials depend on parameters of spin chain.
- Parameters change behavior starting at three-loops.
- Parameters can be adjusted to string sigma model: String chain.
- Three-loop disagreement with parameters for $\mathcal{N} = 4$ SYM.

Conclusions & Outlook

⋆ **Classical Superstrings on $AdS_5 \times S^5$**

- Analytic properties of monodromy of Lax connection.
- Construction of the spectral curve.
- Integral representation.

⋆ **Perturbative $\mathcal{N} = 4$ SYM**

- S-matrix from dilatatation generator.
- Diagonalization by (nested) Bethe ansatz.
- Generalization to the complete model.

⋆ **Outlook**

- Derive S-matrix for the complete model.
- Nested Bethe ansatz for dynamic model.
- Understand three-loop mismatch: Order of limits? Wrappings?