Title: Quantum gravity at astrophysical distances

Date: Mar 16, 2005 02:00 PM

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Abstract: In this talk we assume that Quantum Einstein Gravity (QEG) is the correct theory of gravity on all length scales. We use both analytical results from nonperturbative renormalization group (RG) equations and experimental input in order to describe the special RG trajectory of QEG which is realized in Nature. We identify a regime of scales where gravitational physics is well described by classical General Relativity. Strong renormalization effects occur at both larger and smaller momentum scales. The former are related to the (conjectured) nonperturbative renormalizability of QEG. The latter lead to a growth of Newton's constant at large distances. We argue that this effect becomes visible at the scale of galaxies and could provide a solution to the astrophysical missing mass problem which does not require dark matter. A possible resolution of the cosmological constant problem is proposed by noting that all RG trajectories admitting a long classical regime automatically imply a small cosmological constant.

The assumption:

Quantum Einstein Gravity (QEG), the (presumably) asymptotically safe (= nonperturbatively renormalizable) field theory of the metric tensor, is the correct theory of gravity on all length scales.

The goal:

Derive testable predictions.

The problem:

Find eff. action $\Gamma[g_{\mu\nu}]$, or equivalently:

Determine the Wilsonian renormalization group (RG) flow of QEG from the UV (sub-Planckian distances) to the IR (..., galactic, cosmological scales).

$\rightsquigarrow$ extremely difficult nonpert. problem ($\sim$ QCD)

The strategy:

Use

analytical RG results ("mostly UV")

theory of gravity on all length scales.

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The strategy:

Use
a) analytical RG results ("mostly UV")
b) observational data ("mostly IR")

to characterize the RG trajectory realized in Nature, and to determine its parameters.

3. The UV fixed point: nonperturbative renormalizability

4. With the Einstein-Hilbert truncation towards the IR:
analytical evidence for strong renormalization effects at large distances

5. The RG trajectory realized in Nature:
existence of a classical regime and the cosmological constant problem

6. The deep IR: a phenomenological approach

a) Cosmology without quintessence:
running G and λ "accelerate" the Universe

→ b) Galaxies without dark matter:
flat rotation curves from running G

- Evaluating $\int \mathcal{D}g_{\mu\nu}\, e^{-S[g_{\mu\nu}]}$ is equivalent to finding the "Effective Average Action" $\Gamma_k[g_{\mu\nu}]$.
- k = IR cutoff, only modes with momentum $> k$ are integrated out; $\Gamma_{k\to\infty} = S$, $\Gamma_{k=0} = \Gamma$.
- Γ_k defines effective theory valid at length scale $\ell \approx k^{-1}$, i.e. effective eq. of motion $\delta\Gamma_k[g]/\delta g_{\mu\nu} = 0$ contains all quantum effects relevant at this scale.
- RG trajectory $k \mapsto \Gamma_k[\cdot]$ is given by ERGE:

$$\partial_t \Gamma_k = \frac{1}{2}\,\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1} \partial_t R_k\right], \quad t \equiv \ln k$$

Nonperturbative approximations are obtained by truncating the space of action functionals; example:

$$\Gamma_k[g_{\mu\nu}] = -\frac{1}{16\pi G(k)} \int d^4x \sqrt{g}\,\{R - 2\Lambda(k)\} + \beta(k) \int d^4x \sqrt{g}\, R^2 + \cdots$$

a

A Wilson-type Effective Action for Gravity

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running Newton constant: $G(k)$, $g(k) \equiv k^2 G(k)$

running cosmological constant: $\Lambda(k)$, $\lambda(k) \equiv \Lambda(k)/k^2$

dimensionless

Flow on the g-λ-plane (Einstein–Hilbert truncation)

M.R., 1996

$$\begin{cases} \partial_t g = \beta_g(g,\lambda) \equiv [d-2+\eta_N]\, g \\ \partial_t \lambda = \beta_\lambda(g,\lambda) \end{cases}$$

anomalous dimension: $\partial_t G = \eta_N G$

$$\eta_N = \frac{g\, B_1(\lambda)}{1-g\, B_2(\lambda)}$$

$$B_1(\lambda) \equiv \tfrac{1}{3}(4\pi)^{1-\frac{d}{2}}\Big[d(d+1)\,\Phi^1_{\frac{d}{2}-1}(-2\lambda) - 6d(d-1)\,\Phi^2_{\frac{d}{2}}(-2\lambda) - 4d\,\Phi^1_{\frac{d}{2}-1}(0) - 24\,\Phi^2_{d/2}(0)\Big]$$

$$B_2(\lambda) \equiv -\tfrac{1}{6}(4\pi)^{1-\frac{d}{2}}\Big[d(d+1)\,\tilde{\Phi}^1_{\frac{d}{2}-1}(-2\lambda) - 6d(d-1)\,\tilde{\Phi}^2_{\frac{d}{2}}(-2\lambda)\Big]$$

$$\beta_\lambda = -(2-\eta_N)\lambda + \tfrac{1}{2}(4\pi)^{1-\frac{d}{2}} g \Big[2d(d+1)\,\Phi^1_{\frac{d}{2}}(-2\lambda) - 8d\,\Phi^1_{d/2}(0) - d(d+1)\,\eta_N\,\tilde{\Phi}^1_{d/2}(-2\lambda)\Big]$$

Threshold functions

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Threshold functions

$$\Phi^p_n(w) \equiv \frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}\,\frac{R^{(0)}(z) - z\,R^{(0)\prime}(z)}{\left[z + R^{(0)}(z) + w\right]^p}$$

have singularities at $w = -1 \;\hat{=}\; \lambda = \frac{1}{2}$!

$\Rightarrow$ Flow on half-plane $\lambda < \frac{1}{2}$.

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4

$d = 4$, sharp cutoff

$$\boxed{\begin{aligned} \partial_t \lambda &= -(2-\eta_N)\lambda - \frac{g}{\pi}\left[5 \ln(1-2\lambda) - \varphi_2 + \frac{5}{4}\eta_N\right] \\ \partial_t g &= (2+\eta_N) g \\ \eta_N &= -\frac{2g}{6\pi + 5g}\left[\frac{18}{1-2\lambda} + 5\ln(1-2\lambda) - \varphi_1 + 6\right] \end{aligned}}$$

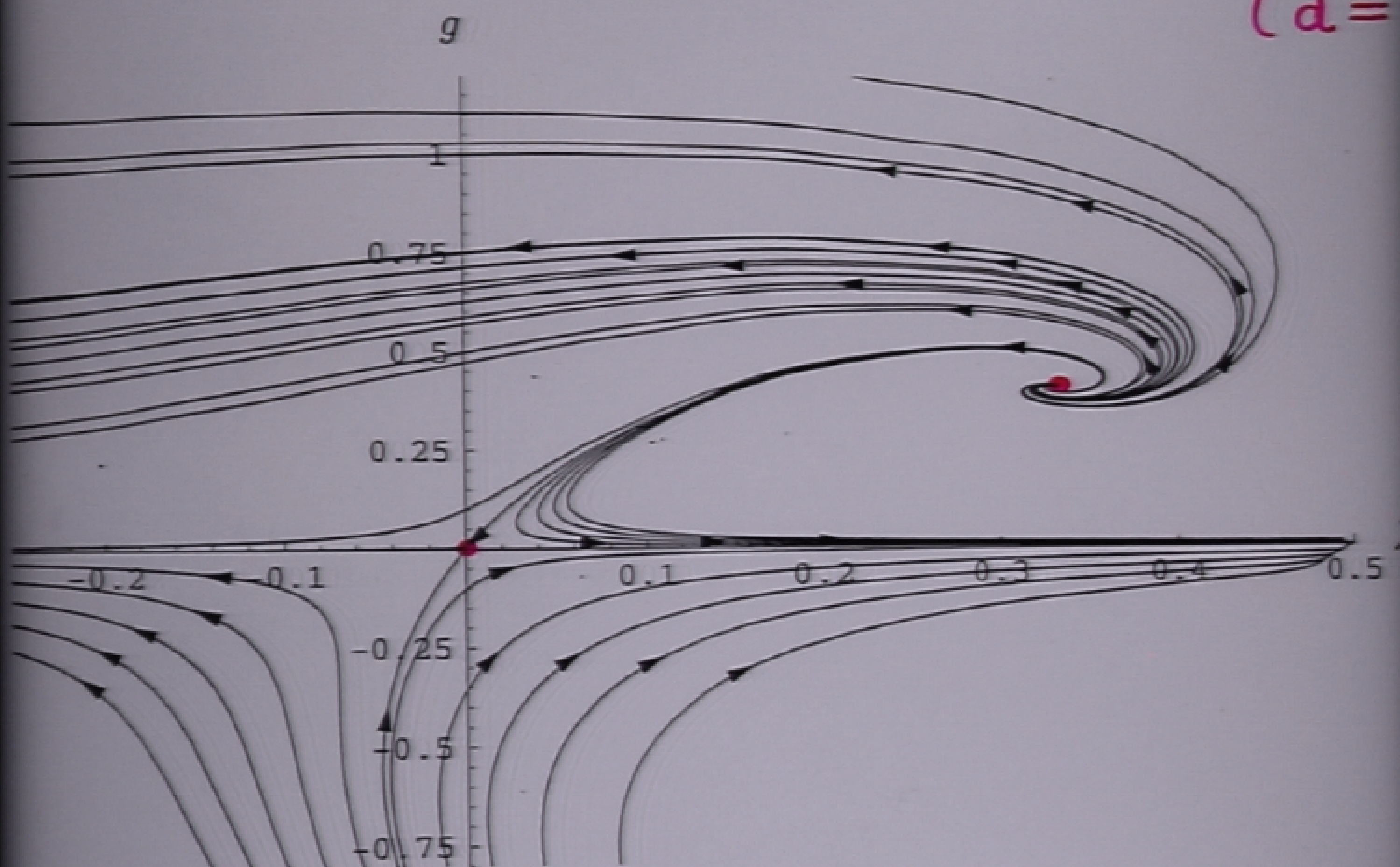
$\varphi_1 = \zeta(2)$, $\varphi_2 = 2\zeta(3)$

$\rightsquigarrow$ β - fcts. grow enormously

for $\lambda(k) \nearrow \frac{1}{2}$ or $\Lambda(k) \nearrow \frac{1}{2} k^2$!

RG - Flow in the Einstein - Hilbert Truncation

(d=4



M.R., F. Saueressi

RG - Flow in the Einstein - Hilbert Truncation

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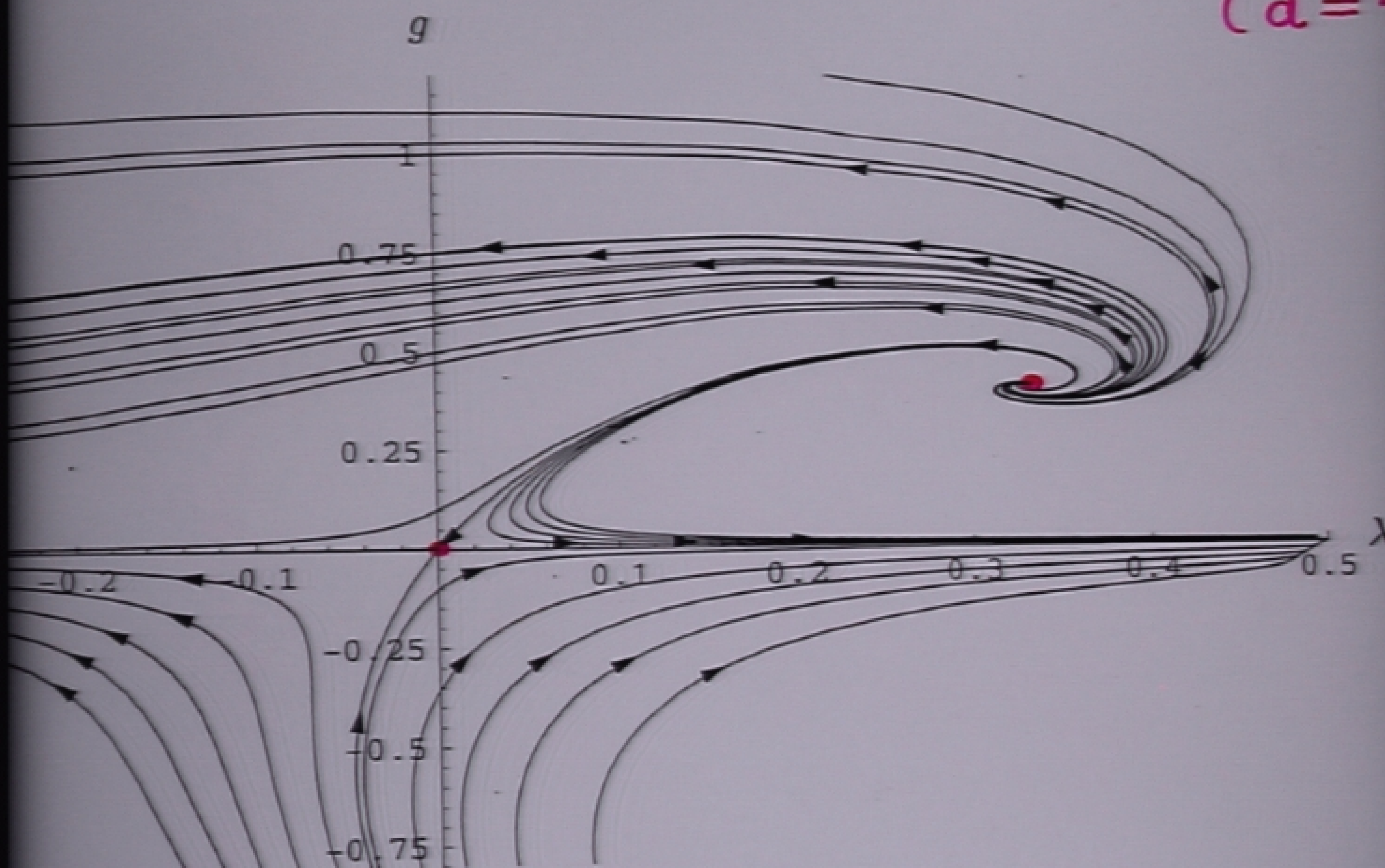

M.R., F. Saueressig,

- Gaussian FP ; $g_* = \lambda_* = 0$

- non-Gaussian FP; $g_*, \lambda_* > 0$

$$G(k) \approx \frac{g_*}{k^2} \quad , \quad \underbrace{\Lambda(k)}_{= \bar{\lambda}_k} \approx \lambda_* \, k^2 \quad \Longrightarrow$$

$G(k \to \infty) = 0$: asymptotic freedom !

Nonperturbative construction of QEG :

If the non-Gaussian FP exists in the exact theory, 4-dimensional Quantum Einstein Gravity is (most probably) nonperturbatively renormalizable ("asymptotically safe").

$d = 2 + \varepsilon$: Weinberg (1979)

Quantum theory defined by RG trajectory with

initial point $= \Gamma_{k \to \infty}$ = bare action $S \stackrel{!}{=}$ fixed point

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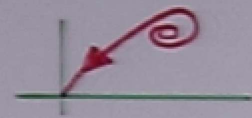
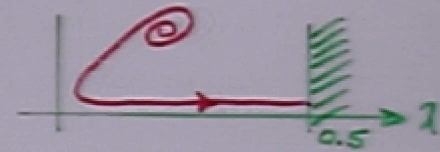
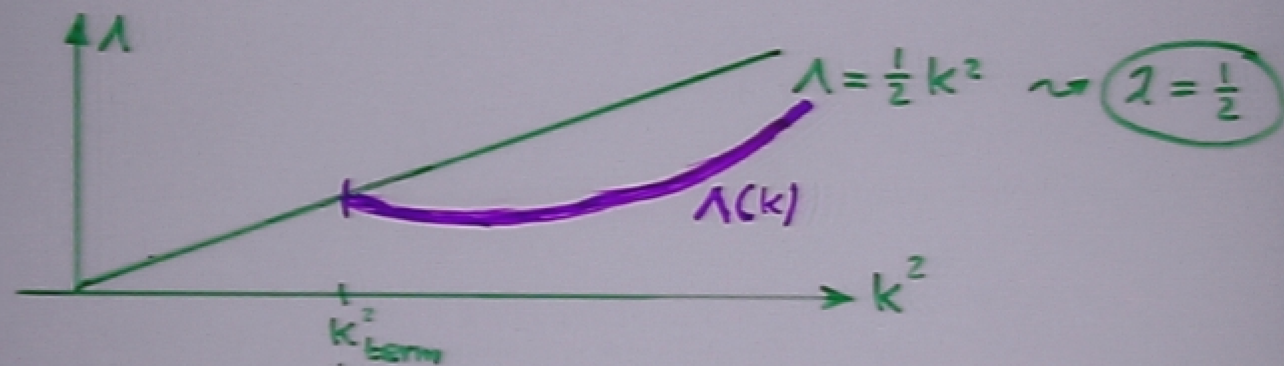
Quantum theory defined by RG trajectory with

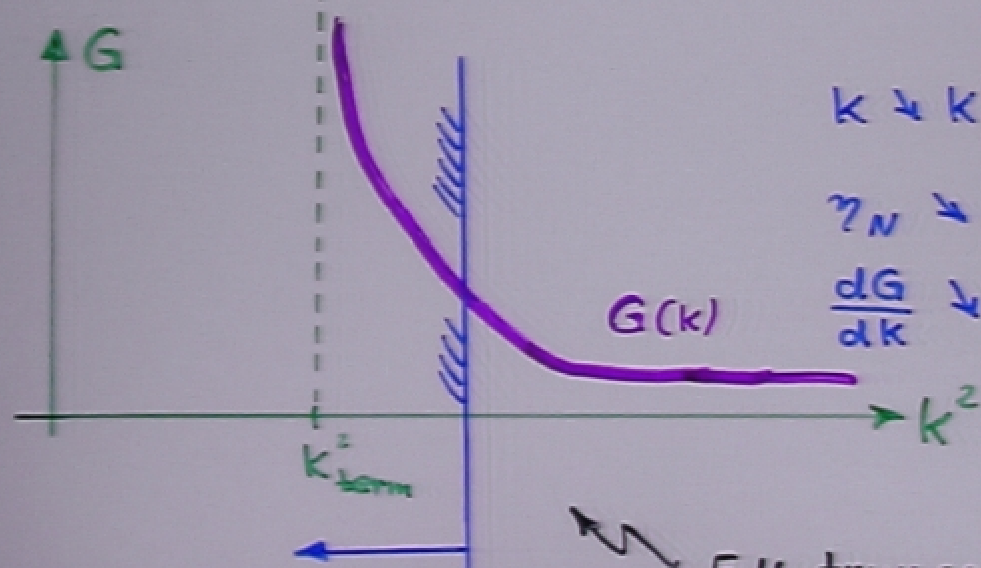
initial point $= \Gamma_{k \to \infty} \equiv$ bare action $S \overset{!}{=}$ fixed point

end point $= \Gamma_{k=0} \equiv \Gamma \equiv$ ordinary eff. action and of Type IIa  extend (reliably) to $k=0$, yielding $\Lambda(k=0) \leq 0$.

Trajectories of Type IIIa 

terminate at a finite scale $k_{term} > 0$:





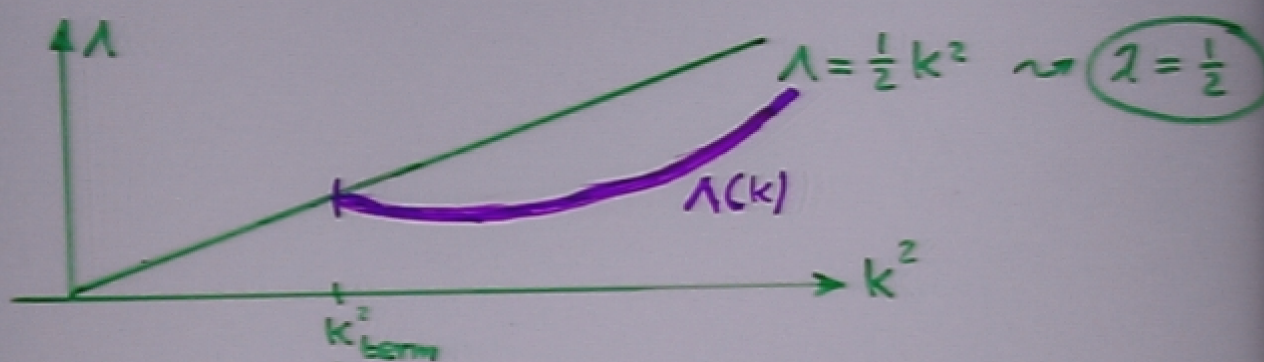
$k \searrow k_{term} \Rightarrow$

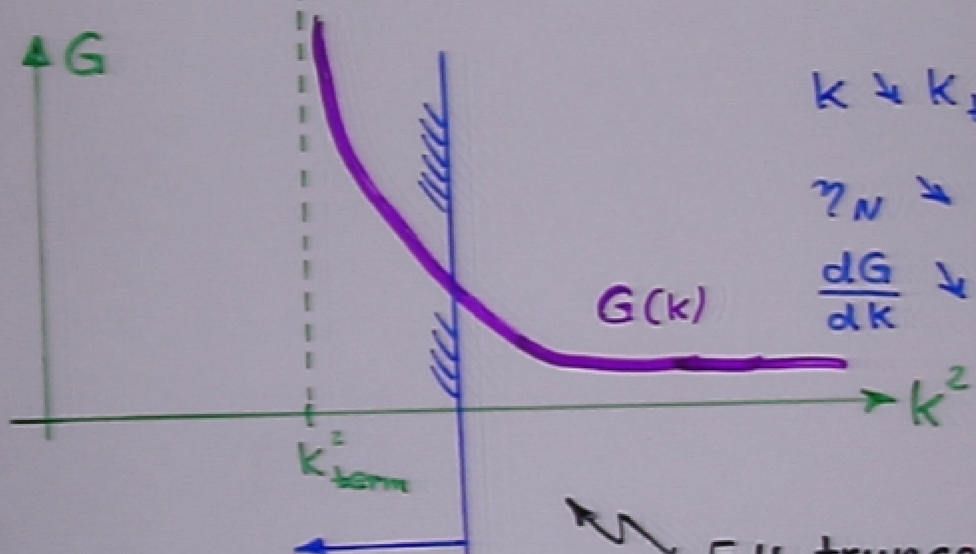
$\eta_N \searrow -\infty$,

$\frac{dG}{dk} \searrow -\infty$

EH truncation suggests

terminate at a finite scale k_{term} ...





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E.H. truncation sugg[ests] strong IR-growth of $G(k)$!

E.H. truncation insufficient, include further (nonlocal) invariants

$\rightsquigarrow$ extremely complicated nonperturbative problem

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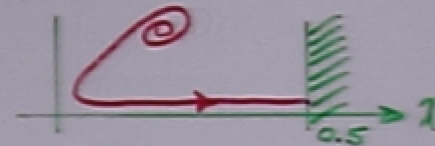
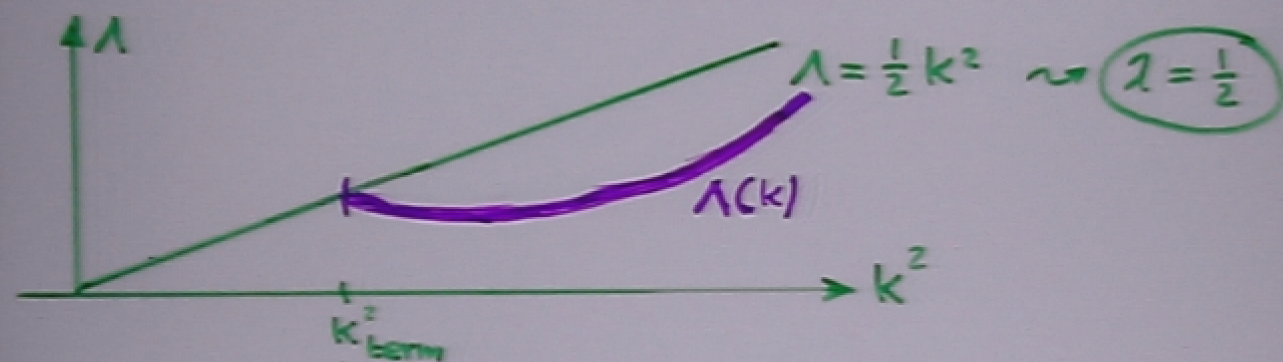
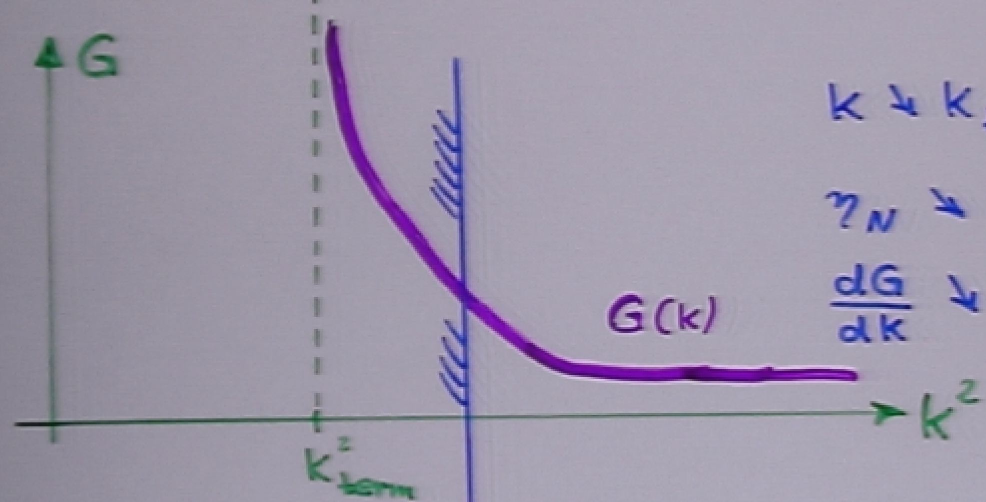
Trajectories of Type IIIa



terminate at a finite scale $k_{term} > 0$:





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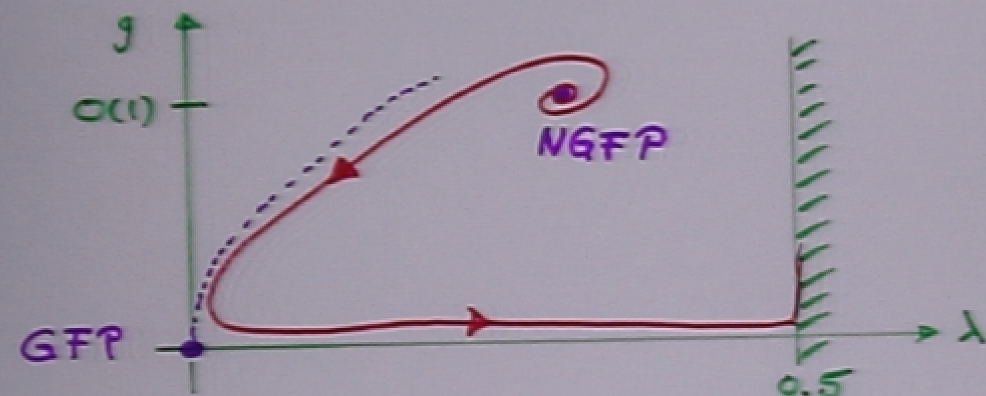
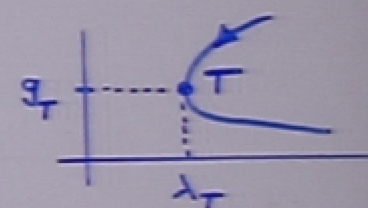
E.H. truncation suggests strong IR-growth of $G(k)$!

E.H. truncation insufficient, include further (nonlocal?) invariants to the GFP, k_{term} can be made arbitrarily small.

The trajectory spends a very long RG time near the GFP then.



Turning point T, passed at $k = k_T$:



$$\partial_t \lambda = 0 \Rightarrow \lambda_T = \left(\frac{\varphi_2}{2\pi}\right) g_T \Rightarrow \boxed{\lambda_T \approx g_T}$$

Flow linearized about GFP:

$$\begin{cases} g(k) = g_T \left(\frac{k}{k_T}\right)^2 \\ \lambda(k) = \frac{1}{2} \lambda_T \left(\frac{k_T}{k}\right)^2 \left[1 + \left(\frac{k}{k_T}\right)^4\right] \end{cases}$$

Selecting a trajectory which gets very close to the GFP, k_{term} can be made arbitrarily small.

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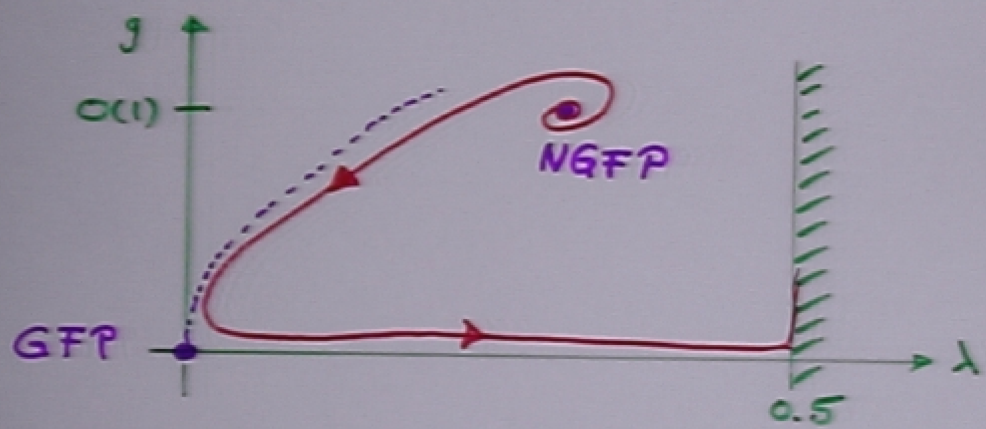

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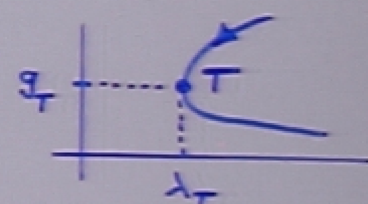

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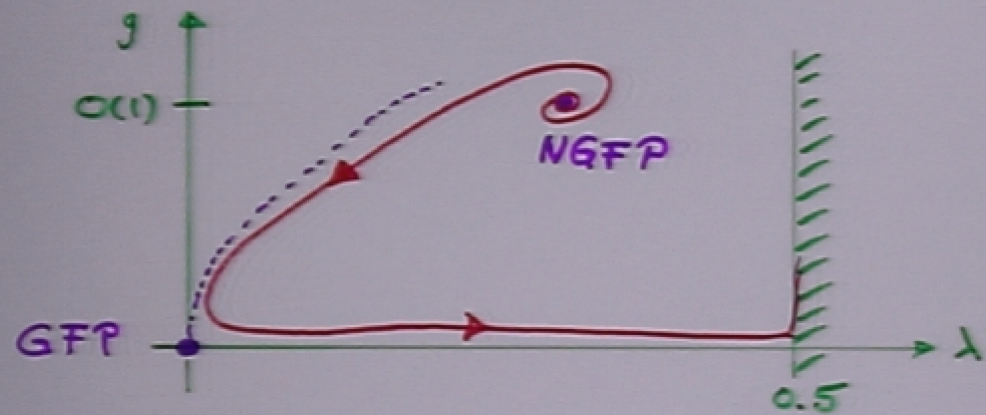

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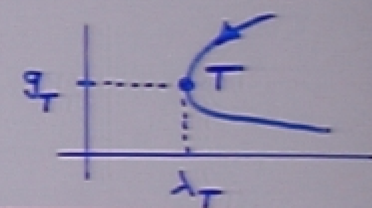

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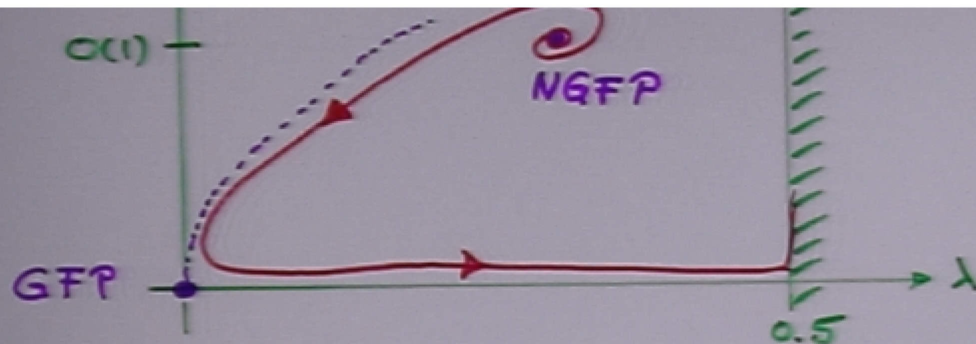

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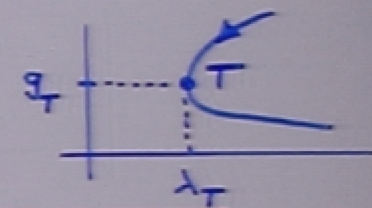

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$$\begin{cases} G(k) = g_T / k_T^2 = \text{const} \\ \Lambda(k) = \frac{1}{2}\lambda_T k_T^2 \left[1 + \left(\frac{k}{k_T}\right)^4\right] \end{cases}$$

The RG trajectory realized in Nature

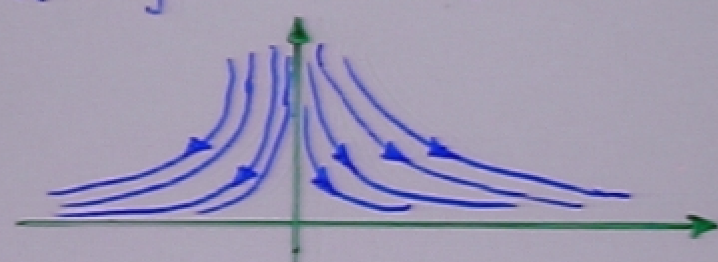
Observations: $\Lambda(k \approx H_0) > 0$

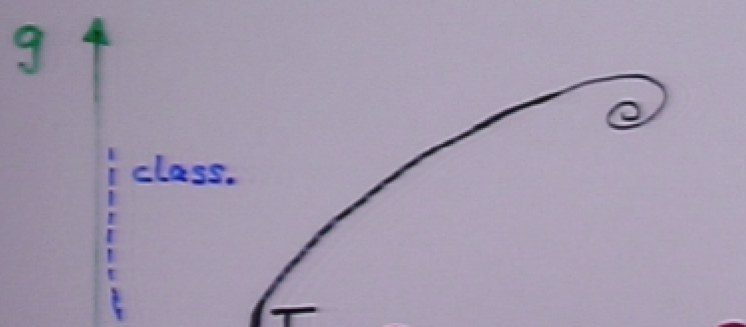
$\rightsquigarrow$ trajectory is of (extended) Type $\mathrm{III}\,a$!

Recovering classical General Relativity (GR):

classical (canonical) flow:

$$\left.\begin{array}{l} G \equiv g/k^2 = \text{const} \\ \Lambda \equiv \lambda\, k^2 = \text{const} \end{array}\right\} \Rightarrow \lambda \sim \frac{1}{g} \sim k^{-2}$$





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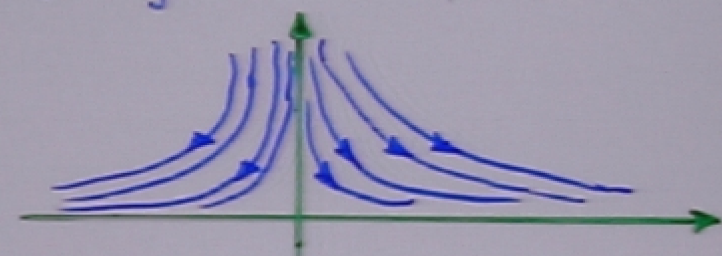

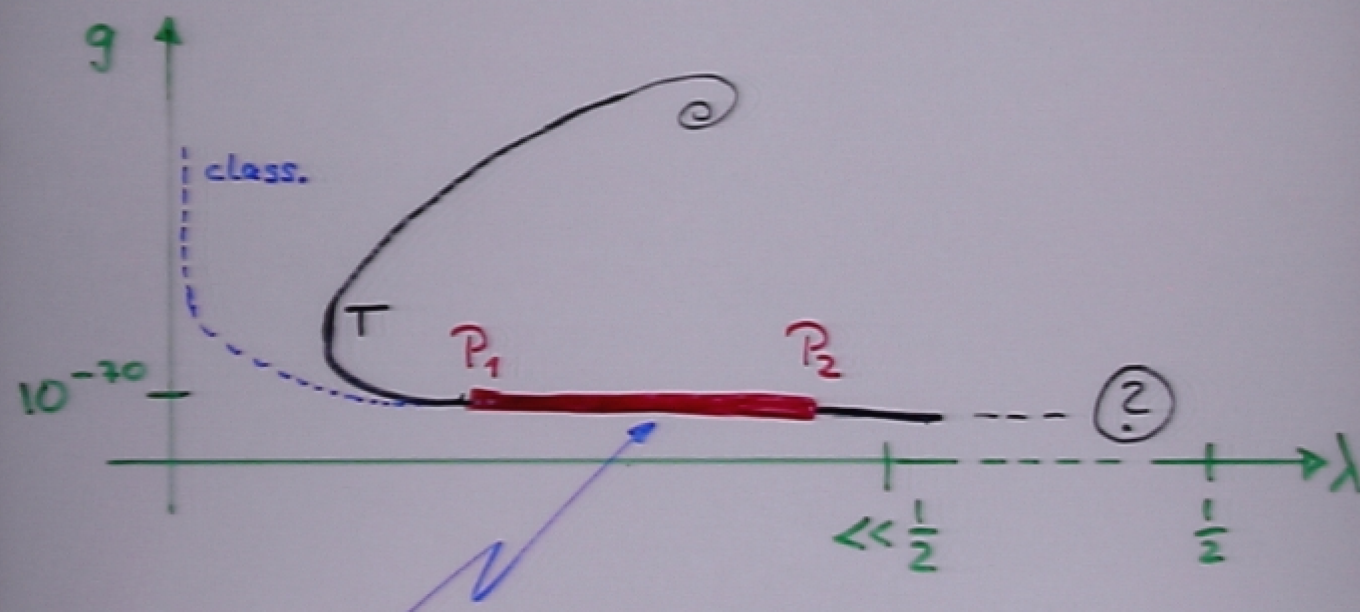

GR regime $\subset$ linear regime

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Fixing the parameters of Nature's RG trajectory:

measure $G \equiv G(k_{lab})$, $\Lambda \equiv \Lambda(k_{lab})$ in a "Laboratory" of linear dimension k_{lab}^{-1}, calculate

$$g(k_{lab}) = k_{lab}^2\, G(k_{lab}) \equiv \left(\frac{k_{lab}}{m_{Pl}}\right)^2 \equiv \left(\frac{\ell_{Pl}}{k_{lab}^{-1}}\right)^2$$

$$\lambda(k_{lab}) = \Lambda(k_{lab}) / k_{lab}^2$$

$$\ell_{Pl} \equiv m_{Pl}^{-1} \equiv \sqrt{G(k_{lab})}$$

The pair $\left(g(k_{lab}), \lambda(k_{lab})\right)$ fixes trunc. traj. uniquely.

"Laboratory" in which GR was successfully tested: terrestrial scales $(\approx 1\,m)$ .... solar system $(\approx 1\,AU)$

$G(k_{lab}) \approx 6.67 \cdot 10^{-11}\, m^3\, kg^{-1} s^{-2} \equiv G_{lab}$ throughout

$$\Longrightarrow \quad \boxed{\begin{array}{ll} g(k) \approx 10^{-70} & \text{at } k^{-1} = 1\,m \\ g(k) \approx 10^{-92} & \text{at } k^{-1} = 1\,AU \end{array}}$$

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Do not use measured $\Lambda(H_0)$ here;
scale $k = H_0$ is probably outside the domain of validity of the E.-H. truncation.

But it is difficult to measure Λ in the "Laboratory" ...

A bound for λ:

$\Lambda(k_{lab}) \Longrightarrow$ spacetime has radius of curvature

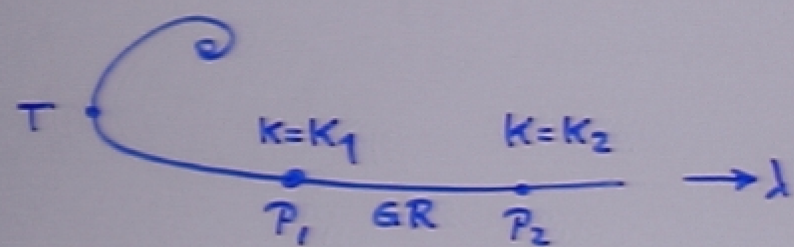
$$r_c \approx \Lambda(k_{lab})^{-\frac{1}{2}} = \lambda(k_{lab})^{-\frac{1}{2}} k_{lab}^{-1} \gg k_{lab}^{-1}$$

observation

$$\Rightarrow \quad \lambda(k_{lab}) \ll 1$$

$$\Rightarrow \quad \boxed{g(k),\ \lambda(k) \ll 1 \quad \forall k \in \text{GR regime}}$$

$$\Rightarrow \quad -\eta_N \Big|_{\text{GR regime}} \approx 10^{-70} \ \cdots\ 10^{-92}$$



$$g_T \approx \lambda_T < \lambda(k_1) < \lambda(k_2) \ll 1$$

$$\Rightarrow \quad \boxed{g_T \approx \lambda_T \ll 1} \qquad \left(\underset{\text{later}}{\approx} 10^{-60}\right)$$

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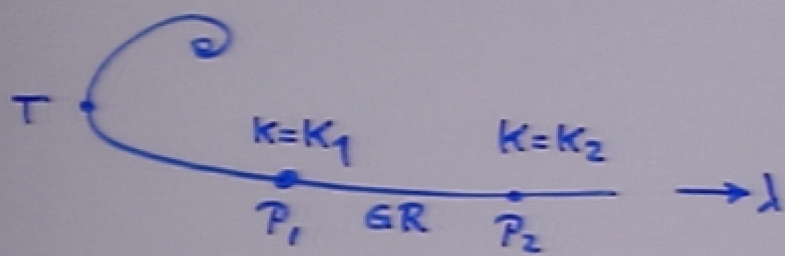

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$\Rightarrow$

RG trajectory picked by Nature gets "unnaturally" close to the GFP, and spends a very long RG time there.

$\Longrightarrow$ Termination $\equiv$ Onset of strong IR quantum effects is extremely delayed : $k_{\text{term}} \ll m_{Pl}$

$$\underbrace{\qquad}_{\ll 1} \quad \Rightarrow \quad \text{hierarchy } k_T \ll m_{Pl}$$

An estimate for k_{term}:

$$\lambda(k_{term}) \underset{\substack{\text{lin.}\\ \text{regime}}}{=\!=} \frac{\lambda_T}{2}\left(\frac{k_T}{k_{term}}\right)^2\left[1+\underbrace{\left(\frac{k_{term}}{k_T}\right)^4}_{\approx 0}\right] \overset{!}{=} \frac{1}{2}$$

$$\Rightarrow \boxed{k_{term} = \sqrt{g_T \lambda_T}\, m_{Pl} \approx g_T\, m_{Pl}}$$

$g_T \lll 1$ implies symmetric "double hierarchy":

$$\boxed{\frac{k_{term}}{k_T} = \sqrt{g_T} \ll 1 \,, \quad \frac{k_T}{m_{Pl}} = \sqrt{g_T} \ll 1}$$

$\Rightarrow$ The trajectory <u>must</u> get "unnaturally" close to the GFP in order to give rise to a long classical regime !

"Generic" initial conditions $\Rightarrow k_{term} \approx k_T \approx m_{Pl}$

$\Rightarrow$ classical GR valid nowhere !

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GR regime and its "cosmological constant problem"

$$G(k) = G_{lab}$$

$$\Lambda(k) = \tfrac{1}{2}\lambda_T k_T^2 \left[1 + \underbrace{\left(\frac{k}{k_T}\right)^4}\right]$$

negligible for $k < k_1$,
by def. of the "GR regime"

$\left(k_1 = \frac{k_T}{10}, \text{ say}\right)$

Cosmological constant in the classical theory:

$$\Lambda = \tfrac{1}{2}\lambda_T k_T^2 = \tfrac{1}{2}\lambda_T g_T m_{Pl}^2 \approx g_T^2 m_{Pl}^2$$

$$\Rightarrow \boxed{\left.\frac{\Lambda}{m_{Pl}^2}\right|_{\text{GR regime}} \approx g_T^2 \ll 1}$$

Conclusion:

Classical GR applicable

$\Rightarrow \exists$ Long classical regime

$\Rightarrow g_T \ll 1$

$\Rightarrow \Lambda_{lab} \ll m_{Pl}^2 \equiv G_{lab}^{-1}$

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Conclusion:

Classical GR applicable

$\Rightarrow \exists$ Long classical regime

$\Rightarrow g_T \ll 1$

$\Rightarrow \Lambda_{lab} \ll m_{Pl}^2 \equiv G_{lab}^{-1}$

The Λ in a "Lab" is automatically small.

The smallness of $\Lambda(k_{lab})$ poses no naturalness

$$\lambda(k) \underset{\substack{GR \\ \text{regime}}}{\equiv} \frac{\Lambda_{lab}}{k^2} \overset{!}{=} O(\tfrac{1}{2}) \quad \text{at } k = k_{term}$$

$$\Rightarrow \quad \boxed{k_{term} \approx \sqrt{\Lambda_{Lab}}}$$

<u>Is the Hubble scale within the GR regime</u> ?

Observations : $\rho_{vac} \approx \rho_{crit}$

$$\Rightarrow \quad \Lambda(k = H_0) \approx H_0^2$$

interpret as a "lab" value:

$$\boxed{k_{term} \approx H_0}$$

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Difference between λ_{lab}, G_{lab} and $\lambda(H_0)$, $G(H_0)$ is comparatively small. ($\lesssim$ 1–2 orders of mgn.)

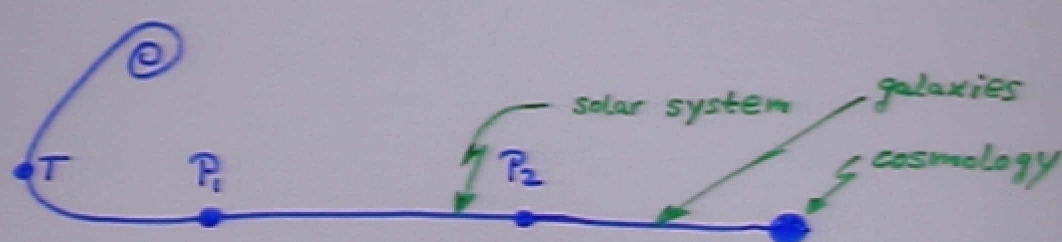
$$\rightsquigarrow \quad g_T^2 = \frac{\lambda_{lab}}{m_{Pl}^2} \approx \left(\frac{H_0}{m_{Pl}}\right)^2 = 10^{-120}$$

$$\rightsquigarrow \quad \boxed{g_T \approx \lambda_T \approx 10^{-60}}$$

Double hierarchy:

$$\boxed{\frac{k_T}{m_{Pl}} = \frac{k_{term}}{k_T} = \sqrt{g_T} = 10^{-30}}$$

or: $\quad k_T^{-1} = 10^{30}\, \ell_{Pl} = 10^{-3}\,\mathrm{cm}$



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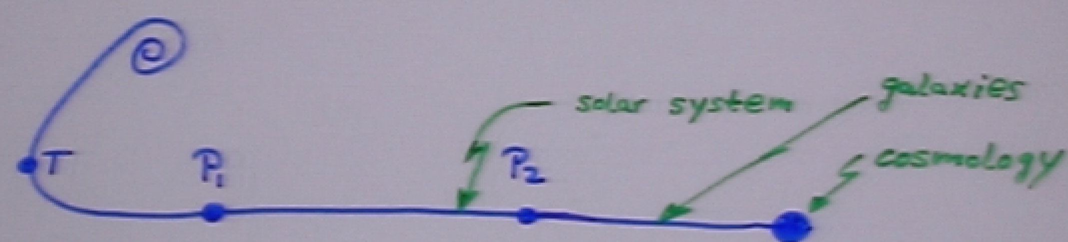

On the g-λ-plane, the E.-H. truncation breaks down near the point

$$g(H_0) \approx 10^{-120}, \quad \lambda(H_0) = O(1)$$

Somewhere between solar system and cosmological distances we should see the scales : a phenomenological argument

Bentivegna, Bonanno, M.R.

The IR fixed point cosmology

Assume (!) IR effects lead to the formation of an IR – NGFP $(g_*^{IR}, \lambda_*^{IR})$:

$$\left.\begin{array}{l} G(k) = g_*^{IR}/k^2 \\ \Lambda(k) = \lambda_*^{IR}\, k^2 \end{array}\right\} \xrightarrow[\frac{1}{k} \sim t \;=\; \text{cosmolog. time}]{} \left\{\begin{array}{l} G(t) \sim t^2 \\ \Lambda(t) \sim \frac{1}{t^2} \end{array}\right.$$

RG improvement of cosmolog. Einstein (= Friedmann) eqs., $G \longrightarrow G(t)$, $\Lambda \longrightarrow \Lambda(t)$, leads to a very attractive ASYMPTOTIC cosmology. Predictions:

(i) accelerated expansion $a \sim t^{4/3}$

(ii) $\Omega_M \approx \Omega_\Lambda$

(iii) $\lambda_*^{IR} = O(1)$, $g_*^{IR} = 10^{-120}$

(iv) $G_{cosmo}/G_{lab} \lesssim O(10)$

Precisely the point where the EH truncation breaks down !

UV-FP cosmology
(mimics inflation)

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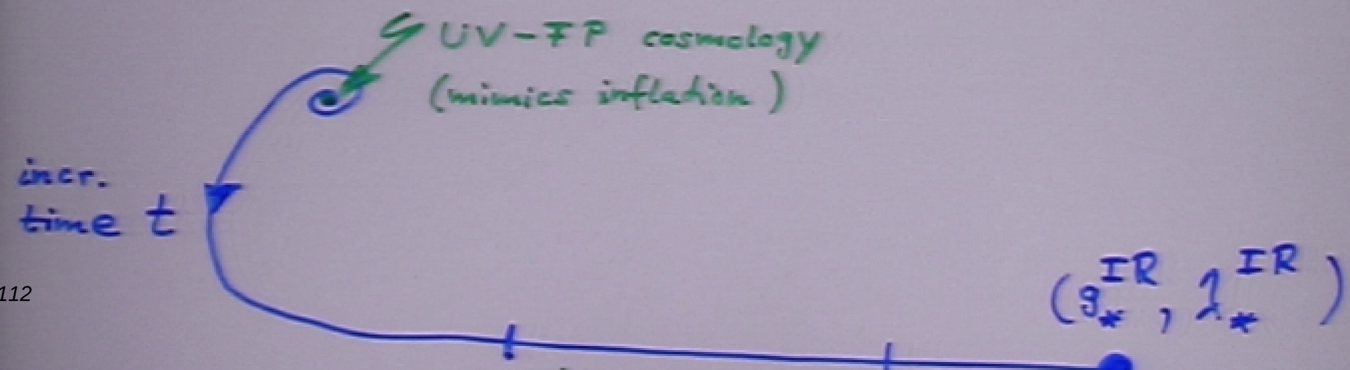
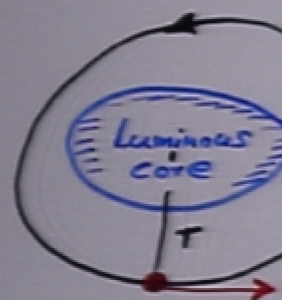
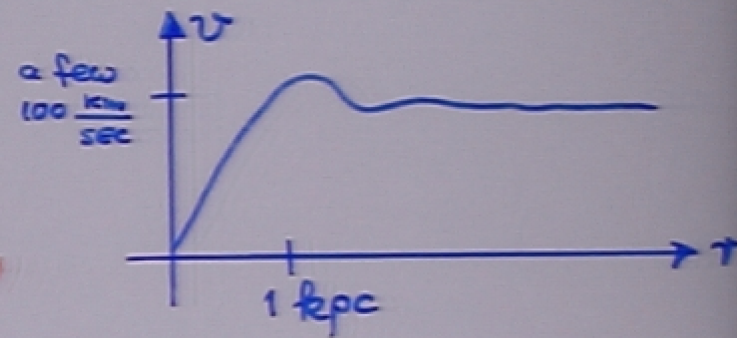


Missing Mass [?] in Galactic Systems

Observation: luminous mass $\ll$ mass infered from motions;
e.g. flat rotation curves of spiral galaxies



$v^2(r) = r\,\phi'(r)$



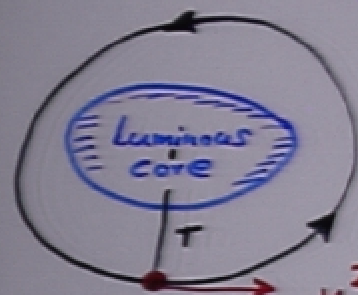
$\longrightarrow$ invention of dark matter

Conjecture: Flat rotation curves are actually due to a modification of gravity at large distances rather than extra matter.

(Milgrom "MOND", Mannheim $C^2_{\mu\nu\rho\sigma}$, ...)

The (formidable) task: derive/solve $\dfrac{\delta}{\delta g_{\mu\nu}} \Gamma^{QEG}_{k=0}[g] = 0$

An "educated guess" for capturing the essence e.g. flat rotation curves of spiral galaxies



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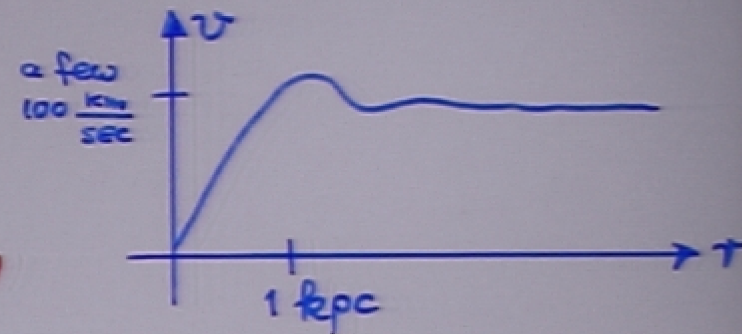

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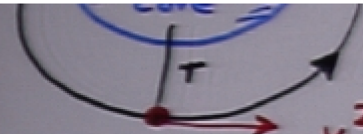
here: $k(r) = \dfrac{\xi}{d(r)} \underset{m.N.}{\approx} \dfrac{\xi}{r}$



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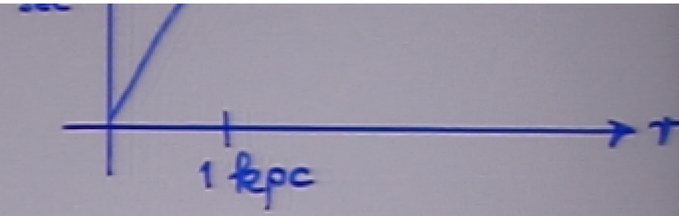

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The RG trajectory

$$\{ G(k), \underbrace{\Lambda(k), \cdots}_{\text{probably irrelevant}} \}$$

must be extracted from the data, for the time being.

The mathematically simplest, and at the same time phenomenologically most promising, assumption:

$\eta_N = \text{const} \equiv -q, \quad q > 0$

$\partial_t G = \eta_N G \rightsquigarrow$ power law

$$\boxed{G(k) \sim k^{\eta_N} = \frac{1}{k^q}}$$

for $k^{-1} = 1\ \text{kpc} \cdots 100\ \text{kpc}$, say.

The Improved Action Approach

$$S = \frac{1}{16\pi}\int d^4x\,\sqrt{-g}\left[\frac{R}{G(x)} - 2\frac{\Lambda(x)}{G(x)}\right]$$

+ matter action + action for scalars $G(\cdot), \Lambda(\cdot)$ (opt.)

$\Rightarrow$

$$G_{\mu\nu} = -\Lambda(x)\, g_{\mu\nu} + 8\pi G(x)\left[T_{\mu\nu} + \Delta T_{\mu\nu} + \theta_{\mu\nu}\right]$$

$$\Delta T_{\mu\nu} \equiv \frac{1}{8\pi}\left[D_\mu D_\nu - g_{\mu\nu} D^2\right]\frac{1}{G(x)}$$

$D^\mu G_{\mu\nu} = 0 \quad \Longrightarrow$ consistency condition:

$$8\pi G\, D^\mu\left[\Delta T_{\mu\nu} + \theta_{\mu\nu}\right] + 8\pi\left[T_{\mu\nu} + \Delta T_{\mu\nu} + \theta_{\mu\nu}\right]\partial^\mu G$$
$$- \partial_\nu \Lambda = 0$$

Example: $T_\mu{}^\mu = 0,\ \Lambda = 0 \Rightarrow \exists$ unique identical solution:

$$\theta^{BD}_{\mu\nu} = \left(-\frac{3}{2}\right)\frac{1}{8\pi G^3}\left[D_\mu G\, D_\nu G - \frac{1}{2} g_{\mu\nu} (DG)^2\right]$$

exactly the Brans-Dicke tensor for $\omega = -\frac{3}{2}$!

The modified Newtonian Limit

$G(\vec{x}) = \bar{G}\,[1+\mathcal{N}(\vec{x})]$ $\bar{G} = \text{const}\ (\equiv G_{lab})$

$\vec{v}^2, \phi, \mathcal{N} \ll 1$, retain only terms of 1st order.

$ds^2 = -[1+2\phi(\vec{x})]\,dt^2 + [1-2\omega(\vec{x})]\,d\vec{x}^2$

$T_{\mu\nu} = \text{diag}[\rho, 0, 0, 0]$ "dust" ; $\Theta_{\mu\nu}$: irrelevant

Geodesic eq. $\rightsquigarrow$ $\ddot{\vec{x}} = -\nabla\phi$ with ϕ satisfying

$$\boxed{\begin{aligned} \nabla^2\phi &= 4\pi\bar{G}\,\rho_{eff} \\ \rho_{eff} &\equiv \rho + (8\pi\bar{G})^{-1}\,\nabla^2\mathcal{N}(\vec{x}) \\ &= \rho + (8\pi\bar{G}^2)^{-1}\,\nabla^2 G(\vec{x}) \end{aligned}}$$

ordinary matter mimics dark matter

Solution to this Poisson equation:

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Solution to this Poisson equation:

$$\boxed{\phi = \hat{\phi} + \tfrac{1}{2}\mathcal{N}\,, \quad \nabla^2\hat{\phi} = 4\pi\bar{G}\,\varrho}$$

$\hat{\phi}$ = potential due to "ordinary" matter alone

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<u>Point mass:</u> $\phi(r) = -\dfrac{\bar{G}M}{r} + \tfrac{1}{2}\mathcal{N}(r)$

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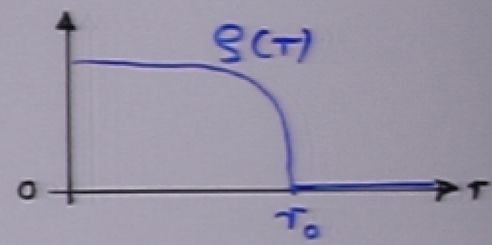
A spherically symmetric model "galaxy"

$$\Phi(r) = \int^{r} dr' \, \frac{\bar{G}\mathcal{M}(r')}{r'} + \frac{1}{2}\mathcal{N}(r)$$

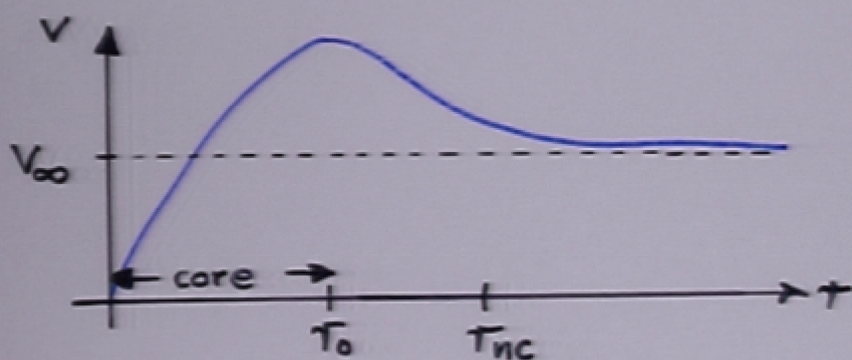


$$\mathcal{M}(r) \equiv 4\pi \int_0^{r} dr' \, r'^2 \rho(r')$$

$$v^2(r) = \frac{\bar{G}\mathcal{M}(r)}{r} + \frac{1}{2} r \frac{d\mathcal{N}}{dr}$$

Rotation curve qualitatively correct for the k^{-q}- trajectory:

$$G \sim k^{-q} \sim r^{q} = 1 + q \ln(r) + \cdots \Rightarrow \mathcal{N}(r) = q \ln(r)$$



In the "halo" $(r > r_0)$:

$$\mathcal{M}(r) = \mathcal{M}(r_0) \equiv \mathcal{M}_0 \quad ; \quad \Phi(r) = -\frac{\bar{G}\mathcal{M}_0}{r} + \frac{1}{2} q \ln(r)$$

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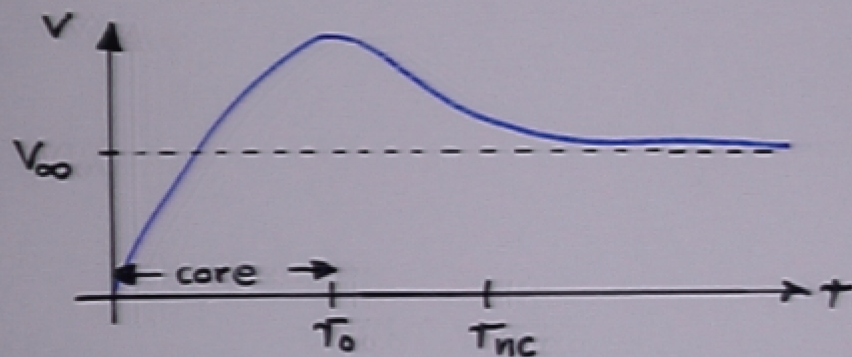

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$$\boxed{v^2(r) = \frac{\bar{G}\mathcal{M}_0}{r} + \frac{q}{2} \xrightarrow[r\to\infty]{} v_\infty^2 = \frac{q}{2}}$$

typical plateau velocities: $v_\infty = 100 \cdots 300$ Km/sec

$$\Rightarrow \boxed{q \approx 10^{-6}}$$

Exact (nonlinear) Solutions

$\Lambda = 0, \; T_{\mu\nu} = 0, \; \Theta_{\mu\nu} = \Theta_{\mu\nu}^{BD}$ ⟶ Weyl technique available

$$G(k) = \bar{G} \cdot \begin{cases} 1 & \text{for } k \geq k_{tr} \\ (k_{tr}/k)^{q} & \text{for } k < k_{tr} \end{cases}$$

Self-consistent cutoff identification: $k(r) = \xi / d_g(r)$

$$\boxed{\begin{aligned} ds^2 &= w(\rho)\left[-f(\rho)dt^2 + f(\rho)^{-1}d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right] \\ G(\rho) &= w(\rho)\,\bar{G} \quad , \quad f(\rho) \equiv 1 - 2\bar{G}M/\rho \end{aligned}}$$

$$w(\rho) = \begin{cases} 1 & \text{for } \rho \leq \rho_{tr} \\ \left[(1-q/2)\,x\, I_f(\rho) + q/2\right]^{\frac{2q}{2-q}} & \text{for } \rho > \rho_{tr} \end{cases}$$

$$I_f(\rho) \equiv \pi \bar{G}M + 2\bar{G}M \ln\left[\sqrt{\rho/2\bar{G}M} + \sqrt{\rho/2\bar{G}M - 1}\right] + \sqrt{\rho(\rho - 2\bar{G}M)}$$

Solution $g_{\mu\nu}$ is conformal to Schwarzschild- $(M \neq 0)$

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Solution $g_{\mu\nu}$ is conformal to Schwarzschild- $(M \neq 0)$ or Minkowski- $(M = 0)$ metric.

$M = 0$ solution: pure "dark matter" halo; perhaps relevant in the early stages of structure formation.

Conclusion

The IR sector of QEG is a difficult nonperturbative problem which is in many respects comparable to its QCD counterpart.

The impact of its potentially strong large-distance quantum effects on astrophysics and cosmology needs to be understood before the necessity of the various "standard extensions" (dark matter, inflation, quintessence, ...) can be assessed.