Title: An Update on AdS/CFT: Sasaki-Einstein Metrics, Toric Quivers and Z-Minimisation

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Abstract: In this talk I will summarise the recent progress in AdS/CFT due to the construction of the new infinite family of Sasaki-Einstein metrics Y^{p,q}, and their dual superconformal gauge theories. I will review some aspects of Sasaki-Einstein geometry and the main features of the Y^{p,q} metrics. I will then discuss the use of toric geometry to obtain a description of the corresponding Y^{p,q} Calabi-Yau singularities. I will explain how the AdS/CFT dual N=1 supersymmetric gauge theories were constructed using the combined information obtained from the metrics and the toric singularities. A crucial check on the consistency of the construction is provided by the field theory technique of a-maximisation. In the last part of the talk I will briefly discuss the recently formulated geometric dual of this, i.e. "Z-minimisation".

Perimeter Institute, 12 April, 2005

Dario Martelli
CERN

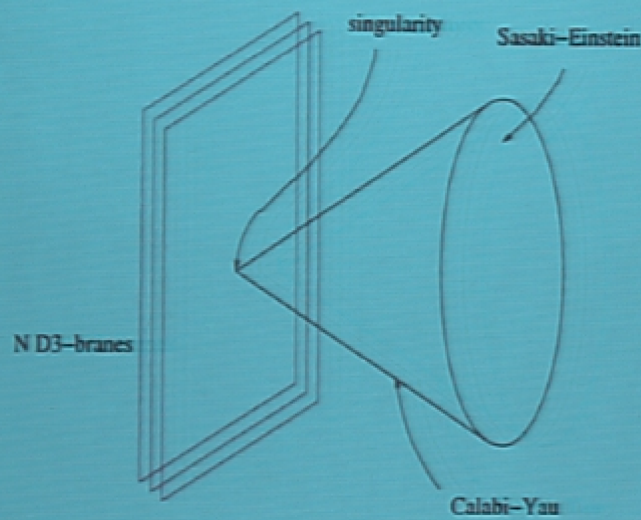
An update on AdS/CFT: Sasaki-Einstein metrics, toric quivers, and Z-minimisation

D.M., J. Sparks +

S. Benvenuti, S. Franco, J. Gauntlett, A. Hanany, D. Waldram, S. -T. Yau

hep-th/0403002, hep-th/0411238, hep-th/0411264, hep-th/0503183

Branes at Calabi–Yau singularities



- N D3-branes $\rightarrow SU(N)$ super-Yang–Mills
- backreacted geometry is $AdS_5 \times S^5$

AdS/CFT: $AdS_5 \times S^5$ is *dual* to $\mathcal{N} = 4$ SYM

$$\underbrace{0123}_{D3}\underbrace{456789}_{\mathbb{R}^6 \rightarrow CY}$$

- D3 branes transverse to CY preserve SUSY
- if CY is cone $\rightarrow$ backreacted geometry is $AdS_5 \times Y_5$

$$ds^2(\mathrm{CY}) = dr^2 + r^2 ds^2(Y_5)$$

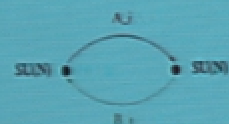
- take this as the *definition* of Sasaki–Einstein

The conifold theory

1) cone(S^5) $= \mathbb{C}^3 \to \mathcal{N} = 4$ SYM ✓

2) cone($T^{1,1}$) = conifold:

- $\mathcal{N} = 1$ $SU(N) \times SU(N)$ quiver gauge theory. With bi-fundamental fields: A_i, B_i $i = 1, 2$



- $U(1)$ isometry of $T^{1,1} \to U(1)_R$ R–symmetry

- $SU(2) \times SU(2)$ isometry of $T^{1,1} \to$ flavour symmetry

- $T^{1,1} \simeq S^2 \times S^3 \to U(1)_B$ baryonic symmetry from KK reduction on S^3 of the four-form potential C_μ

- central charge $a = \frac{16}{27} = \frac{\pi^3}{4\,\mathrm{vol}(T^{1,1})}$

- superpotential $W = \epsilon^{ij}\epsilon^{kl} A_i B_k A_j B_l$ must respect the symmetries

<u>Bottom line</u>: AdS/CFT allows us to study a large number of supersymmetric gauge theories purely in terms of Sasaki–Einstein geometry !

...wd be nice to have more Sasaki–Einstein, besides $T^{1,1}$...

Sasaki–Einstein metrics and AdS/CFT

1979 Tanno: existence of a SE metric on $T^{1,1} \simeq S^2 \times S^3$

⋮

1984 Romans: construction of $T^{1,1}$ metric

⋮

1989 Candelas-de la Ossa: cone over $T^{1,1}$ is the conifold

⋮

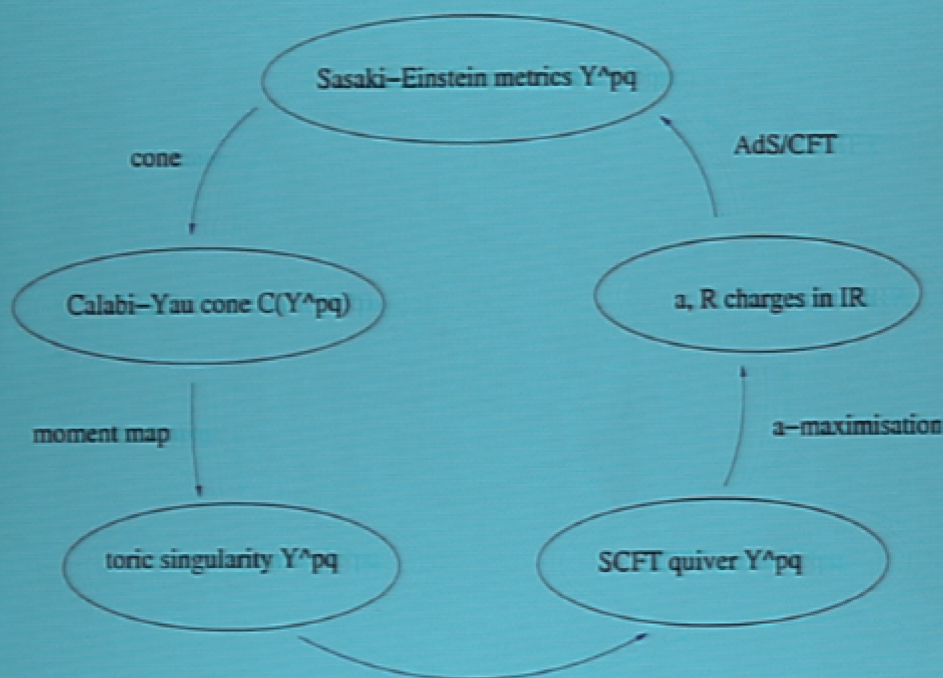
1998 Klebanov-Witten: gauge theory dual of $T^{1,1}$

⋮

Feb 2004 GMSW: Sasaki–Einstein *metrics* $Y^{p,q}$

Nov 2004 MS: cones over $Y^{p,q}$ are *toric Calabi–Yau*'s

Nov 2004 BFHMS: *gauge theory* duals of $Y^{p,q}$


Sasaki–Einstein metrics Y^pq
cone
AdS/CFT
Calabi–Yau cone C(Y^pq)
a, R charges in IR
moment map
a–maximisation
toric singularity Y^pq
SCFT quiver Y^pq

The $Y^{p,q}$ metrics

[Gauntlett,DM,Sparks,Waldram]

$Y^{p,q}$ is an infinite family of inhomogeneous and irregular Sasaki–Einstein metrics:

$$
\begin{aligned}
ds^2(Y^{p,q}) &= \frac{1-y}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1}{w(y)q(y)}dy^2 \\
&+ \frac{q(y)}{9}[d\psi - \cos\theta d\phi]^2 + w(y)\ell^2\,[d\gamma + A]^2
\end{aligned}
$$

Functions $w(y), q(y)$ depend on a discretized parameter

$a = \frac{1}{2} - \frac{p^2-3q^2}{4p^3}\sqrt{4p^2-3q^2}$ $\qquad \ell = \frac{q}{3q^2-2p^2+p(4p^2-3q^2)^{1/2}}$

Isometry: $SU(2) \times U(1) \times U(1)$

Topology: $Y^{p,q} \simeq S^2 \times S^3$ (p, q co-prime)

Reeb: $K = 3\frac{\partial}{\partial\psi} - \frac{1}{2\ell}\frac{\partial}{\partial\gamma}$ ("$U(1)_R$" in the SCFT)

The volume of $Y^{p,q}$ is given by

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\mathrm{vol}(Y^{p,q}) = \frac{q^2[2p + (4p^2-3q^2)^{1/2}]}{3p^2[3q^2-2p^2+p(4p^2-3q^2)^{1/2}]}\pi^3
$$

- It is generically quadratic irrational and it ranges in

$$
\mathrm{vol}(T^{1,1}/\mathbb{Z}_p) > \mathrm{vol}(Y^{p,q}) > \mathrm{vol}(S^5/\mathbb{Z}_2 \times \mathbb{Z}_p)
$$

Sasaki–Einstein geometry

• Geometric data of a SE structure are

◦ a unit-norm Killing vector $K = \frac{\partial}{\partial \psi'}$ (Reeb) and its dual one-form η

◦ a two-form J_4 such that $d\eta = 2J_4$

Finer classification of SE manifolds

• locally $$ds^2(\mathrm{SE}) = ds_4^2 + (d\psi' + \eta)^2$$

with (ds_4^2, J_4) Kähler-Einstein

1) *Regular*: (ds_4^2, J_4) is a KE manifold

2) *Quasi-regular*: (ds_4^2, J_4) is a KE orbifold

3) *Irregular*: (ds_4^2, J_4) is not globally defined

• Only in cases 1) and 2) can the SE be defined as $U(1)$ bundle over the respective KE base manifold/orbifold.

• In case 3) one cannot quotient by $\partial/\partial\psi'$: the orbits of the Reeb don't close, but densely fill a torus T^k ($k \equiv$rank of the SE).

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Calabi–Yau cones $C(Y^{p,q})$

- Metric on the CY cones:

$$ds^2(C(Y^{p,q})) = dr^2 + r^2 ds^2(Y^{p,q})$$

Local KE data $J_4, \Omega_4, \eta \quad \Rightarrow \quad$ CY data J, Ω

$$J = r^2 J_4 + r\mathrm{d}r \wedge (\tfrac{1}{3}\mathrm{d}\psi' + \eta)$$

$$\Omega = e^{i\psi'} r^2 \Omega_4 \wedge \left[\mathrm{d}r + ir(\tfrac{1}{3}\mathrm{d}\psi' + \eta)\right]$$

- Use J to find susy 3-submanifolds in $Y^{p,q}$:

$1/2 J \wedge J$ is the calibrating form for Kähler 4-submanifolds (divisors):

$$\frac{1}{2} J \wedge J|_{M_4} \leq \mathrm{vol}(M_4) \quad \forall \quad M_4$$

$$1/2 J \wedge J|_{C(\Sigma_i)} = \mathrm{vol}(C(\Sigma_i)) \quad \Rightarrow \quad C(\Sigma_i) \text{ are complex}$$

$\Rightarrow \quad \Sigma_1, \Sigma_2, \Sigma_3$ are supersymmetric 3-submanifolds in $Y^{p,q}$

$$\text{e.g.:} \quad \mathrm{vol}(\Sigma_2) = 2\pi^2 \ell^q_p$$

- Use J to find out about toric singularity..

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Toric symplectic manifolds

M symplectic manifold, w/ symplectic form J. A $U(1)$ action on M is symplectic if generated by V preserving J:

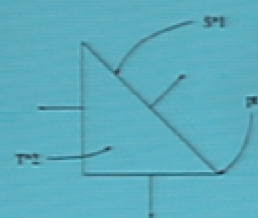
$$\mathcal{L}_V J = d(i_V J) + i_V dJ = d(i_V J) = 0$$

Take μ such that $d\mu = i_V J$ (Hamiltonian)

$\mu : M \to S^1$ is a moment map

- (M, J), $\dim(M) = 2n$ is symplectic toric if there is a $T^n = U(1)^n$ action.
- Result: $\mu(M) = \Delta$ ia a convex polytope in $\mathbb{R}^n$

Example: $\mu(\mathbb{C}P^2) =$



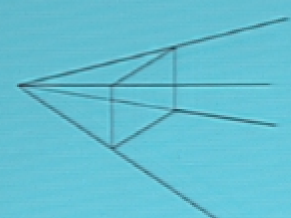
T^2 fibered over Δ

The S^1 collapsing over the edges is generated by the vector field specified by the normals v_i.

If M is a symplectic toric cone

$\mu(M) = \mathcal{C}$ is a polyhedral cone in $\mathbb{R}^n$ with d facets

Example:



4-faceted $\mathcal{C}$ in $\mathbb{R}^3$

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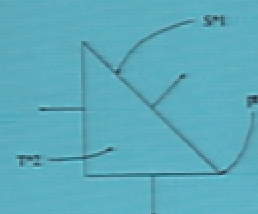
Example: $\mu(\mathbb{C}P^2) =$  T^2 fibered over Δ

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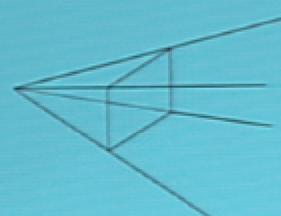
Example:  4-faceted $\mathcal{C}$ in $\mathbb{R}^3$

$C(Y^{p,q})$ are symplectic toric cones

[DM,Sparks]

$$J = \mathrm{d}\phi \wedge \mathrm{d}\left[r^2 \tfrac{1-y}{6} \cos\theta\right] + \mathrm{d}\psi \wedge \mathrm{d}\left[-r^2 \tfrac{1-y}{6}\right] + \mathrm{d}\gamma \wedge \mathrm{d}\left[\ell r^2 y\right]$$

$T^3 = [\partial/\partial\phi, \partial/\partial\psi, \partial/\partial\gamma] \subset SU(2) \times U(1) \times U(1)$ isometry

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- Compute the moment map from J. The image is

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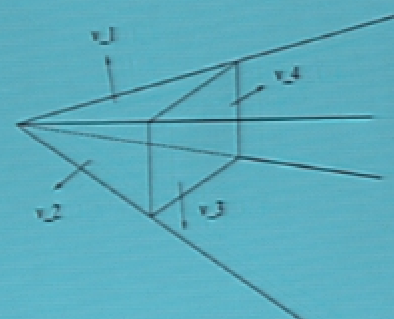

4 facets correspond to toric divisors, which are cones over

$\Sigma_1 \simeq S^3/\mathbb{Z}_{p-q}, \qquad \Sigma_3 \simeq S^3/\mathbb{Z}_{p+q}, \qquad \Sigma_2 \simeq \Sigma_4 \simeq S^3/\mathbb{Z}_p$

- $\{v_i\}$ span $\mathbb{Z}^3$ over $\mathbb{Z}$ $\Rightarrow$ $\pi_1(Y^{p,q}) \simeq \mathbb{Z}^3/ < v_i >_{\mathbb{Z}} \simeq 0$
- $\pi_2(Y^{p,q}) \simeq \mathbb{Z}_{d-n} = \mathbb{Z}$ $\Rightarrow$ $b_2(Y^{p,q}) = 1$

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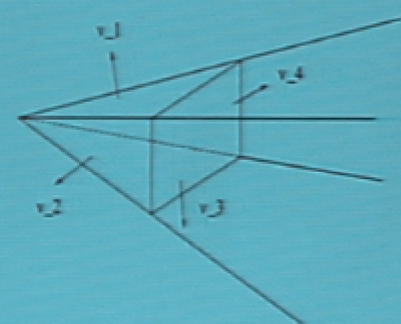

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- $\{v_i\}$ span $\mathbb{Z}^3$ over $\mathbb{Z}$ $\quad \Rightarrow \quad \pi_1(Y^{p,q}) \simeq \mathbb{Z}^3/ < v_i >_{\mathbb{Z}} \simeq 0$
- $\pi_2(Y^{p,q}) \simeq \mathbb{Z}_{d-n} = \mathbb{Z} \quad \Rightarrow \quad b_2(Y^{p,q}) = 1$

Toric geometry and $Y^{p,q}$

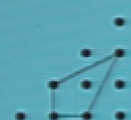
[DM,Sparks]

- Normals $\{v_i\} \rightarrow$ gauged linear sigma model

$$C(Y^{p,q}) \simeq \mathbb{C}^4//U(1) \qquad Q = (p, p, -p+q, -p-q)$$

Toric diagrams: because of the CY condition, the information of the moment cone in $\mathbb{R}^3$ is entirely encoded in polyhedrons in $\mathbb{R}^2$. The normals v_i "end" on a plane:

Projecting v_i onto this plane we get diagrams



$Y^{2,1}$

$Y^{3,1}$

- Each $Y^{p,q}$ diagram is embedded in that of $\mathbb{C}^3/\mathbb{Z}_{p+1} \times \mathbb{Z}_{p+1}$
- Special limits: $p = q$ and $q = 0$

$Y^{3,3}$

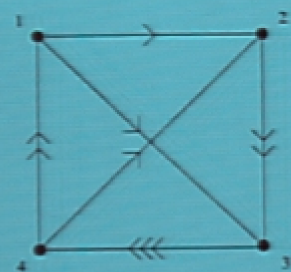
this is $\mathbb{C}^3/\mathbb{Z}_{2p}$ with action
$(z_1, z_2, z_3) \rightarrow (\omega_{2p} z_1, \omega_{2p} z_2, \omega_{2p}^{-2} z_3)$

$Y^{3,0}$

this is conifold$/\mathbb{Z}_p$ with action
$B_1 \rightarrow \omega_p B_1, \quad B_2 \rightarrow \omega_p^{-1} B_2$ where
$|A_1|^2 + |A_2|^2 = |B_1|^2 + |B_2|^2$

Quivers and toric quivers

Consider example (dP_1 theory) : $U(N)^4$ theory



Nodes $\bullet = U(N)$ gauge group

${}_i \longrightarrow_j =$ bifundamental X_{ij} in $(\mathbf{N}, \bar{\mathbf{N}})$ of $U_i(N) \times U_j(N)$

Multiple arrows are assigned to a rep. of the flavour symmetry

Moduli space (consider $U(N) \to U(1)$)

D–terms: $\vec{\mu} = d \cdot |\vec{X}|^2 = \vec{\xi}_{\rm FI}$ (d is "incidence matrix")

F–terms: $\partial W / \partial X = 0$ $\rightarrow$ "toric quiver" gives monomial equations

$F + D$ terms are repackaged in terms of some auxiliary fields $p_\alpha, \alpha = 1, \ldots c$, to produce a Kähler quotient

$$\mathbb{C}^c // U(1)^{c-3}$$

which is a 3-dim$_{\mathbb{C}}$ toric variety. By construction, it is also Calabi–Yau.

Constructing the quivers

[Benvenuti,Franco,Hanany,DM,Sparks]

"Partial resolutions" of $\mathbb{C}^3/\mathbb{Z}_{p+1} \times \mathbb{Z}_{p+1} \rightarrow$ not computationally practical..

- Constraints from the geometry: topology and isometry.

\# gauge groups = # of cycles (in the resolved geometry!) on which D3, D5, D7 branes can wrap, i.e. 0-, 2-, 4-cycles.

$= b_0 + b_2 + b_4 =$ Euler characteristic $\qquad (b_{odd} = 0)$

$= \#$ vertices in the polytope in $\mathbb{R}^3 = 1 + p + (p-1) = 2p$

- From here.. "educated guess"!

$$Y^{p,p} = S^5/\mathbb{Z}_{2p}, \quad Y^{p,0} = T^{1,1}/\mathbb{Z}_p, \quad Y^{2,1} = C_{\mathbb{C}}(dP_1)$$

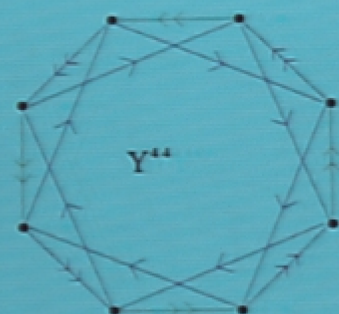
$SU(2)$ flavour symmetry is a strong constraint

Susy 3-submanifolds $\Sigma_i \rightarrow$ baryonic operators

<u>Final check</u>: compute R-charges with a-maximisation

Iterative procedure starting with $Y^{p,p} = \mathbb{C}^3/\mathbb{Z}_{2p}$

- Quiver is constructed using standard orbifold techniques



Gauge group $SU(N)^{2p}$ with $4p + 2p = 6p$ bifundamentals

$SU(2) \times U(1)$ isometry $\Rightarrow$ global flavor symmetry

$2p$ Y_i singlets, p U_i^α doublets, p V_i^α doublets

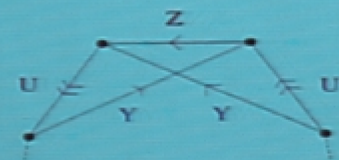
$$W = \sum_{i=1}^{p} \epsilon_{\alpha\beta}(U_i^\alpha V_i^\beta Y_{2i+2} + V_i^\alpha U_{i+1}^\beta Y_{2i+3})$$

- $2p$ loops $\times$ (3 legs in each loop) $\times$ (2 from $\epsilon_{\alpha\beta}$) $= 12p$ $= 6p \times 2$ fields, as expected from toric varieties

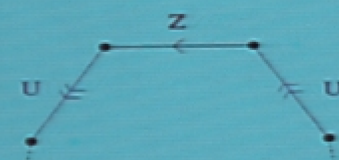
- Now apply a 3-step procedure which preserves symmetries and "toricity"

Pick an (arbitrary) arrow corresponding to a V doublet

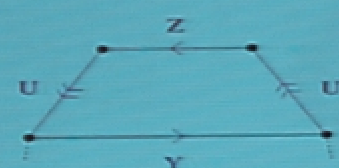
1) replace a V doublet with a new singlet Z



2) remove the two diagonal singlets Y



3) add a new singlet Y to close the new loop

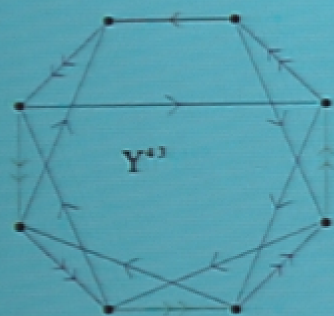


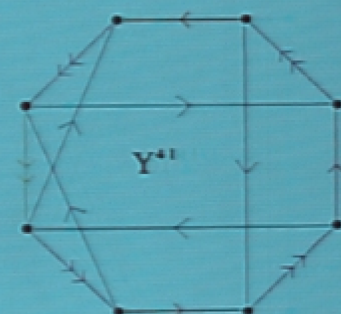
- # fields $6p \to 6p-2$
- In W: -2 cubic terms $\to$ +1 quartic $\Rightarrow$ $12p-4$ fields
- Repeating the procedure $p-q$ times gives a $Y^{p,q}$ *quiver*

fields $= 6p - 2(p-q) = 4p + 2q = p$ U doublets $+ q$ V doublets $+ (p-q)$ Z singlets $+ (p+q)$ Y singlets

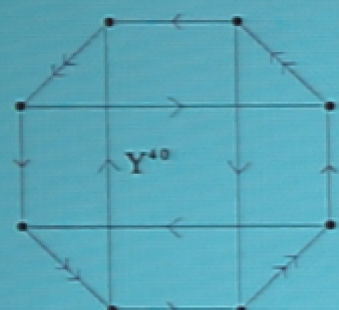
$$W = \sum^{2q} UVY + \sum^{p-q} ZUYU$$

Some examples





$Y^{p,0}$ =conifold/$\mathbb{Z}_p$



- For $Y^{p,0}$: # fields $= 4p =$ (4 "conifold fields")$\times p$
$= p\ U$ doublets $+$ ($p\ Z$ singlets $+\ p\ Y$ singlets)

$W = \Sigma^p$ (quartic terms)

a-maximisation

- Any 4d $\mathcal{N}=1$ SCFT must have a $U(1)_R$ R-symmetry, which is part of the superconformal algebra $SU(2,2|1)$.

- Knowledge of exact R-charges determines the central charges

$$a = \frac{3}{32}(3\mathrm{Tr}R^3 - \mathrm{Tr}R) \qquad c = \frac{1}{32}(9\mathrm{Tr}R^3 - 5\mathrm{Tr}R)$$

$$< T^\mu_\mu > = c(\mathrm{Weyl})^2 + a(\mathrm{Euler})^2$$

- The issue: the full symmetry group of a SCFT may contain additional global flavor symmetries. The abelian part can mix with $U(1)_R$.

- Resolution: consider a "trial" R-symmetry (subject to consistency conditions)

$$R_{\mathrm{trial}} = R_0 + \sum_I s^I F_I$$

F_I are generators of flavour $U(1)$'s; $s^I \in \mathbb{R}$ (a priori)

- The exact R-charges are those that *maximise* the central charge a_{trial} over s^I. And $a = a_{max}$.

The generic charges in 4d $\mathcal{N}=1$ SCFTs are *algebraic irrational* numbers.

R-charges of $Y^{p,q}$ quivers using a-maximisation

- Obtain a parameterization (the s^I) of the R-charges

- vanishing of β-functions at each node

- terms in W have R-charge 2

$R[Z]=x,\ R[Y]=y,\ R[V]=1+\frac{1}{2}(x-y),\ R[U]=1-\frac{1}{2}(x+y)$

$a(x,y)=2p+(p-q)(x-1)^3+(p+q)(y-1)^3-\frac{p}{4}(x+y)^3+\frac{q}{4}(x-y)^3$

- Maximising $a(x,y)$ gives the R-charges and a central charge of the $Y^{p,q}$ quiver theory at its IR fixed point.

These agree with values obtained from the $Y^{p,q}$ metric!

$$a_{max}=\frac{\pi^3}{4\mathrm{vol}(Y^{p,q})}=\frac{3p^2[3q^2-2p^2+p(4p^2-3q^2)^{1/2}]}{4q^2[2p+(4p^2-3q^2)^{1/2}]}$$

Field	$R-$charge	$U(1)_B$
Y	$(-4p^2+3q^2+2pq+(2p-q)\sqrt{4p^2-3q^2})/3q^2$	$p-q$
Z	$(-4p^2+3q^2-2pq+(2p+q)\sqrt{4p^2-3q^2})/3q^2$	$p+q$
U	$(2p(2p-\sqrt{4p^2-3q^2}))/3q^2$	$-p$
V	$(3q-2p+\sqrt{4p^2-3q^2})/3q$	q

Baryons and 3-cycles

There are four types of baryonic operators

$$\mathcal{B}[A] = \epsilon^{\alpha_1 \ldots \alpha_N} A_{\alpha_1}^{\beta_1} \ldots A_{\alpha_N}^{\beta_N} \epsilon_{\beta_1 \ldots \beta_N}$$

$\mathcal{B}[Z], \mathcal{B}[Y]$ are *singlets* of $SU(2)$

$\mathcal{B}[V], \mathcal{B}[U]$ transform in the $\mathbf{N+1}$ of $SU(2)$

• From the a-maximization calculation we know the respective R-charges – these are simply $N \cdot R[A]$

• In the geometry baryons are D3 wrapped on susy 3-submanifolds Σ, and their R-charges are given by

$$R[\mathcal{B}_i] = \frac{\pi}{3\,\mathrm{vol}(Y^{p,q})} \mathrm{vol}(\Sigma_i)$$

Recall we identified 3 susy cycles in $Y^{p,q}$:

$$\begin{array}{rcc} \Sigma_1 \simeq S^3/\mathbb{Z}_{p-q} & \rightarrow & \mathcal{B}[Z] \\ \Sigma_2 \simeq \Sigma_4 \simeq S^3/\mathbb{Z}_p & \rightarrow & \mathcal{B}[U] \\ \Sigma_3 \simeq S^3/\mathbb{Z}_{p+q} & \rightarrow & \mathcal{B}[Y] \\ \Sigma_5 \equiv -\Sigma_1 - \Sigma_3 & \rightarrow & \mathcal{B}[V] \end{array}$$

• Their volumes match the field theoretical computation of R-charges.

• $\Sigma_2, \Sigma_4 \rightarrow S^2$ degeneracy $\rightarrow N+1$ degeneracy of states (quantization of zero-modes of D3 in N units of flux)

R-charges of $Y^{p,q}$ quivers using a-maximisation

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A geometric dual of a-maximisation

[DM,Sparks,Yau]

Toric singularity $\{v_i\}$ → quiver gauge theory →

$$\rightarrow a_{\{v_i\}}(\text{R-symmetry}) \stackrel{a-\text{maximisation}}{\longrightarrow} \textit{exact}\ \text{R-symmetry} \rightarrow$$

$$\rightarrow a(R_{\text{exact}}), R_i(R_{\text{exact}}) \stackrel{\text{AdS/CFT}}{\longrightarrow} \text{vol(SE)}, \text{vol}(\Sigma_i)$$

• Can we compute the volumes of SE manifolds without knowing the metric?

A geometric dual of a-maximisation

[DM,Sparks,Yau]

Toric singularity $\{v_i\}$ $\rightarrow$ quiver gauge theory $\rightarrow$

$\rightarrow a_{\{v_i\}}$(R-symmetry) $\stackrel{a-\text{maximisation}}{\longrightarrow}$ *exact* R-symmetry $\rightarrow$

$\rightarrow a(R_{\text{exact}}), R_i(R_{\text{exact}}) \stackrel{\text{AdS/CFT}}{\longrightarrow}$ vol(SE), vol(Σ_i)

- Can we compute the volumes of SE manifolds without knowing the metric?

- If Sasaki–Einstein is regular: $U(1)_R \rightarrow SE \rightarrow KE$

$$\text{vol}(SE) = \text{vol(fibre)} \cdot \text{vol}(KE) = \text{const.} \int_{KE} c_1^2$$

- What should we extremise over?

Key: K Reeb vector $\leftrightarrow$ $U(1)_R$ R-symmetry

$$\mathcal{L}_K \Omega \neq 0 \quad \Rightarrow \quad \mathcal{L}_K \epsilon \neq 0$$

Z-minimisation

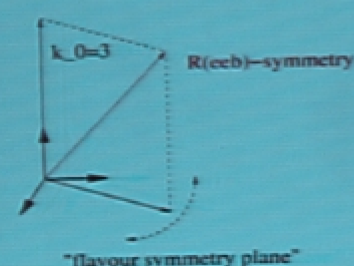
- Idea: consider enlarged space of Sasakian metrics

Sasakian = cone is Kähler, but not Ricci-flat

- Reeb vector still exists!

Given a toric singularity $\{v_i\}$, we define a function $Z_{\{v_i\}}(k_i)$ over the space of Reeb vectors K:

$$K \in \mathbb{T}^3 \quad \Rightarrow \quad K = k_i \frac{\partial}{\partial \phi_i}$$



Why minimise Z? Because $Z \sim \int R + \Lambda$

$\frac{\partial Z}{\partial k_i} = 0 \Rightarrow$ Einstein metric!

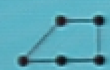
Toric singularity $\{v_i\} \rightarrow Z_{\{v_i\}}(k_i) \xrightarrow{Z\text{-minimisation}}$

$\rightarrow$ " *exact*" Reeb $\rightarrow Z(k_i^{\min}) = \mathrm{vol(SE)}, \mathrm{vol}(\Sigma_i)$

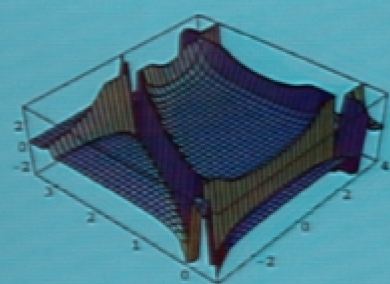
- Reeb, volumes are algebraic numbers

Example: Z-min vs. a-max for SPP

- The "suspended pinch point" is a toric CY singularity

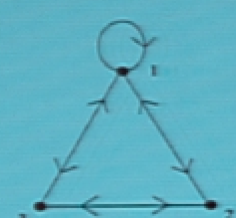


$$Z_{\{n\}}[x,y,t] = \frac{(x-2)(2x-t)}{2t(t-x)(t-x-y)(x-y)} \qquad k_i = (x,y,t)$$



$$dZ = 0 \quad \Rightarrow \quad \mathrm{vol}(Y_{SPP}) = \tfrac{2}{9}\sqrt{3}\pi^3$$

- Consider now the quiver gauge theory and a-maximisation



$$\begin{aligned} a(x,w,z,s) = 3 &+ (x-1)^3 + (w-1)^3 + (z-1)^3 + (s-1)^3 \\ &+ (x+w-1)^3 + (1-x-w-z)^3 + (1-x-w-s)^3 \end{aligned}$$

$$da = 0 \quad \Rightarrow \quad a(Y_{SPP}) = \frac{3}{8}\sqrt{3} = \frac{\pi^3}{4 \cdot \mathrm{vol}(Y_{SPP})}$$