Title: Geometry of Fast Moving Strings

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Abstract: It was conjectured that the N=4 Yang-Mills perturbation theory in the sector with large R charges corresponds to considering the classical string worldsheets in $AdS_5imes S^5$ as perturbations of the null-surfaces. We discuss this perturbation theory with a special emphasis on a hidden symmetry.

Geometry of fast moving strings.

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Strings from single trace operators.

We want to describe gauge theory ($\mathcal{N} = 4$ Yang-Mills) in terms of stringy degrees of freedom. As a step in this direction, I will descuss here the interpretation of single trace operators with the large R-charge as semiclassical strings in $AdS_5 \times S^5$.

We will consider the operators of the form

$$\text{tr}\ \underbrace{\phi\phi\ldots\partial\ldots\partial\phi\ldots}_{L}$$

where ϕ is a scalar field of the $\mathcal{N} = 4$ Yang-Mills. We will discuss a conjecture that there is a limit involving large L in which the dynamics of the corresponding states becomes classical and is equivalent to the dynamics of classical strings in $AdS_5 \times S^5$.

The talk is based on the following papers:

- S. Frolov, A.A. Tseytlin, "Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$", hep-th/0204226.
- D. Mateos, T. Mateos, P.K. Townsend, "Supersymmetry of Tensionless Rotating Strings in $AdS_5 \times S^5$ and Nearly-BPS Operators", hep-th/0309114.
- M. Kruczenski, "Spin chains and string theory", hep-th/0311203.
- M. Kruczenski, A.V. Ryzhov, A. Tseytlin, "Large spin limit of $AdS_5 x S^5$ string theory and low energy expansion of ferromagnetic spin chains", hep-th/0403120.
- M. Kruczenski, A. Tseytlin, "Semiclassical relativistic strings in S^5 and long coherent operators in N=4 SYM theory", hep-th/0406189.
- G. Arutyunov, M. Staudacher, "Matching Higher Conserved Charges for Strings and Spins", hep-th/0310182.
- J. Engquist, "Higher Conserved Charges and Integrability for Spinning Strings in AdS_5 x S^5", hep-th/0402092.
- A.M., hep-th/0311019, 0402067, 0404173, 0409040, 0411178, 0502097

Plan of the talk.

- Frolov-Tseytlin solutions.
  - fast moving "rigid" strings.
  - the energy reproduces the anomalous dimension on the field theory side.
- "Speeding strings".
  - Null-surfaces. Fast moving strings are perturbations of the null-surfaces.
  - Conjecture: the Yang-Mills perturbation theory corresponds to the perturbation theory around the null-surfaces.
- Classification of the null-surfaces. Supersymmetric null-surfaces. $U(1)_L$ symmetry.
- The role of $U(1)_L$ as the length of the field theory operator; $U(1)_L$ as the "fast degree of freedom"; continuous limit of the spin chain as the Hamiltonian reduction of the classical string in $AdS_5 \times S^5$ by $U(1)_L$.
- Extension of $U(1)_L$ symmetry from the null-surfaces to the nearly-degenerate extremal surfaces; interpretation of $U(1)_L$ as an action variable; the role of Pohlmeyer charges.
- $U(1)_L$ in the BMN limit. Explicit expression for the Hamiltonian of $U(1)_L$ in terms of the Pohlmeyer charges.
- Application of the action variable: anomalous dimension is the sum of local charges.

Frolov-Tseytlin solutions.

Classical string worldsheet theory on $AdS_5 \times S^5$ has so-called rigid solutions. These are the solutions for which the shape of the string does not depend on the time. This means: the profile of the string at a time $T = T_0$ is related to the profile of the string at $T = 0$ by an isometry of $AdS_5 \times S^5$.

The simplest examples of the rigid solutions are those worldsheets which project to the timelike geodesic in AdS_5. These strings "move only in S^5". Let us parametrize S^5 by three complex numbers Z_1, Z_2, Z_3 satisfying $|Z_1|^2 + |Z_2|^2 + |Z_3|^2 = 1$. The profile of the rigid string is given by the equation:

$$Z_I(\tau, \sigma) = e^{iw_i t} Z_I^{(0)}(\sigma)$$

where w_i, $i = 1, 2, 3$ are some real constants and $Z_I^{(0)}(\sigma)$ solves the differential equation:

$$\partial_\sigma^2 Z_I^{(0)} + Z_I^{(0)} \sum_{J=1}^{3} |\partial_\sigma Z_J|^2 = -w_I^2 Z_I^{(0)} + Z_I^{(0)} \sum_{J=1}^{3} w_J^2 |\partial_\sigma Z_J|^2$$

subject to the constraint

$$\sum_{J=1}^{3} w_J \overline{Z}_J \partial_\sigma Z_J = 0$$

$Z_I(\sigma)$ should be periodic modulo the "overall phase" ϕ:

$$Z_J(\sigma = 2\pi) = e^{iw_J \phi} Z_J(\sigma = 0)$$

Properties of the rigid strings.

For each set $(w_1, w_2, w_3) \in \mathbf{R}^3$ there will be a discreet set of the periodic (modulo the "overall phase") trajectories, therefore a discreet set of string worldsheets.

For each worldsheet we can compute the momenta of $U(1)^3 \subset U(3) \subset SO(6)$, and parametrize the solution by the momentum (J_1, J_2, J_3). It turns out that the energy is given by

$$E = J\left[1 + \frac{\lambda}{J^2}c_1 + \left(\frac{\lambda}{J^2}\right)^2 c_2 + \ldots\right]$$

Frolov and Tseytlin conjectured that this expansion in powers of $\frac{\lambda}{J^2}$ correpsonds to the Yang-Mills perturbative expansion, and $c_1, c_2, \ldots$ are the coefficients of the anomalous dimension. They depend on the ratios $\frac{J_1}{J_2}, \frac{J_2}{J_3}$. These rigid strings correspond to very special operators in the $N = 4$ SYM: operators which extremize the anomalous dimension in the sector with the given charges.

It turns out that the Frolov-Tseytlin conjecture can be generalized to a more general class of solutions of the classical string worldsheet theory.

Speeding strings.

One of the lessons from the BMN paper is that the perturbation theory for the long operators is often organized in powers of λ/L^2 (rather than powers of λ).

On the AdS side, to use the classical worldsheet theory, we need $\lambda >> 1$. For the YM perturbation theory to work, we need $\lambda/L^2 << 1$. Therefore, we need very large L to have an "overlap". Large number of partons means that the state has large R-charge, or from the point of view of the string theory the large momentum in S^5. When $L >> \sqrt{\lambda}$ the string is moving very fast. In the limit $L = \infty$ (for fixed large λ) every point on the string moves with the speed of light. The string worldsheet becomes a degenerate surface. A special class of degenerate surfaces obtained in this way is known as null-surfaces.

Null-surfaces are "degenerate" string worldsheets, and fast moving strings are "nearly degenerate". It turns out that there is a perturbation theory in powers of $\sqrt{1-v^2}$ for the fast moving string as a perturbation (or "resolution") of the null-surface.

Conjecture: The Yang-Mills perturbation theory for the operators with the large R-charge corresponds to considering the worldsheet of the fast moving string as a perturbation of the null-surface. (Null-surfaces themselves correspond to operators in the free theory.)

Degenerate surfaces and null surfaces.

The surface is called degenerate if the induced metric is degenerate. When the string moves very fast, the worldsheet becomes a degenerate surface. The inverse is not quite true: not every degenerate surface can be obtained as a limit of a string worldsheet. Only the so-called null surfaces.

Indeed, let us introduce on the string worldsheet the coordinates ξ^+, ξ^- such that the induced metric is $\rho(\xi^+, \xi^-)d\xi^+ d\xi^-$. Let us denote $x^\mu(\xi^+, \xi^-)$ the embedding functions. Since the surface is extremal the embedding is a harmonic map:

$$\frac{D}{\partial\xi^+}\frac{\partial x^\mu}{\partial\xi^-} = 0 \tag{1}$$

In the limit when the string moves with the speed of light, the surface becomes isotropic. In this limit the two null-directions $\frac{\partial}{\partial\xi^+}$ and $\frac{\partial}{\partial\xi^-}$ coincide, and Eq (1) implies that the limiting null-directions are null-geodesics (= light rays).

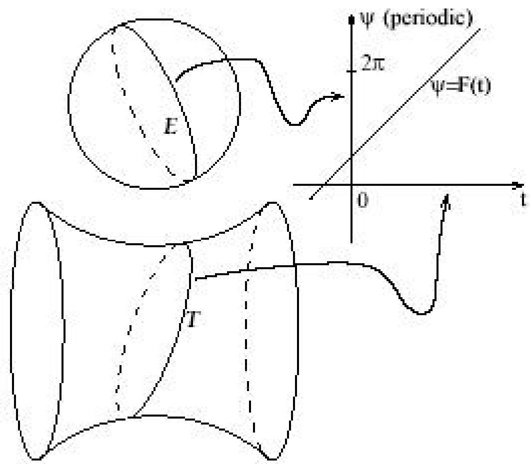
Degenerate surfaces and null surfaces.

Definition. A null-surface is a degenerate surface ruled by the light rays.

There are two types of light rays in $AdS_5 \times S^5$, therefore there are two types of the null-surfaces. The first type are null-surfaces ruled by the light rays totally inside AdS_5. Such null-surfaces extend to the boundary of AdS_5. We will not discuss this type of the null-surfaces here.

What we need now is the second type of null-surfaces. They are generated by the light rays which are obtained as a diagonal in the product of a timelike geodesic in AdS_5 and an equator in S^5.

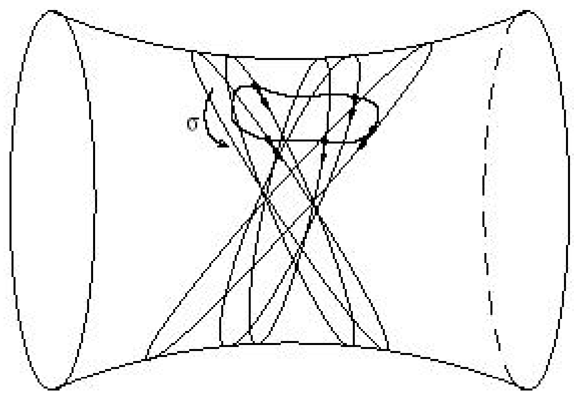
Classification of the null-geodesics of the second type in $AdS_5 \times S^5$:

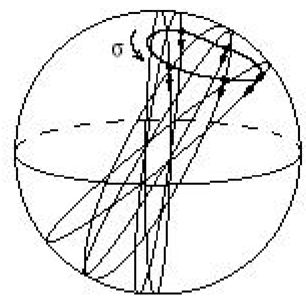


The moduli space of null-geodesics:

$$\frac{SO(2,4)}{SO(2)\times SO(4)} \times \frac{SO(6)}{SO(2)\times SO(4)} \widetilde{\times} S^1$$

Classification of the null-surfaces of the second type in $AdS_5 \times S^5$:

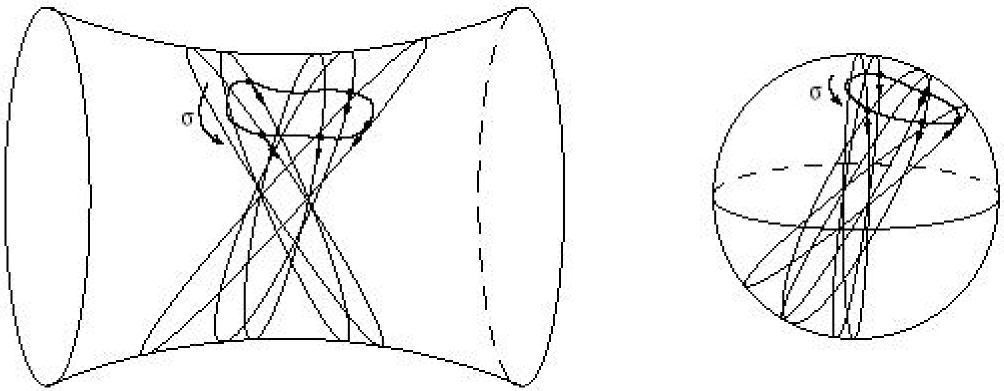




The moduli space of collections of light rays:

$$\frac{\text{Map}\left(S^1,\ \ \frac{SO(2,4)}{SO(2)\times SO(4)} \times \frac{SO(6)}{SO(2)\times SO(4)} \widetilde{\times} S^1\right)}{\text{Diff}(S^1)}$$

Classification of the null-surfaces of the second type in $AdS_5 \times S^5$:



The moduli space of null-surfaces:

$$\frac{\mathrm{Map}_0\left(S^1,\ \ \frac{SO(2,4)}{SO(2)\times SO(4)} \times \frac{SO(6)}{SO(2)\times SO(4)}\right) \widetilde{\times} S^1}{\mathrm{Diff}(S^1)}$$

Notice that: $\frac{SO(2,4)}{SO(2)\times SO(4)} \times \frac{SO(6)}{SO(2)\times SO(4)} \simeq \frac{SU(2,2)}{S(U(2)\times U(2))} \times \frac{SU(4)}{S(U(2)\times U(2))}$

Moduli space of supersymmetric null surfaces:

Turning on the fermionic degree of freedom on the light ray leads to:

$$\frac{SU(2,2)}{S(U(2)\times U(2))}\times\frac{SU(4)}{S(U(2)\times U(2))}\to\frac{PSU(2,2|4)}{PSU(2|2)\times PSU(2|2)\times U(1)^2}$$

In other words:

$$Gr(2,4)\times Gr(2,4)\to Gr(2|2,4|4)$$

This coset space is a super-symmetrization of the *future tube* in the complexified Minkowski space; it is also known as a *(4,2,2) analytic superspace*.

The moduli space of supersymmetric null-surfaces is:

$$\frac{\mathrm{Map}_0\left(S^1,\quad Gr(2|2,4|4)\right)\widetilde{\times}S^1}{\mathrm{Diff}(S^1)}$$

Comparison with the field theory.

Null-surfaces correspond to the long operators in the free Yang-Mills theory, which are "locally $\frac{1}{2}$-BPS". We will see that the moduli space of such operators with the fixed length is

$$\mathrm{Map}_0\left(S^1,\quad Gr(2|2,4|4)\right) \tag{2}$$

We can compare it to the moduli space of the supersymmetric null-surfaces:

$$\frac{\mathrm{Map}_0\left(S^1,\quad Gr(2|2,4|4)\right)\widetilde{\times}S^1}{\mathrm{Diff}(S^1)} \tag{3}$$

There is an apparent difference between (2) and (3). But in fact,

- The fiber S^1 is the degree of freedom dual to the length of the operator.
- we have to consider the *parametrized* null-surfaces; if we consider the null-surface as the limit of the worldsheet of the very fast moving string, it actually comes with a parametrization. This parametrization corresponds to the density of the conserved charge corresponding to the length of the operator. This removes $\mathrm{Diff}(S^1)$.

Field theory side.

Consider the operators of the form

$$\text{tr}\ \underbrace{\phi\phi\ldots\partial\ldots\partial\phi\ldots}_{L}$$

where ϕ is a scalar field of the $\mathcal{N}=4$ Yang-Mills.

It is convenient to consider instead of the single-trace operators tr $\phi\cdots\phi$ the corresponding state in the theory on $\mathbf{R}\times S^3$. This is a chain of one-particle states ("partons"). For the one-loop computations, we can consider partons as 1-particle states in the free theory. These 1-particle states form a representation of the conformal group $SO(2,4)\simeq SU(2,2)$ known as the "singleton representation". The conformal group $SU(2,2)$ together with the R-symmetry group $SU(4)$ form a bosonic part of the super-group of super-conformal transformations, called $PSU(2,2|4)$. The 1-particle states are in the "super-singleton" representation of this group.

Definition of coherent states.

Consider a special class of 1-particle states known as "coherent states". To define them, we first take the following state:

$$\psi_1 = \int_{S^3} d^3\vec{n}(\Phi_1(\vec{n}) - i\Phi_2(\vec{n}))|0>$$

Here Φ_1 and Φ_2 are two of the six scalar fields of the $N = 4$ super-Yang-Mills theory and $|0>$ is the conformally invariant vacuum of this theory. This state ψ_1 can be described as the creation operator of the zero harmonic of the field $\Phi_1 - i\Phi_2$ on S^3, acting on the vacuum.

Let us act on this state ψ_1 by the superconformal group $G = PSU(2,2|4)$. Define ψ_g:

$$\psi_g = g.\psi_1, \qquad g \in PSU(2,2|4)$$

The stabilizer of ψ_1 is $H = PSU(2|2) \times PSU(2|2) \times U(1)^2$, so the coherent states are parametrized by the coset space G/H.

We conclude that the coherent states in the super-singleton representation are parametrized by the coset space:

$$G/H = \frac{PSU(2,2|4)}{PSU(2|2) \times PSU(2|2) \times U(1)^2} = Gr(2|2,4|4)$$

The states ψ_g generate the super-singleton representation. Consider parton chains of the form:

$$\text{tr}\ \psi_{g_1} \otimes \psi_{g_2} \otimes \cdots \otimes \psi_{g_L} = \Psi_{[g(n)]}$$

— the chain of coherent states of the individual partons. Such 'coherent parton chains' generate the space of L-particle states. Consider the quantum evolution of the state $\Psi_{[g(n)]}$. Take the limit:

- $L \to \infty$
- λ/L^2 finite but small (a small parameter of the perturbation theory),
- use $\sigma = n/L$ instead of n; $g(n)$ becomes $g(\sigma)$: a contour in $Gr(2|2,4|4)$.

Conjecture: This is a classical limit, defining the classical continuous spin chain.

Length of the operator on the field theory side.

In the free field theory we can parametrize the single trace operators (or states) by the "words" composed of the "letters" — the elementary fields. With the interactions turned on we cannot parametrize states by the expressions composed of the elementary fields because there are divergencies and there is no conformally invariant regularization. All we can say, is that we have a space of states with the action of $PSU(2,2|4)$. (And in the classical limit, this becomes the classical phase space, a symplectic manifold with the action of $PSU(2,2|4)$.)

But, if we construct the space of states in the perturbation theory, we at least should be able to say for each state what is the *engineering dimension* of the corresponding operator. The space of states is a representation of $PSU(2,2|4)$. For each subrepresentation, the highest weight (of energy) will be an integer plus a correction proportional to λ; this integer is the engineering dimension. In fact, the perturbation theory is usually developed in a sector with a given engineering dimension (the operator mixes under the renormgroup only with the operators of the same engineering dimension).

What can we say about the engineering dimension in the classical limit?

Length of the operator on the field theory side.

Let us first consider the operators which do not have any derivatives, operators composed only of the scalars. The renormgroup would mix such operators with the operators containing derivatives, but not in the continuous limit. In the classical continuous limit, those long operators composed of purely scalar elementary fields will only mix among themselves, the derivatives and spinor fields will not appear.

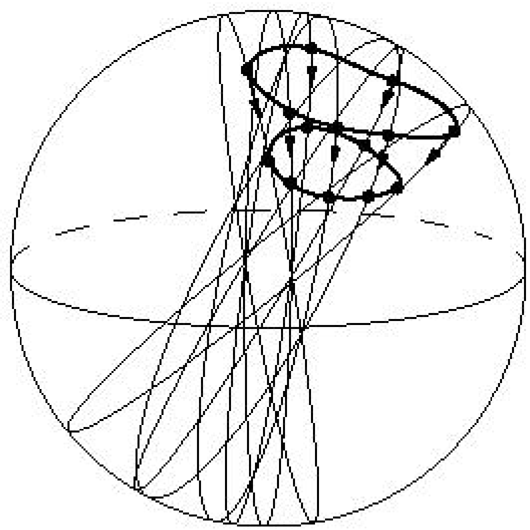
Therefore, for the purely scalar operators in the continuous limit the engineering dimension coincides with the length — the number of elementary fields under the trace.

The purely scalar operators correspond to the classical strings which project to a single timelike geodesic in AdS_5 — all the nontrivial motion is in S^5. The most general classical strings have nontrivial projections to AdS_5; they correspond to the operators which involve the derivatives of the elmentary fields (in fact, coherent superpositions of the derivatives).

These most general quasiclassical states do not have a definite engineering dimension. But, they should have a definite length — the number of elementary fields under the trace. (This is a conjecture.)

The moduli space of the null surfaces has a $U(1)$ symmetry.

Can we find the conserved charge corresponding to the length of the operator on the string theory side? Consider the following $U(1)$ symmetry acting on the null surface:



We will call it $U(1)_L$.

We conclude that the coherent states in the super-singleton representation are parametrized by the coset space:

$$G/H = \frac{PSU(2,2|4)}{PSU(2|2) \times PSU(2|2) \times U(1)^2} = Gr(2|2,4|4)$$

The states ψ_g generate the super-singleton representation. Consider parton chains of the form:

$$\mathrm{tr}\ \psi_{g_1} \otimes \psi_{g_2} \otimes \cdots \otimes \psi_{g_L} = \Psi_{[g(n)]}$$

— the chain of coherent states of the individual partons. Such 'coherent parton chains' generate the space of L-particle states. Consider the quantum evolution of the state $\Psi_{[g(n)]}$. Take the limit:

- $L \to \infty$
- λ/L^2 finite but small (a small parameter of the perturbation theory),
- use $\sigma = n/L$ instead of n; $g(n)$ becomes $g(\sigma)$: a contour in $Gr(2|2,4|4)$.

Conjecture: This is a classical limit, defining the classical continuous spin chain.

Length of the operator on the field theory side.

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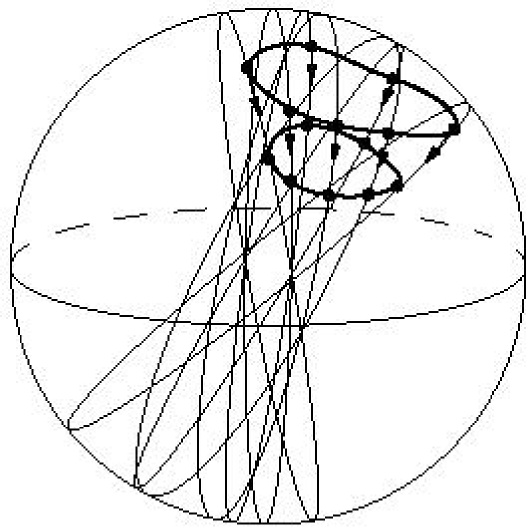


We will call it $U(1)_L$.

The length of the spin chain on the string theory side.

Statement. There is a unique extension of $U(1)_L$ from the null-surfaces to the string phase space (at least to the fast moving strings). The charge of $U(1)_L$ corresponds to $L/\sqrt{\lambda}$ on the field theory side.

The action of $U(1)_L$ on the phase space of a classical string can be defined by the following characteristic properties:

- preserves the symplectic structure.
- has periodic orbits.
- commutes with $PSU(2,2|4)$
- acts on the null surfaces, as we described on the previous slide.
- does not change the projection of the worldsheet to AdS_5; moreover it preserves the projections to AdS_5 of the null-directions on the worldsheet.

Hamiltonian reduction; the statement of equivalence.

Consider the Hamiltonian reduction of the phase space of the classical string by $U(1)_L$ on the level set of $L = L_0$ =const.

- **There is an equivalence:**

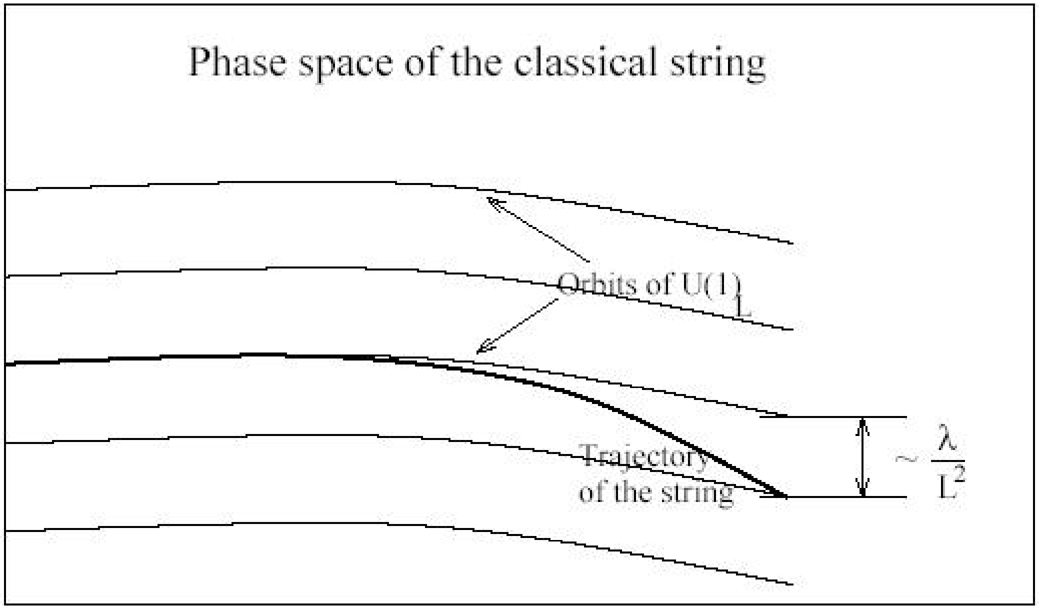
Reduced phase space of the classical string reduced on the level set L_0 of $U(1)_L$.	$\Longleftrightarrow$	Phase space of the classical single trace state of the length L_0 in the Yang-Mills theory.

- **This equivalence commutes with the action of $PSU(2,2|4)$.**
- **The Yang-Mills perturbation theory corresponds to considering fast moving strings as perturbations of the null-surfaces; the small parameter is $\sqrt{1-v^2}$.**

This statement is a **conjecture**.

Why do we need the Hamiltonian reduction by $U(1)_L$?

In the limit $L >> \sqrt{\lambda}$ the string is moving very fast, because $L \simeq \sqrt{\lambda}(1-v^2)^{-1/2}$



The string itself moves very fast, but the fast motion corresponds to the large engineering dimension (more precisely, large length). But it moves almost along the orbit of $U(1)_L$. Therefore the evolution on the space of orbits is slow, of the order $\frac{\lambda}{L^2}$. This should be compared to the evolution of the coherent states on the Yang-Mills theory side.

The construction of $U(1)_L$ in the perturbation theory.

The first Pohlmeyer charge. Suppose that the target space of the classical string is of the form $A \times S$. Then we can define the first Pohlmeyer charge:

$$Q^{[1]} = \oint_C d\tau^+ \sqrt{(\partial_{\tau^+} x_S, \partial_{\tau^+} x_S)}$$

Notations:

C is the closed contour on the string worldsheet;
τ^+, τ^- are the coordinates on the worldsheet; the worldsheet metric is $\simeq d\tau^+ d\tau^-$;
$x_S(\tau_+, \tau_-)$ is the projection of the worldsheet to S

Similarly, we define

$$\widetilde{Q}^{[1]} = \oint_C d\tau^- \sqrt{(\partial_{\tau^-} x_S, \partial_{\tau^-} x_S)}$$

The construction of $U(1)_L$ in the perturbation theory.

If S is the *sphere* S^5 then $Q^{[1]}$ generates periodic trajectories on the null surfaces.
The key properties of $Q^{[1]}$:

- $Q^{[1]}$ commutes with the reparametrizations $(\tau^+, \tau^-) \to (f^+(\tau^+), f^-(\tau^-))$
- $Q^{[1]}$ generates $U(1)_L$ *on the null-surfaces*

These two properties of $Q^{[1]}$ allow us to use $Q^{[1]}$ as a tool for extending $U(1)_L$ from the null-surfaces to the fast moving strings.

The construction of $U(1)_L$ in the perturbation theory.

Let M be the phase space of the classical string in $AdS_5 \times S^5$ It is the space of embeddings $(x_A(\tau^+, \tau^-),\ x_S(\tau^+, \tau^-))$ with the Virasoro constraints $(\partial_+ x_A)^2 + (\partial_+ x_S)^2 = 0$ and $(\partial_- x_A)^2 + (\partial_- x_S)^2 = 0$ satisfying the equations of motion $D_+ \partial_- x_{A,S} = 0$. Modulo conformal reparametrizations. The symplectic form is given by:

$$\omega(\delta_1 x, \delta_2 x) = \oint_C (\delta_1 x, * \overleftrightarrow{d}\ \delta_2 x)$$

Consider the phase space $\widehat{M}$ which consists of the embeddings $(x_A(\tau^+, \tau^-),\ x_S(\tau^+, \tau^-))$, $D_+ \partial_- x_{A,S} = 0$ *without the Virasoro constraints.* (Just harmonic maps.) Define the function K on $\widehat{M}$:

$$K = \int_{\tau=0} d\sigma |\partial_\tau x_S| = \int_{\tau=0} d\sigma |p_S(\sigma)| \qquad \text{(fixed contour } \tau = 0)$$

The Hamiltonian vector field generated by K is manifestly periodic. But K does not preserve the Virasoro constraint. The idea of the construciton of $U(1)_L$ is: try to modify K so that the modified charge is periodic but also commutes with the Virasoro constraints. The modified charge will then descend on M and be the Hamiltonian of $U(1)_L$.

The construction of $U(1)_L$ in the perturbation theory.

Although K is not in the involution with the Virasoro constraints, there is a canonical transformation $F : \widehat{M} \to \widehat{M}$ such that F^*K is in the involution with the Virasoro constraints; then F^*K will be the generator of $U(1)_L$ which we are looking for.

The first Pohlmeyer charge $Q^{[1]}$ is useful for constructing such a canonical transformation, in the following way. Notice that $\{K, Q^{[1]}\} \neq 0$. Let us try to find a canonical transformation F such that $\{F^*K, Q^{[1]}\} = 0$. Then, it turns out that

$$\{F^*K, Q^{[1]}\} = 0 \quad \Rightarrow \quad \{F^*K, \text{Virasoro}\} = 0 \tag{4}$$

Therefore, instead of considering invfinitely many conditions on F that $\{F^*K, \text{Virasoro}\} = 0$ we can consider just one condition $\{F^*K, Q^{[1]}\} = 0$.
The Virasoro constraints will follow.

We will now explain how to construct F such that $\{F^*K, Q^{[1]}\} = 0$ and prove (4).

We can construct F order by order in the perturbation theory. First, use the fact that $Q^{[1]}$ generates $U(1)_L$ on the null-surfaces, the same as K. Therefore:

$$Q^{[1]} = K + q_1 + q_2 + \dots$$

Under the rescaling $p_S \to tp_S$: $K \to tK$, $q_1 \to t^{-1}q_1$, $q_2 \to t^{-3}q_2$, $q_m \to t^{1-2m}q_m$.
The symplectic form is of the degree 1: $\omega \to t\omega$; the Poisson brackets are of the degree -1: $\{,\} \to t^{-1}\{,\}$.
Since K generates periodic trajectories, we can decompose q_m in the Fourier series:

$$q_m = q_{m,0} + \sum_{k\neq 0} q_{m,k} \quad \text{where} \quad \{K, q_{m,k}\} = ikq_{m,k}$$

Now we can take $F_{(1)} = \exp\left[\text{Hamiltonian vector field generated by } \left(\sum_{k\neq 0}\frac{1}{ik}q_{m,k}\right)\right]$ then $F_{(1)}^{-1*}Q^{[1]} = K + q_{1,0} + O(1/|p_S|^3)$ therefore $\{F_{(1)}^*K, Q^{[1]}\} \sim 1/|p_S|^3$. In the same way we can define $F_{(2)}, F_{(3)}, \dots$ so that $\{F_{(m)}^*K, Q^{[1]}\} \sim 1/|p_S|^{2m+1}$.

This procedure gives $F = F_{(\infty)}$, such that F^*K commutes with $Q^{[1]}$.

F^*K commutes with the Virasoro constraints.

After the canonical transformation we have:

$$F^{-1*}(Q^{[1]}) = K + q'_1 + q'_2 + \ldots + q'_m + \ldots$$

where for all k: $\{K,\ q'_k\} = 0$. The reparametrization invariance is manifestly preserved at each order, therefore the resulting charge F^*K will commute with $(p_S, \partial_\sigma x_S)(\sigma)$ for any σ. We can also prove that F^*K commutes with $|p_S|^2(\sigma) + |\partial_\sigma x_S|^2(\sigma)$ for any σ.

Indeed, we know that $F^{-1*}Q^{[1]} = K + q'_1 + q'_2 + \ldots$ commutes with $F^{-1*}(|p_S|^2(\sigma) + |\partial_\sigma x_S|^2(\sigma)) = |p_S|^2 + \phi_0 + \phi_1 + \ldots$; therefore

$$\{K, \phi_0\} = \{|p_S|^2(\sigma), q'_1\} \Rightarrow \{K, \{K, \phi_0\}\} = 0 \Rightarrow \{K, \phi_0\} = 0$$

$$\{K, \phi_1\} + \{q'_1, \phi_0\} = \{|p_S|^2(\sigma), q'_2\} \Rightarrow \{K, \{K, \phi_1\}\} = 0 \Rightarrow \{K, \phi_1\} = 0$$

$etc.$ (here it was important that K generates periodic trajectories)

We see that F^*K commutes with the Virasoro constraints. It is a conserved charge, local at each order in the perturbation theory. It generates the "hidden" symmetry $U(1)_L$.

The role of the higher Pohlmeyer charges.

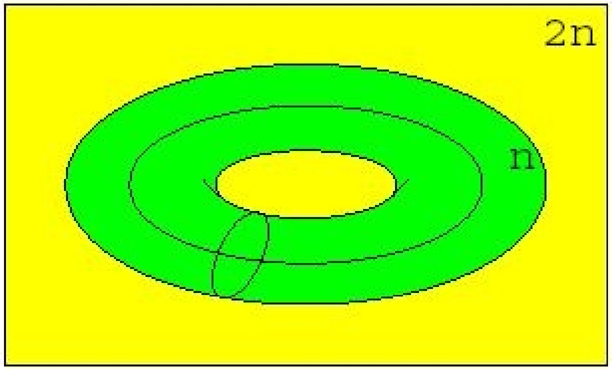
In the special case when S is the sphere there is an infinite tower of the "higher" conserved charges. For example,

$$Q^{[2]} = \oint_C \left[\frac{d\tau^+}{|\partial_+ x_S|} \left(D_+ \frac{\partial_+ x_S}{|\partial_+ x_S|}, D_+ \frac{\partial_+ x_S}{|\partial_+ x_S|} \right) + 2 \frac{d\tau^-}{|\partial_+ x_S|} (\partial_- x_S, \partial_+ x_S) \right]$$

is the second conserved charge. (It is conserved only if S is a sphere.)

This is a part of the general story of integrable systems.

In classical mechanics, a mechanical system with $2n$-dimensional phase space is integrable if there are n functions $F_1, \ldots, F_n$ in involution with each other, and with the Hamiltonian. Then, there are n action variables $I_1, \ldots, I_n$, each generating a $U(1)$ symmetry. These action variables are special combinations of $F_1, \cdots, F_n$.



String theory is an infinite-dimensional integrable system, therefore this geometrical intuition is not directly applicable. But still, there is an infinite tower of the Pohlmeyer charges $Q^{[k]}$, in involution with each other. Can we find combinations of the Pohlmeyer charges generating periodic trajectories?

Notice that $Q^{[1]}$ generates periodic trajectories on the null-surfaces. On the fast moving string, the trajectries of $Q^{[1]}$ are not closed, the deviation from the periodicity is of the order $1/|p_S|$. Can we modify $Q^{[1]}$ by adding to it higher charges, so that the modified charge generates periodic trajectories?

It is certainly possible at the first order in $1/|p_S|$. Consider the following linear combination of the first two Pohlmeyer charges:

$$\frac{1}{16}\left[7(Q^{[1]}-\widetilde{Q}^{[1]})-\frac{1}{2}(Q^{[2]}-\widetilde{Q}^{[2]})\right]=$$

$$=\int d\sigma\left[|p_S|+\frac{1}{4|p_S|}\left[(\partial_\sigma x_S)^2-\left(D_\sigma\frac{p_S}{|p_S|},D_\sigma\frac{p_S}{|p_S|}\right)-\frac{(p_S,\partial_\sigma x_S)^2}{(p_S,p_S)}\right]+\ldots\right]$$

One can see immediately that the trajectories of this linear combination are closed up to the terms of the order subleading to $1/|p_S|$. Indeed, the leading term is just K, it gives periodic trajectories. And the subleading term averages to zero on the periodic trajectories of the leading term. Therefore the trajectories of of this Hamiltonian do not drift at the order $1/|p_S|$; the deviation from the periodicity is of the order $1/|p_S|^3$.

Conjecture: It is possible to add higher Pohlmeyer charges, so that the resulting charge will generate periodic trajectories to all orders in $1/|p_S|$. The resulting charge coincides with the generator of $U(1)_L$ which we described in perturbation theory (as F^*K). In other words the classical string theory on $AdS_5\times S^5$ has an action variable which is local at each order of perturbation theory around the null-surfaces. This action variable corresponds to the length of the operator on the field theory side.

Pohlmeyer charges for rigid solutions were computed by G. Arutyunov, M. Staudacher and J. Engquist. The conserved charges have the following structure:

$$\mathcal{E}_n = \delta_{2,n}\mathcal{J} + \frac{\epsilon_n^{(1)}}{\mathcal{J}} + \frac{\epsilon_n^{(2)}}{\mathcal{J}^3} + \frac{\epsilon_n^{(3)}}{\mathcal{J}^5} + \dots$$

where $\mathcal{J} = J/\sqrt{\lambda}$, and J is a particular combination of the $U(1)^3 \subset U(3) \subset SO(6)$ momenta. The coefficients $\epsilon_n^{(m)}$ depend on what kind of a rigid string is considered (the ratio of spins). But Arutyunov, Staudacher and Engquist noticed that the coefficients $\epsilon_n^{(m)}$ for different values of n are not independent. For all the solutions they considered, they find that:

$$\mathcal{E}_{10} + \frac{74}{7}\mathcal{E}_8 + \frac{1898}{35}\mathcal{E}_6 + \frac{6922}{35}\mathcal{E}_4 + \frac{32768}{35}(\mathcal{E}_2 - \mathcal{J}) \sim \frac{1}{\mathcal{J}^9}$$

This means that up to the terms of the order $1/|p_S|^9$ we should have:

$$\frac{J}{\sqrt{\lambda}} = \mathcal{E}_2 + \frac{6922}{32768}\mathcal{E}_4 + \frac{1898}{32768}\mathcal{E}_6 + \frac{370}{32768}\mathcal{E}_8 + \frac{35}{32768}\mathcal{E}_{10} + \ldots \qquad (5)$$

At first this formula looks rather strange, because it seems to imply that a certain combination of Pohlmeyer charges (which all commute with $SO(6)$) is equal to some component of the angular momentum (which transforms in the adjoint of $SO(6)$). We propose the following resolution of this puzzle. The right hand side of (5) is actually the action variable, which for a particular class of the solutions considered by Arutyunov *et al* happens to be equal to the $SO(6)$ charge J (because these particular solutions correspond to the chiral operators on the field theory side). In other words, this formula should be understood as follows:

$$\text{Generator of } U(1)_L \ = \ \mathcal{E}_2 + \frac{6922}{32768}\mathcal{E}_4 + \frac{1898}{32768}\mathcal{E}_6 + \frac{370}{32768}\mathcal{E}_8 + \frac{35}{32768}\mathcal{E}_{10} + \ldots$$

The BMN limit.

The worldsheet theory becomes quadratic in the Penrose limit:

$$S = \frac{1}{2\pi}\int d\tau d\sigma \sum_{i=1}^{4}\left[(\partial_\tau x_i)^2 + (\partial_\tau y_i)^2 - (\partial_\sigma x_i)^2 - (\partial_\sigma y_i)^2 - p_+^2(x_i^2 + y_i^2)\right]$$

The Hamiltonian is:

$$H = \frac{1}{2\pi}\int d\sigma \sum_{i=1}^{4}\left[p_i^2 + q_i^2 + (\partial_\sigma x_i)^2 + (\partial_\sigma y_i)^2 + p_+^2(x_i^2 + y_i^2)\right]$$

Here $p_i(\sigma)$ and $q_i(\sigma)$ are the momenta conjugate to x_i and y_i respectively:

$$\{p_i(\sigma), x_j(\sigma')\} = 2\pi\delta(\sigma - \sigma'), \quad \{q_i(\sigma), y_j(\sigma')\} = 2\pi\delta(\sigma - \sigma')$$

There is also a constraint:

$$\int d\sigma\, [(p, \partial_\sigma x) + (q, \partial_\sigma y)] = 0$$

The states of the quantum theory are constructed by acting on the vacuum $|0>$ by the operators α_n^i and β_n^i, $n \in \mathbf{Z}$, $i \in \{1,2,3,4\}$:

$$\alpha_n^i = \int_0^{2\pi} d\sigma\ e^{-in\sigma} \left((n^2 + p_+^2)^{1/4}\ x^i + \frac{1}{(n^2+p_+^2)^{1/4}} \frac{\partial x^i}{\partial \tau} \right)$$

$$\beta_n^i = \int_0^{2\pi} d\sigma\ e^{-in\sigma} \left((n^2 + p_+^2)^{1/4}\ y^i + \frac{1}{(n^2+p_+^2)^{1/4}} \frac{\partial y^i}{\partial \tau} \right)$$

Definition. Let us define the BMN-length of the state as the total number of the β-oscillators needed to create this state. For example the BMN-length of $(\beta_1^i)^3\ \beta_7^j\ |0>$ is equal to $3+1=4$.

In other words, the BMN-length is:

$$\mathcal{I} = \sum_{n=-\infty}^{\infty} \beta_n \overline{\beta_n}$$

The explicit expression for the BMN-length in coordinate space (in terms of $y_i(\sigma)$) is:

$$\mathcal{I} = \frac{1}{4\pi} \sum_{i=1}^{4} \int d\sigma \left[y_i(\sigma) \sqrt{p_+^2 - \partial_\sigma^2}\, y_i(\sigma) + q_i(\sigma) \frac{1}{\sqrt{p_+^2 - \partial_\sigma^2}}\, q_i(\sigma) \right]$$

The BMN-length generates the following $U(1)$ symmetry:

$$\{\mathcal{I}, y_j(\sigma)\} = \frac{1}{\sqrt{p_+^2 - \partial_\sigma^2}}\, q_j(\sigma), \quad \{\mathcal{I}, q_j(\sigma)\} = -\sqrt{p_+^2 - \partial_\sigma^2}\, y_j(\sigma)$$

Therefore the BMN-length $\mathcal{I}$ is an action variable.

The length can be expanded in the inverse powers of p_+:

$$\mathcal{I} = \frac{1}{4\pi} \sum_{i=1}^{4} \int d\sigma \left[p_+ y_i y_i - \sum_{k=1}^{\infty} p_+^{1-2k} \frac{(2k-3)!!}{2^k k!} \, y_i \partial_\sigma^{2k} y_i + \right.$$

$$\left. + \sum_{k=1}^{\infty} p_+^{1-2k} \frac{(2k-3)!!}{2^{k-1}(k-1)!} \, q_i \partial_\sigma^{2k-2} q_i \right] = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2} \, \frac{I_{k+1}}{p_+^{2k+1}} \qquad (6)$$

where I_k are the local conserved charges of the free massive theory:

$$I_k = \frac{1}{4\pi} \sum_{i=1}^{4} \int d\sigma \left[q_i \partial_\sigma^{2k-2} q_i - y_i \partial_\sigma^{2k} y_i + p_+^2 y_i \partial_\sigma^{2k-2} y_i \right] =$$

$$= (-1)^{k+1} \times \sum_n n^{2k-2} \omega_n \, \beta_n \overline{\beta_n}$$

Eq. (6) shows that the BMN length $\mathcal{I}$ can be expanded in local conserved charges. This is the Penrose limit of the Arutyunov-Staudacher-Engquist formula.

Arutyunov-Staudacher-Engquist formula and the plane wave limit.

ASE used the original definition of the Pohlmeyer charges (due to Pohlmeyer) using the Bäcklund transformation. Given the solution $Y(\tau,\sigma)$ of the sigma-model with the target space S^5 one constructs another solution $Y'(\tau,\sigma)$ defined by the equations:

$$\partial_+(Y'+Y) = \frac{1}{2}(1+\gamma^{-2})(Y',\partial_+Y)(Y'-Y)$$
$$\partial_-(Y'-Y) = -\frac{1}{2}(1+\gamma^{2})(Y',\partial_-Y)(Y'+Y)$$

where γ is a constant parameter. We will also write $Y(\gamma)$ instead of Y'. The generating function for the local conserved charges is:

$$\mathcal{E}(\gamma) = \frac{1}{2\pi}\int d\sigma \left[\gamma(Y(\gamma),\partial_+Y) + \gamma^3(Y(\gamma),\partial_-Y)\right]$$

One can also define $\widetilde{\mathcal{E}}(\gamma)$ by the same formulas but with $\partial_+ \leftrightarrow \partial_-$. Let us define $\mathcal{E}^{even}(\gamma) = \frac{1}{2}(\mathcal{E}(\gamma)+\widetilde{\mathcal{E}}(\gamma))$.

Let us define the "modified" charges $\mathcal{G}_k$ by reexpanding $\mathcal{E}^{even}(\gamma)$ in powers of $\frac{\gamma^2}{(1+\gamma^2)^2}$:

$$\mathcal{E}^{even}(\gamma) = -\epsilon^2\gamma^2 \sum_{k=0}^{\infty} \left[\frac{4\gamma^2}{(1+\gamma^2)^2}\right]^k \mathcal{G}_k$$

Applying the definition through the Bäcklund transformations in the plane wave limit we have found that in the plane wave limit

$$\mathcal{G}_k = p_+^{-1-2k} I_{k+1} + \delta_{k,0} J$$

This equation with (6) imply that

$$J + \sum_{n=-\infty}^{\infty} \beta_n \overline{\beta_n} = \sum_{k=0}^{\infty} \frac{1}{2^{2k}} \frac{(2k)!}{(k!)^2}\, \mathcal{G}_k \qquad (7)$$

The right hand side should be the generator of $U(1)_L$ not just in the plane wave limit.

Summary of the discussion of $U(1)_L$.

We argued that there is a "hidden symmetry" $U(1)_L$ whose generator is an infinite linear combination of Pohlmeyer charges. Let us summarize our arguments.

1. It is possible to build the generator of $U(1)_L$ order by order in the perturbation theory. In perturbation theory $U(1)_L$ can be defined as $U(1)$ commuting with the first Pohlmeyer charge. It will then automatically commute with the Virasoro constraints.
2. We see in perturbation theory that $U(1)_L$ is local in each order of the perturbation theory. We have not really used the integrability yet. The possibility of constructing $U(1)$ commuting with the Hamiltonian *in the perturbation theory* does not require integrability: it is enough that the Hamiltonian is a small perturbation of the periodic system. But:
3. There are higher Pohlmeyer charges; we see that $U(1)_L$ commutes with them.
4. The set of Pohlmeyer charges is presumably a complete set of mutually commuting local conserved charges. Therefore we conjecture that it should be possible to construct the generator of $U(1)_L$ as a linear combination of the Pohlmeyer charges.

We then use the explicit computation to the first nontrivial order in the perturbation theory and known results for rigid strings to reinforce our conjecture. We use the explicit computation in the plane wave limit to fix the coefficients of the higher Pohlmeyer charges in the expansion of the generator of $U(1)_L$.

Anomalous dimension and local charges.

As an example of the application of the action variable, we will now answer the following question: given the classical string worldsheet, how to compute the anomalous dimension of the corresponding operator?

The anomalous dimension is usually defined as the deformation of the particular generator of the conformal algebra — the dilatation operator. But in fact the anomalous dimension parametrizes the deformation of the *representation* of the conformal algebra, rather than the deformation of a particular generator. It is natural to characterize the deformation of the representation by the action of the center of the group. Therefore we will define the anomalous dimension through the action of the center of the conformal group. This definition is manifestly conformally invariant.

The center of the superconformal group.

The group $PSU(2,2|4)$ is not simply connected; the superconformal group of the conformal field theory is actually a covering group which we will denote $\widetilde{PSU}(2,2|4)$. The bosonic part of $\widetilde{PSU}(2,2|4)$ is $[\widetilde{SU}(2,2) \times SU(4)]/\mathbf{Z}_2$ where $\widetilde{SU}(2,2)$ is the universal covering of $SU(2,2)$. Let c denote the generator of the center. The action of c can be understood in the following way. Consider the conformal field theory on $\mathbf{R} \times S^3$ where $\mathbf{R}$ is the time and the radius of S^3 is 1. Let t denote the time and $\vec{n}$ denote the unit vector parametrizing S^3. Then c acts as the combination of the conformal transformation:

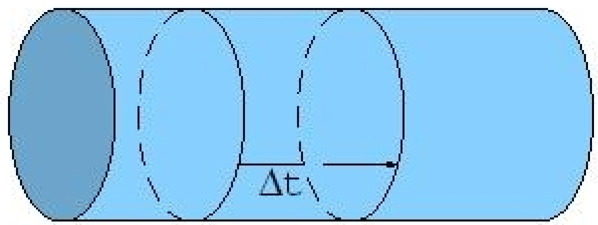
$$c: \ (t, \vec{n}) \to (t + \pi, -\vec{n})$$

with the R-symmetry $i\mathbf{1} \in SU(4)$. This transformation commutes with the generators of $so(2,4)$ and therefore it is in the center of the conformal group. It also commutes with the fermionic generators of the supersymmetry, therefore it is in the center of $\widetilde{PSU}(2,2|4)$.

We will define the anomalous dimension through the action of c^2:

$$c^2 = e^{2\pi i \Delta}$$

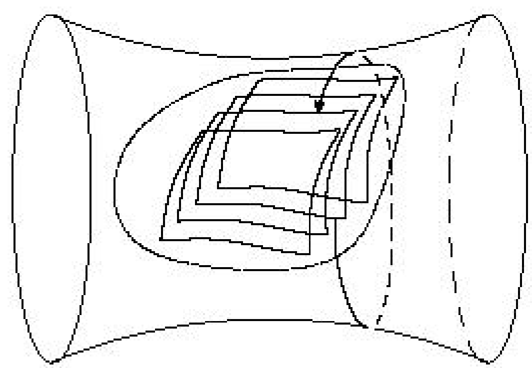
c^2 acts as a shift in time by $\Delta t = 2\pi$ (radius of S^3) and $(-1)^F$:



In the free theory the action of the centre is trivial; but in the interacting theory $c \neq 1$; the logarithm $\log c$ is called the anomalous dimension. We can expand $\log c$ in powers of λ: $\log c = \lambda d_1 + \lambda^2 d_2 + \ldots$.

Center of $PSU(2,2|4)$: the string theory side.

The AdS space is the universal covering space of the hyperboloid and c^2 acts as a deck transformation exchanging the sheets. We can visualize the action of this deck transformation on the string phase space in the following way:



The string worldsheet looks locally like a deck of cards. Going around the noncontractible cycle in the hyperboloid exchanges the sheets. This is how c^2 acts on the phase space of the classical string.

Deck transformation and local charges.

We have seen that the conserved charge corresponding to the length of the operator is an infinite linear combination of the Pohlmeyer charges:

$$L = \sqrt{\lambda}\,[\mathcal{E}_2 + a_1\mathcal{E}_4 + a_2\mathcal{E}_6 + a_3\mathcal{E}_8 + \ldots]$$

The corresponding Hamiltonian vector field ξ_L has periodic trajectories:

$$e^{2\pi\xi_L} = \text{identical transformation}$$

In this expansion $\mathcal{E}_{2k}$ are the Pohlmeyer charges for the S^5 sigma-model. But the classical string in $AdS_5 \times S^5$ essentially splits into two systems: the sigma-model with the target space AdS_5 and the sigma-model with the target space S^5. The AdS_5 sigma-model also has Pohlmeyer charges. Let us denote them $\mathcal{F}_{2k}$. How can we use them?

Consider the conserved charge $M = \sqrt{\lambda}\,[\mathcal{F}_2 + a_1\mathcal{F}_4 + a_2\mathcal{F}_6 + a_3\mathcal{F}_8 + \ldots]$ defined with the same coefficients a_k.

Notice that $e^{-2\pi\xi_M}$ acts as the deck transformation: $e^{-2\pi\xi_M} = c^2$

Since ξ_M commutes with ξ_L, we can also write:

$$c^2 = e^{2\pi(\xi_L - \xi_M)}$$

But $\mathcal{E}_2 = \mathcal{F}_2$ because of the Virasoro constraints. Therefore we can identify

$$\frac{1}{2\pi}\log c^2 = \sqrt{\lambda}\,[a_1(\mathcal{E}_4 - \mathcal{F}_4) + a_2(\mathcal{E}_6 - \mathcal{F}_6) + a_3(\mathcal{E}_8 - \mathcal{F}_8) + \ldots] \qquad (8)$$

This expression is a perturbative expansion of the anomalous dimension of the fast moving string in the perturbation theory around the null surface. The small parameter is the relativistic factor $\sqrt{1-v^2} \sim \frac{\sqrt{\lambda}}{L}$, where v is the typical velocity of the string. One can define the local conserved charges in such a way that $\mathcal{E}_{2k},\quad \mathcal{F}_{2k} \;\simeq (1-v^2)^{k-3/2}$. Therefore (8) is an expansion in powers of $\frac{\lambda}{L^2}$. The first term is of the order $\frac{\lambda}{L}$, the second term is of the order $\frac{\lambda^2}{L^3}$ and so on.