Title: Quantum Dynamical Semigroups and Entanglement

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Abstract:

QUANTUM DYNAMICAL SEMI–GROUPS
AND
ENTANGLEMENT

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OUTLINE :

- ENTANGLEMENT AND THE PHYSICS OF COMPLETE POSITIVITY
- SEMIGROUPS : DISSIPATIVE QUANTUM DYNAMICS
- ENTANGLEMENT AND POSITIVITY : SQUARE LATTICE STATES ARENA

BIPARTITE ENTANGLEMENT

- A, B , d-DIM. SYSTEMS : HILBERT SPACES $\mathbb{C}^d \ni |\psi\rangle$
- COMPOUND SYSTEM A + B : HILBERT SPACE $\mathbb{C}^d \otimes \mathbb{C}^d \ni |\psi\rangle_A \otimes |\phi\rangle_B$
- STATES (DENSITY MATRICES): ρ_A OF A, ρ_B OF B, ρ_{A+B} OF A+B

EXAMPLE : $d = 2$

$$\mathbb{C}^2 \ni |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longmapsto \rho^{(1)}_{A,B} = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbb{C}^2 \ni |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longmapsto \rho^{(2)}_{A,B} = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho^{(11)}_{A+B} = \rho^{(1)}_A \otimes \rho^{(1)}_B = |0\rangle\langle 0| \otimes |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho^{(22)}_{A+B} = \rho^{(2)}_A \otimes \rho^{(2)}_B = |1\rangle\langle 1| \otimes |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- **SEPARABLE STATES** : Classical Correlations

$$\lambda_{ij} \geq 0,\ \sum_{i,j} \lambda_{ij} = 1 \quad \rho^{sep}_{A+B} = \sum_{i,j} \lambda_{ij}\ \rho^i_A \otimes \rho^j_B$$

- **ENTANGLED STATES** : **NOT sums of tensor products**

Nonclassical Correlations

EXAMPLE : $d = 2, \quad |\widehat{\Psi}^2_+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \qquad |ij\rangle := |i\rangle \otimes |j\rangle$

$$\widehat{P}^2_+ = \frac{1}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + +|1\rangle\langle 1| \otimes |1\rangle\langle 1| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 0|)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = |\hat{\Psi}^2_+\rangle\langle\hat{\Psi}^2_+|$$

ENTANGLEMENT : From Quantum Riddle to Physical Resource

D. Brüss quant-ph/0110078

- *Einstein-Podolski-Rosen* : An entangled wavefunction does not describe the physical reality in a complete way
- *Schrödinger* : For an entangled state the best possible knowledge of the whole does not include the best possible knowledge of its parts
- *Bell* : a correlation that is stronger than any classical correlation
- *Mermin* : a correlation that contradicts the theory of elements of reality
- *Peres* : a trick that quantum magicians use to produce phenomena that cannot be imitated by classical magicians
- *Bennet* : a resource that enables quantum teleportation
- *Shor* : a global structure of the wavefunction that allows for faster algorithms
- *Ekert* : a tool for secure communication
- *Horodecki's* : the need for first applications of positive maps in physics

ENTANGLEMENT : PHYSICAL RESOURCE

- Quantum Teleportation
- Quantum Superdense Coding
- Quantum Cryptography
- Quantum Computation by Measurement Based Quantum Computers

1. ENTANGLEMENT MUST BE DETECTED
2. ENTANGLEMENT IS DECREASED BY NOISE: $\widehat{P}_+^2 \longmapsto \rho$
3. MIXED ENTANGLEMENT NEED BE DISTILLED

HOW TO DETECT ENTANGLEMENT? BY POSITIVE MAPS

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HOW TO DETECT ENTANGLEMENT? BY POSITIVE MAPS

- POSITIVE OBSERVABLES : $M_d(\mathbb{C}) \ni X = X^\dagger \geq 0 \Leftrightarrow$ Positive Spectrum
- POSITIVE MAPS : $\Lambda : M_d(\mathbb{C}) \mapsto M_d(\mathbb{C})$ such that $X \geq 0 \Rightarrow \Lambda[X] \geq 0$

EXAMPLE : Transposition $\mathsf{T} : M_d(\mathbb{C}) \mapsto M_d(\mathbb{C})$, $\mathsf{T}\left[|a\rangle\langle b|\right] = |b\rangle\langle a|$, positive map

EXAMPLE : Partial (Lifted) Transposition $\mathrm{id} \otimes \mathsf{T} : M_d(\mathbb{C}) \otimes M_d(\mathbb{C}) \mapsto M_d(\mathbb{C}) \otimes M_d(\mathbb{C})$

- positive on separable states

$$\mathrm{id} \otimes \mathsf{T}\left[\rho^{sep}_{A+B}\right] = \sum_{i,j} \lambda_{ij}\, \rho^i_A \otimes \mathsf{T}[\rho^j_B]$$

- nonpositive on entangled states

$$\mathrm{id} \otimes \mathsf{T}[\widehat{P}^2_+] = \frac{1}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| + |0\rangle\langle 1| \otimes |1\rangle\langle 0| + |1\rangle\langle 0| \otimes |0\rangle\langle 1|)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{negative eigenvalues } -\tfrac{1}{2}$$

$$\rho_{sep} = \sum_{ij} \lambda_{ij}\, \rho_i^A \otimes \rho_j^B$$

STATISTICAL INTERPRETATION OF QUANTUM MECHANICS

- SPECTRUM OF $\rho = \sum_j r_j \, |r_j\rangle\langle r_j| : \quad \{r_j\}$ PROBABILITY DISTRIBUTION
- PHYSICAL TRANSFORMATIONS MAPS STATES INTO STATES

- **Transposition** : T maps states of A into states
- **Partial Transposition** : $\mathrm{id} \otimes \mathsf{T}$ **does not** map states of $A + B$ into states

Not only linear maps Λ must be positive on $M_d(\mathbb{C})$
Because of Entanglement $\mathrm{id} \otimes \Lambda$ must be positive on $M_d(\mathbb{C}) \otimes M_d(\mathbb{C})$

- Λ POSITIVE on $M_d(\mathbb{C})$ AND $\mathrm{id} \otimes \Lambda$ POSITIVE on $M_d(\mathbb{C}) \otimes M_d(\mathbb{C})$

IFF Λ COMPLETELY POSITIVE

PHYSICAL OPERATIONS MUST BE COMPLETELY POSITIVE MAPS

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$$\mathrm{id}\otimes\mathsf{T}[\widehat{P}^2_+] = \frac{1}{2}\big(|0\rangle\langle 0|\otimes|0\rangle\langle 0| + |1\rangle\langle 1|\otimes|1\rangle\langle 1| + |0\rangle\langle 1|\otimes|1\rangle\langle 0| + |1\rangle\langle 0|\otimes|0\rangle\langle 1|\big)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

negative eigenvalues $-\frac{1}{2}$

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IFF Λ COMPLETELY POSITIVE

PHYSICAL OPERATIONS MUST BE COMPLETELY POSITIVE MAPS

COMPLETELY POSITIVE MAPS

UNLIKE POSITIVE MAPS, CP MAPS HAVE A FIXED STRUCTURE

THEOREM : Λ completely positive **on** $M_d(\mathbb{C})$ iff

$$\Lambda[X] = \sum_{j\in J} G_j^\dagger X G_j\,, \quad G_j : \mathbb{C}^d \mapsto \mathbb{C}^d$$ **Kraus-Stinespring**

$$\Lambda[1] = \sum_{j\in J} G_j^\dagger G_j \quad \Big(= 1 \Longrightarrow \Lambda \quad \text{UNITAL}\Big)$$

EXAMPLES

- Measurement processes (POVM's) : $A = A^\dagger = \sum_j a_j\ P_j$, P_j **spectral projections**

$$\rho \longmapsto \sum_j\ P_j\ \rho\ P_j$$

- Totally depolarizing channel : **Tr the trace operation**

$$\rho \longmapsto \mathbf{Tr}[\rho] := \mathrm{Tr}(\rho)\ \mathbb{1}_d$$

- Unitary Time-Evolution : $\dfrac{\mathrm{d}}{\mathrm{d}t}\rho_t = -i\big[H\,,\,\rho_t\big]$

$$\mathcal{U}_t\ :\ \rho \longmapsto \rho_t = \mathcal{U}_t[\rho] = U_t\ \rho\ U_t^\dagger\,, \quad U_t = \exp(-i\,t\,H)$$

POSITIVE MAPS

POSITIVE MAPS CANNOT CONSISTENTLY DESCRIBE PHYSICAL OPERATIONS

WHAT IS THEN THEIR ROLE IN PHYSICS?

THEY OPERATE AS ENTANGLEMENT DETECTORS

THEOREM:

$\rho_{A+B} \in M_d(\mathbb{C}) \otimes M_d(\mathbb{C})$ entangled **IFF there exists** positive $\Lambda \ : \ M_d(\mathbb{C}) \mapsto M_d(\mathbb{C})$ **such that**

$$\mathrm{Tr}(\mathrm{id}_d \otimes \Lambda[\widehat{P}^d_+]\rho_{A+B}) < 0$$

(**Horodecki, PLA 1996**)

$$\widehat{P}^d_+ = |\Psi^d_+\rangle\langle\Psi^d_+| \ , \quad |\Psi^d_+\rangle = \frac{1}{d}\sum_{i=1}^{d} |i\rangle \otimes |i\rangle$$

STRUCTURE OF POSITIVE MAPS

Unlike completely positive maps, NO GENERAL STRUCTURE OF positive maps

DEFINITION : Λ positive on $\Lambda : M_d(\mathbb{C}) \mapsto M_d(\mathbb{C})$ are decomposable if

$$\Lambda = \Lambda_1 + \Lambda_2 \circ \mathsf{T}\,, \qquad \Lambda_{1,2} \text{ completely positive}$$

THEOREM (Woronowicz, RMP 1976) : All positive $\Lambda : M_2(\mathbb{C}) \mapsto M_{2,3}(\mathbb{C})$ are decomposable

COROLLARY : $\rho_{A+B} \in M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ separable IFF
Positive under Partial Transposition (PPT) : $\mathrm{id} \otimes \mathsf{T}[\rho^{sep}_{A+B}] \geq 0$

- (Woronowicz, RMP 1976; Choi, Lin. Al. 1975; P. Horodecki, PLA 1997 :)

$d \geq 3 \Rightarrow$ there exist PPT Entangled States (PPTES)

THEOREM (Horodecki's PRL 1998):

PPTES cannot be distilled $\Longrightarrow$ there exists bound entanglement

PHYSICALLY IMPORTANT :

1. Entanglement **detection**
2. Bound Entanglement **recognition**

- $d \geq 3$: decomposable positive Λ cannot **detect** PPTES

$$\begin{cases} \rho \text{ PPTES} \\ \\ \Lambda = \Lambda_1 + \Lambda_2 \circ \mathsf{T} \end{cases} \Longrightarrow \begin{cases} \mathrm{id}_d \otimes \Lambda_1[\rho] \geq 0 \\ \mathrm{id}_d \otimes \mathsf{T}[\rho] \geq 0 \\ \mathrm{id}_d \otimes \Lambda_2[\mathrm{id}_d \otimes \mathsf{T}[\rho]] \geq 0 \end{cases}$$

PROPOSITION : ρ_{A+B} PPT

1. Λ decomposable $\Longrightarrow$ **dual** Λ^* decomposable $\quad \mathrm{Tr}(\Lambda[X]\rho) = \mathrm{Tr}(X\Lambda^*[\rho])$
2. **IF** $\mathrm{Tr}(\mathrm{id} \otimes \Lambda[\widehat{P}^d_+]\ \rho_{A+B}) = \mathrm{Tr}(\widehat{P}^d_+\ \mathrm{id} \otimes \Lambda^*[\rho_{A+B}]) < 0$ **THEN** ρ_{A+B} PPTES **AND** Λ indecomposable

QUANTUM DYNAMICAL SEMIGROUPS

Time-evolution of Subsystems S weakly coupled to their Environment E

- **Reversible time-evolution of $S + E$**

$$\rho_{S+E} \longmapsto \mathcal{U}_t^{S+E}[\rho_{S+E}] \; = \; U_t^{S+E} \; \rho_{S+E} \; (U_t^{S+E})^\dagger$$

- $\rho_{S+E} = \rho_S \otimes \rho_E \Longrightarrow$ **Dissipative time-evolution for S :**

$$\rho_S := \mathrm{Tr}_E[\rho_{S+E}] \longmapsto \mathcal{G}_t[\rho_S] \; = \; \mathrm{Tr}_E[\mathcal{U}_t^{S+E}[\rho_{S+E}]]$$

$\mathcal{G}_t$ **are completely positive on $M_d(\mathbb{C})$, NOT a semigroup :**
$\mathcal{G}_t \circ \mathcal{G}_s \neq \mathcal{G}_{t+s}$, $s, t \geq 0$

- **Markovian Approximations : From $\mathcal{G}_t$ TO Semigroups**
$\gamma_t : M_d(\mathbb{C}) \mapsto M_d(\mathbb{C})$

$$\rho_S \longmapsto \gamma_t[\rho_S] \; , \qquad \gamma_t \circ \gamma_s = \gamma_s \circ \gamma_t = \gamma_{t+s} \quad s, t \geq 0$$

Bloch-Redfield maps γ_t : not even positive on $M_d(\mathbb{C})$

Physically consistent γ_t must map states ρ into states $\gamma_t[\rho]$ preserving the positivity of eigenvalues

- **Generators of hermiticity-preserving semigroups on $M_d(\mathbb{C})$:**

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_t[\rho] = L[\rho] , \qquad \gamma_t = \mathrm{e}^{t\,L}$$

$$L[\rho] = -i[H , \rho_t] + \sum_{i,j=1}^{d^2-1} C_{ij}(F_i\rho_t F_j^\dagger - \frac{1}{2}\{F_j^\dagger F_i , \rho_t\})$$

Kossakowski Matrix : $C_{ij} = C_{ij}^*$

$$\mathrm{Tr}(F_i^\dagger F_j) = \delta_{ij} , \; F_{d^2} = \frac{\mathbb{1}}{\sqrt{d}} : \quad \textbf{ONB in } M_d(\mathbb{C})$$

- **NO GENERAL RULES for C_{ij} to generate POSITIVE γ_t**

THEOREM (Gorini et al., JMP 1976: Lindblad, CMP 1976) :

γ_t completely positive **IFF** $[C_{ij}] \geq 0$

- γ_t **COMPLETELY POSITIVE** $\Longrightarrow$ $\mathrm{id}\otimes\gamma_t$ **POSITIVE**

IS COMPLETE POSITIVITY NECESSARY?

CAN ONE DO WITH POSITIVITY ONLY?

- IF γ_t only positive THEN there exists an ancilla S_{anc} and an entangled state $\rho_{S_{anc}+S}$ such that

$$\mathrm{id}\otimes\gamma_t[\rho_{S_{anc}+S}] \quad \text{has negative eigenvalues}$$

- Complete Positivity imposes a hierarchy of decay times

EXAMPLE : $d=2$, $\gamma_t \ : \ M_2(\mathbb{C})\mapsto M_2(\mathbb{C})$, $\rho=\begin{pmatrix}\rho_1 & \rho_3\\ \rho_3^* & \rho_2\end{pmatrix}$

Kossakowski matrix : $C=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

$$L[\rho] \;=\; \frac{1}{2}(\sigma_1\,\rho\,\sigma_1 \;+\; \sigma_2\,\rho\,\sigma_2 \;+\; \sigma_3\,\rho\,\sigma_3 \;-3\,\rho)$$

$$\gamma_t \;=\; \mathrm{e}^{-2t}\mathrm{id}_2+\frac{1-\mathrm{e}^{-2t}}{2}\mathrm{Tr}_2$$

THEOREM (B., Floreanini, Romano, JPA 2002) :

$$\Gamma_t = \gamma_t \otimes \gamma_t \text{ positive } \textbf{IFF } \gamma_t \text{ completely positive}$$

[illegible] Floreanini, Piani, PRL 2005: Γ_t positive **IF**

$$[illegible]\ \rho G_t^{(i)} - \frac{1}{2}\{(G_t^i)^2, \rho\}) \quad \text{with}$$

[illegible] for only one k, [illegible] $\geq |c_2^k|$ all p

EXAMPLE : $d = 4$, $\Gamma_t = \gamma_t^1 \otimes \gamma_t^2 : M_4(\mathbb{C}) \mapsto M_4(\mathbb{C})$.

[illegible]kowski matrices: $C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

ONE-SYSTEM SEMIGROUPS :

$$\gamma_t^1 = e^{-2t}\mathrm{id}_2 + \frac{1 - e^{-2t}}{2}\mathrm{Tr}_2, \qquad \gamma_t^2 = \frac{1 + e^{-2t}}{2}\mathrm{id}_2 + \frac{1 - e^{-2t}}{2}\mathrm{T}_2$$

THEOREM (B., Floreanini, Romano, JPA 2002) :

$$\Gamma_t = \gamma_t \otimes \gamma_t \text{ positive } \textbf{IFF } \gamma_t \text{ completely positive}$$

THEOREM (B., Floreanini, Piani, PRL 2003) : Γ_t positive **IF**

$$L_i[\rho] = \sum_{\ell=1}^{d^2-1} c_i^\ell (G_\ell^{(i)} \rho G_\ell^{(i)} - \frac{1}{2}\{(G_\ell^i)^2, \rho\}) \qquad \textbf{with}$$

$c_2^k = -|c_2^k| < 0$, for only one k , $c_2^\ell \geq |c_2^k|$, $\ell \neq k$; $c_1^p \geq |c_2^k|$ all p

EXAMPLE : $d = 4$, $\Gamma_t = \gamma_t^1 \otimes \gamma_t^2 : M_4(\mathbb{C}) \mapsto M_4(\mathbb{C})$,

Kossakowski matrices: $C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

ONE-SYSTEM SEMIGROUPS :

$$\gamma_t^1 = e^{-2t}\mathrm{id}_2 + \frac{1-e^{-2t}}{2}\mathsf{Tr}_2 , \qquad \gamma_t^2 = \frac{1+e^{-2t}}{2}\mathrm{id}_2 + \frac{1-e^{-2t}}{2}\mathsf{T}_2$$

- **TWO-SYSTEM SEMIGROUP:** $\Gamma_t : \ M_4(\mathbb{C}) = M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \mapsto M_4(\mathbb{C})$

$$\begin{aligned} \Gamma_t &= \gamma_t^1 \otimes \gamma_t^2 \\ &= e^{-2t}\frac{1+e^{-2t}}{2}\mathrm{id}_4 + \frac{1+e^{-4t}}{4}\mathsf{Tr}_2 \otimes \mathrm{id}_2 \\ &+ \frac{1-e^{-2t}}{2}\left(e^{-2t}\mathsf{T}_2 \otimes \mathrm{id}_2 + \frac{1-e^{-2t}}{2}\mathsf{Tr}_2 \otimes \mathrm{id}_2\right) \circ \mathsf{T}_4 \end{aligned}$$

where $\mathsf{T}_4 = \mathsf{T}_2 \otimes \mathsf{T}_2 \ , \quad \mathsf{Tr}_2 \circ \mathsf{T}_2 = \mathsf{Tr}_2$

ARE THE Γ_t DECOMPOSABLE ?

DO THEY DETECT BOUND ENTANGLEMENT?

SQUARE LATTICE STATES ARENA

- **Lattice** : $L_{16} = \{(\alpha,\beta) : \alpha\,,\ \beta = 0,1,2,3\}$
- **Tensor products of Pauli matrices** : $\sigma_{\alpha\beta} := \sigma_\alpha \otimes \sigma_\beta \in M_4(\mathbb{C})$
- **Orthonormal basis of maximally entangled vectors in** $\mathbb{C}^{16}$ **:**

$$|\Psi_{\alpha\beta}\rangle = (\mathrm{id}_4 \otimes \sigma_{\alpha\beta})|\widehat{\Psi}^4_+\rangle\ , \quad |\widehat{\Psi}^4_+\rangle = \frac{1}{2}\sum_{a,b=0}^{1} |a\otimes b\rangle \otimes |a\otimes b\rangle$$

- **Orthogonal 1-dimensional projections :**

$$P_{\alpha\beta} := |\Psi_{\alpha\beta}\rangle\langle\Psi_{\alpha\beta}| = (\mathrm{id}_4 \otimes \sigma_{\alpha\beta})\,\widehat{P}^4_+\,(\mathrm{id}_4 \otimes \sigma_{\alpha\beta})\ , \qquad P_{\alpha\beta}\,P_{\gamma\epsilon} = \delta_{\alpha\gamma}\,\delta_{\beta\epsilon}\ P_{\alpha\beta}$$

- **Square–Lattice States:** $I \subseteq L_{16},\quad N_I = \#(I),$

$$\rho_I = \frac{1}{N_I}\sum_{(\alpha,\beta)\in I} P_{\alpha\beta}$$

THEOREM (F.B., Floreanini, Piani, OSIP 2004) :

NECESSARY AND SUFFICIENT CONDITION FOR PPT

ρ_I PPT IFF for any $(\alpha,\beta) \in L_{16}$ the corresponding column and row DO NOT contain more than $\frac{N_I}{2}$ elements of I, (α,β) excluded

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SUFFICIENT CONDITION FOR BOUND ENTANGLEMENT

ρ_I PPTES IF there exists a column and a row containing ONLY ONE element of I not coinciding with their intersection

• **EXAMPLE :** crosses indicate contributing sites

$N_I = 4$:

3				×
2			×	
1		×		
0	×			
	0	1	2	3

3				
2		×		
1	×			
0	×	×		
	0	1	2	3

$N_I = 6$:

3				
2				
1	×	×	×	
0	×	×	×	
	0	1	2	3

3			×	×
2	×			×
1				×
0		×		
	0	1	2	3

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PROOF OF

1. **THE INDECOMPOSABILITY OF** Γ_t, $0 < t < \frac{\log 3}{2}$

2. **THE ENTANGLEDNESS OF** ρ_6 **and** ρ_{10}

$$\rho_6 = \frac{1}{6}\Big(P_{02} + P_{11} + P_{23} + P_{31} + P_{32} + P_{33}\Big) \,, \quad I_6 =$$

	0	1	2	3
3			×	×
2	×			×
1		×		×
0				

$$\rho_{10} = \frac{1}{10}\Big(P_{02} + P_{11} + P_{12} + P_{13} + P_{21} + P_{22} + P_{23} + P_{31} + P_{32} + P_{33}\Big) \,,$$

$I_{10} =$

	0	1	2	3
3		×	×	×
2	×	×	×	×
1		×	×	×
0				

.

USE Γ_t AND COMPUTE

$$\mathrm{id}_4 \otimes \Gamma_t[\widehat{P}^4_+] = e^{-2t}\frac{1+e^{-2t}}{2}P_{00} + e^{-2t}\frac{1-e^{-2t}}{4}\sum_{\nu=0}^{3}\xi_{0\nu}P_{0\nu}$$

$$+ \frac{1-e^{-4t}}{8}\sum_{\mu=0}^{3}P_{\mu 0} + (\frac{1-e^{-2t}}{4})^2\sum_{\mu,\nu=0}^{3}\xi_{0\nu}P_{\mu\nu}$$

$$\xi_{\beta\delta} = 1 - 2\delta_{|\beta-\delta|,2}$$

$$\xi_{0\nu} = 1 - 2\delta_{\nu,2} = \begin{cases} 1 & \nu \neq 2 \\ -1 & \nu = 2 \end{cases}$$

FOR $0 < t < \frac{\log 3}{2}$:

$$\frac{1-e^{-2t}}{48}(1-3e^{-2t}) < 0\ , \qquad \frac{1-e^{-2t}}{80}(1-3e^{-2t}) < 0$$

$$= \mathrm{Tr}\left(\mathrm{id}_4 \otimes \Gamma_t[\hat{P}^4_+]\,\rho_6\right) \qquad = \mathrm{Tr}\left(\mathrm{id}_4 \otimes \Gamma_t[\hat{P}^4_+]\,\rho_{10}\right)$$

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	0	1	2	3
3			×	×
2	×			×
1		×		×
0				

$$\rho_{10} = \frac{1}{10}\Big(P_{02} + P_{11} + P_{12} + P_{13} + P_{21} + P_{22} + P_{23} + P_{31} + P_{32} + P_{33}\Big)\ ,$$

$I_{10} =$

	0	1	2	3
3		×	×	×
2	×	×	×	×
1		×	×	×
0				

.

USE Γ_t AND COMPUTE

$$\mathrm{id}_4 \otimes \Gamma_t[\widehat{P}_+^4] = e^{-2t}\frac{1+e^{-2t}}{2} P_{00} + e^{-2t}\frac{1-e^{-2t}}{4}\sum_{\nu=0}^{3} \xi_{0\nu} P_{0\nu}$$

$$+ \quad \frac{1-e^{-4t}}{8}\sum_{\mu=0}^{3} P_{\mu 0} + (\frac{1-e^{-2t}}{4})^2 \sum_{\mu,\nu=0}^{3} \xi_{0\nu} P_{\mu\nu}$$

$$\xi_{\beta\delta} = 1 - 2\delta_{|\beta-\delta|,2} \qquad \xi_{0\nu} = 1 - 2\delta_{\nu,2} = \begin{cases} 1 & \nu \neq 2 \\ -1 & \nu = 2 \end{cases}$$

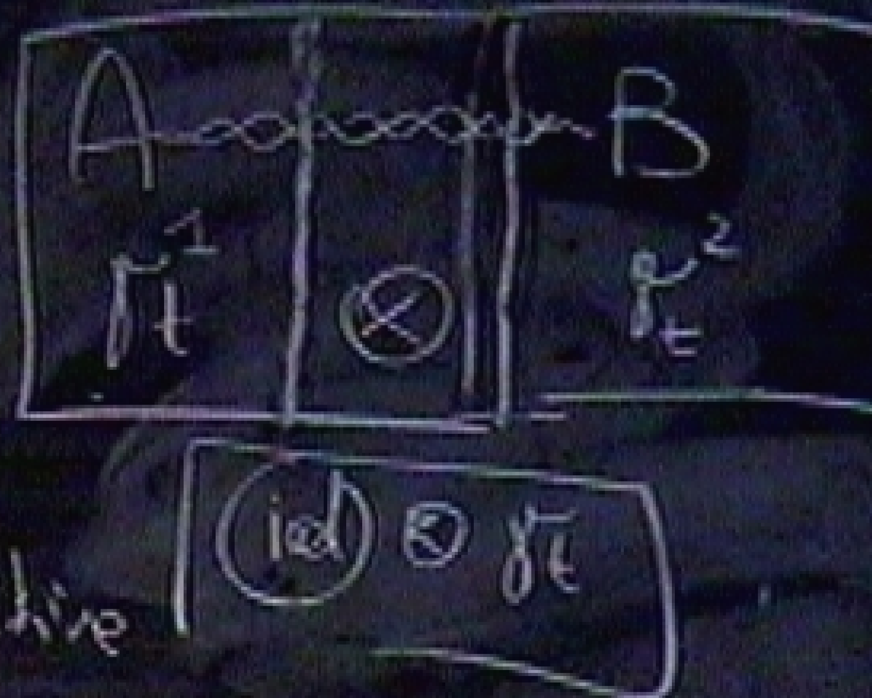
FOR $0 < t < \frac{\log 3}{2}$:

$$\frac{1-e^{-2t}}{48}(1-3e^{-2t}) < 0 \,, \qquad \frac{1-e^{-2t}}{80}(1-3e^{-2t}) < 0$$

$$\| \qquad\qquad\qquad \|$$

$$\mathrm{Tr}\left(\mathrm{id}_4 \otimes \Gamma_t[\hat{P}_+^4]\,\rho_6\right) \qquad \mathrm{Tr}\left(\mathrm{id}_4 \otimes \Gamma_t[\hat{P}_+^4]\,\rho_{10}\right)$$

$\rho_{sep} = \sum_{ij} \alpha_{ij} \, \rho_i^A \otimes \rho_j^B$



Rcut

P: $\Lambda : M_d \to M_d$ $\quad \Lambda$ positive

$id \otimes \Lambda$ positive

$$\Gamma_t \circ \Gamma_s = \Gamma_{t+s} = \Gamma_s \circ \Gamma_t \qquad s,t \geq 0$$