Title: Fermions from Half-BPS Supergravity

Date: Feb 23, 2005 11:00 AM

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Abstract: We discuss collective coordinate quantization of the half-BPS geometries of Lin, Lunin and Maldacena (hep-th/0409174). The LLM geometries are parameterized by a single function u on a plane. We treat this function as a collective coordinate. We arrive at the collective coordinate action as well as path integral measure by considering D3 branes in an arbitrary LLM geometry. The resulting functional integral is shown, using known methods (hep-th/9309028), to be the classical limit of a functional integral for free fermions in a harmonic oscillator. The function u gets identified with the classical limit of the Wigner phase space distribution of the fermion theory which satisfies u * u = u. The calculation shows how configuration space of supergravity becomes a phase space (hence noncommutative) in the half-BPS sector. Our method sheds some new light on counting supersymmetric configurations in supergravity.

Fermions from $\frac{1}{2}$ BPS Supergravity

GM 0502104

Recently LLM studied IIB SUGRA solutions which are $\frac{1}{2}$-BPS, asymptotically $AdS_5 \times S_5$ and preserving $O(4) \times O(4)$ symmetry.

$$\underset{\uparrow}{AdS_5} \times \underset{\uparrow}{S_5}$$
$$O(4) \times O(4)$$

- They found all such (regular) solutions.
- All solutions are completely characterized by a single function on a plane:

$$u(x_1, x_2, 0)$$

(2)

- $$ds^2 = -\frac{y}{\sqrt{u(1-u)}}\left(dt + V_i\, dx_i\right)^2 + \frac{\sqrt{u(1-u)}}{y}\left(dy^2 + dx_i\, dx_i\right) \quad \left.\right\} \; t,\ x_1,\ x_2,\ y$$

$$+\, y\sqrt{\frac{1-u}{u}}\, d\Omega_3^2$$

$$+\, y\sqrt{\frac{u}{1-u}}\, d\tilde{\Omega}_3^2$$

- $$F^{(5)} = F^{(5)}_{el} + F^{(5)}_{mag}$$

$$F^{(5)}_{el} = \left(d\left(-\frac{y^2}{4}\,\frac{1-u}{u}\,(dt+V)\right) - \frac{y^3}{4} *_3 d\left(\frac{1-u}{y^2}\right)\right)\wedge d^3\Omega$$

$$F^{(5)}_{mag} = \left(d\left(-\frac{y^2}{4}\,\frac{u}{1-u}\,(dt+V)\right) + \frac{y^3}{4} *_3 d\left(\frac{u}{y^2}\right)\right)\wedge d^3\tilde{\Omega}$$

$$u = u(x_1, x_2, y)$$

$$V_i = V_i(x_1, x_2, y) \qquad i = 1, 2$$

- $$u(x_1, x_2, y) = \frac{y^2}{\pi} \int \frac{\boxed{u(x_1', x_2', 0)}\, dx_1' dx_2'}{\left((\vec{x} - \vec{x}')^2 + y^2\right)^2}$$

- $$V_i(x_1, x_2, y) = -\frac{\epsilon_{ij}}{\pi} \int \frac{\boxed{u(x_1', x_2', 0)}\,(x_j - x_j')\, dx_1' dx_2'}{\left((\vec{x} - \vec{x}')^2 + y^2\right)^2}$$

To avoid possible singularities at $y=0$ we need

$$u(x_1, x_2, 0) = 0 \quad \text{or} \quad 1$$

$$ds^2 = -\frac{y}{\sqrt{u(1-u)}}\left(dt + V_i\, dx_i\right)^2 + \frac{\sqrt{u(1-u)}}{y}\left(dy^2 + dx_i\, dx_i\right) + y\sqrt{\frac{1-u}{u}}\, d\Omega_3^2 + y\sqrt{\frac{u}{1-u}}\, d\tilde{\Omega}_3^2$$

$$\left.\begin{matrix} t \\ \left.\begin{matrix} x_1 \\ x_2 \end{matrix}\right\} \begin{matrix} r \\ \phi \end{matrix} \\ y \end{matrix}\right.$$

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$$+\, y\sqrt{\frac{1-u}{u}}\, d\Omega_3^2$$

$$+\, y\sqrt{\frac{u}{1-u}}\, d\tilde{\Omega}_3^2$$

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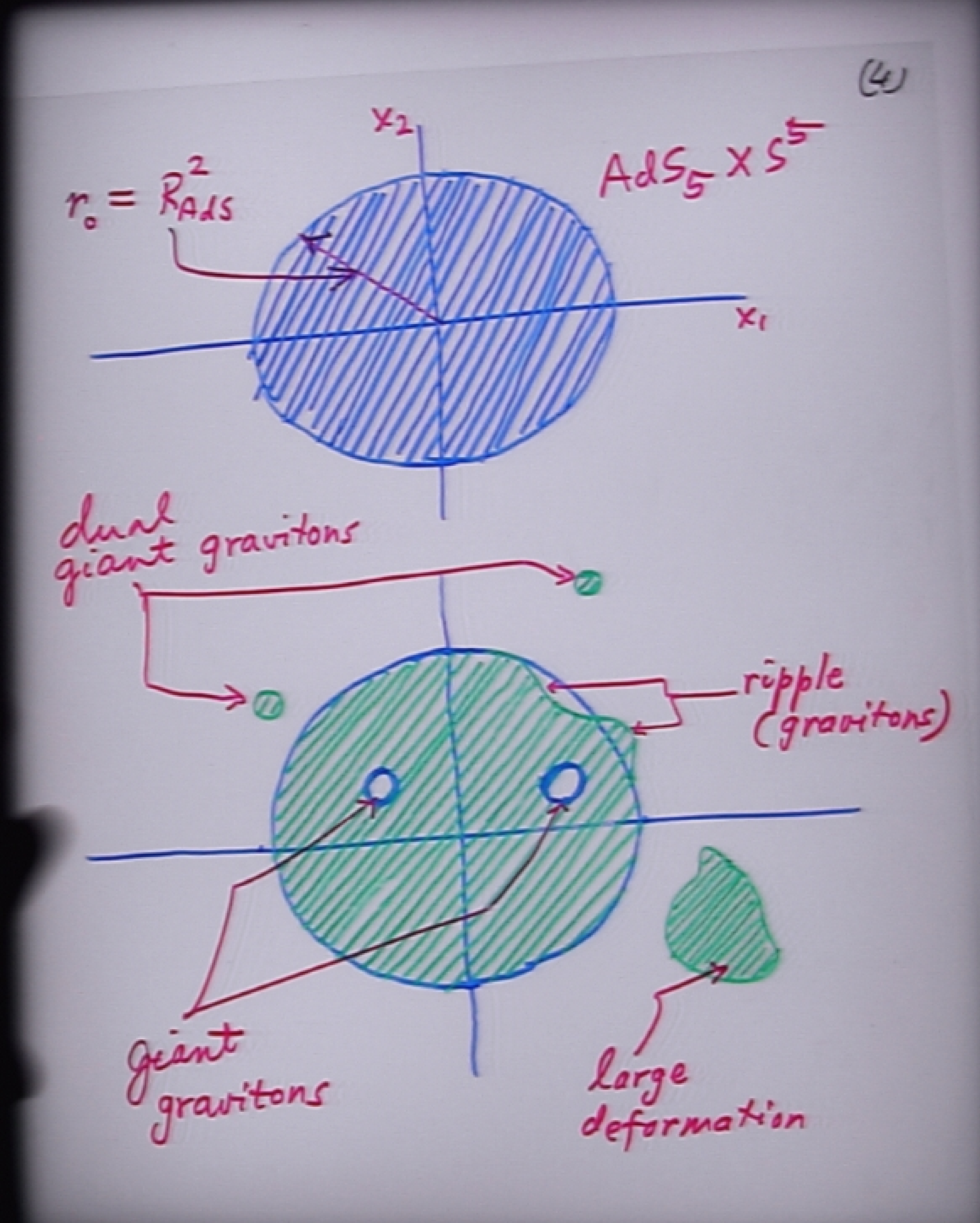
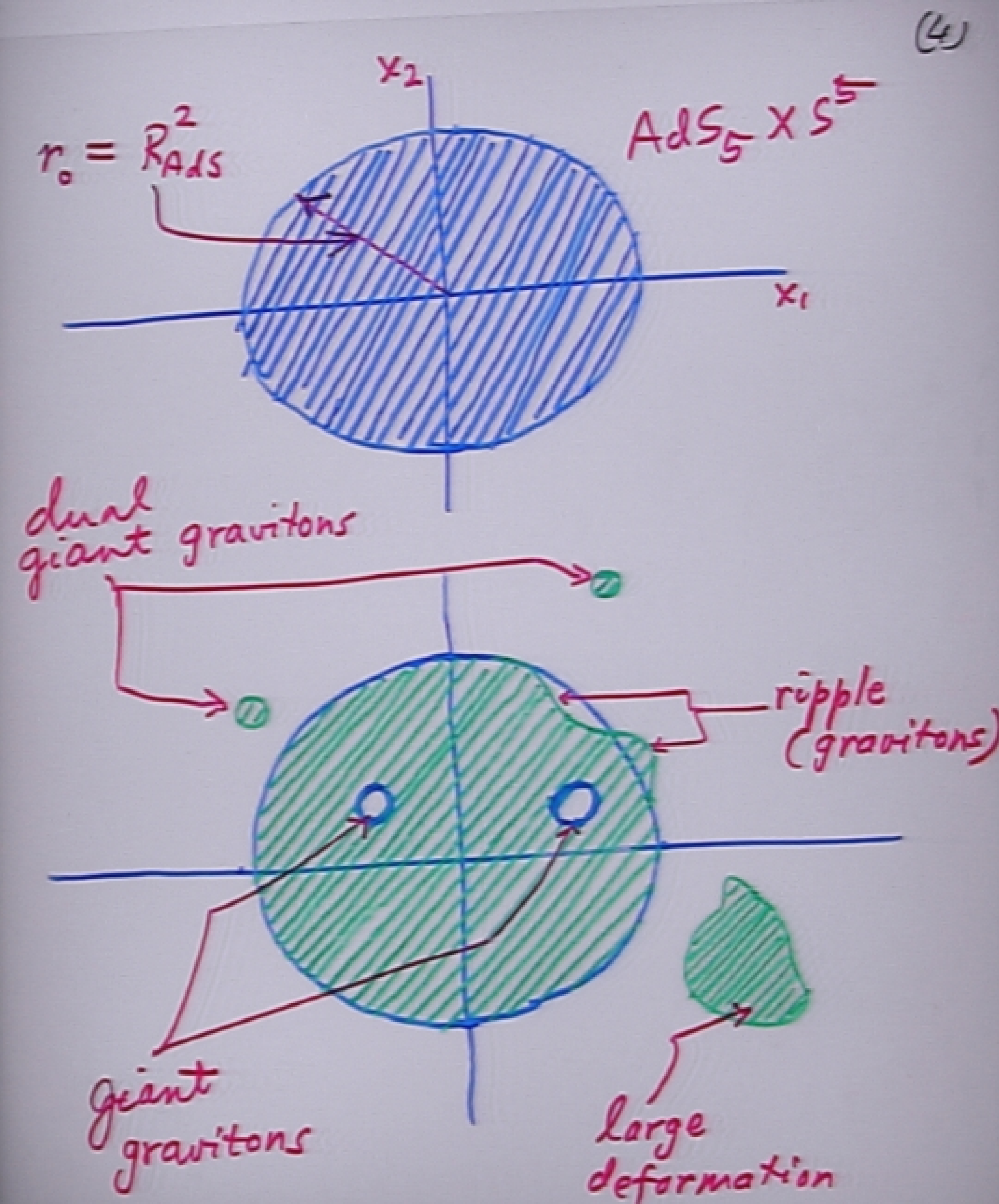

(4)
x_2
$AdS_5 \times S^5$
$r_0 = R^2_{AdS}$
x_1
dual giant gravitons
ripple (gravitons)
giant gravitons
large deformation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$r_0 = R_{AdS}^2$
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(4a)

Relation to global coordinates:

global $AdS_5 \times S^5$

$$ds^2 = R^2\left[-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2 + \cos^2\theta\, d\tilde{\phi}^2 + d\theta^2 + \sin^2\theta\, d\tilde{\Omega}_3^2\right] \quad (1)$$

LLM:

$$ds^2 = -\frac{y}{\sqrt{u(1-u)}}(dt + V_i dx_i)^2 + \frac{\sqrt{u(1-u)}}{y}(dy^2 + dx_i dx_i) + y\sqrt{\frac{1-u}{u}}\, d\Omega_3^2 + y\sqrt{\frac{u}{1-u}}\, d\tilde{\Omega}_3^2 \quad (2)$$

(1) = (2) if $u(x_1, x_2 0) = \theta(r_0 - r)$

$r_0 = R^2$

and

$$\left.\begin{array}{l} y = r_0 \sinh\rho \sin\theta \\ r = r_0 \cosh\rho \cos\theta \end{array}\right\} \; r \pm iy = r_0 \cos(\theta \pm i\rho)$$

$$\phi = \tilde{\phi} + t$$

(5)

Total area constraint

- $AdS_5 \times S^5$ solution:

$$u(x_1, x_2, 0) = \theta(r_0 - r) \qquad r_0 = R_{AdS}^2$$

Area of blob

$$= \pi r_0^2 = \pi R_{AdS}^4 = \pi . \ 4\pi g_s \ell_s^4 N$$

$$\therefore \ \text{Area} = 4\pi^2 \ell_p^4 N \qquad \ell_p^4 = g_s \ell_s^4$$

$$\equiv 2\pi \hbar N$$

where $\hbar = 2\pi \ell_p^4$

$$\therefore \int_{\mathbb{R}^2} \frac{dx_1\, dx_2}{2\pi\hbar} u(x_1, x_2, 0) = N$$

- Also true of arbitrary LLM solutions which are only asymptotically $AdS_5 \times S^5$

$\frac{1}{2}$ BPS states of $\mathcal{N}=4$ SYM $(S^3 \times \mathbb{R})$

Corley, Jevicki, Ramgoolam 0111222
Berenstein 0403110

$$Z = \phi^1 + i\phi^2$$

Operators associated with the $\frac{1}{2}$ BPS states (with $O(4) \times O(4) \times U(1)_{\Delta - J}$) are $\prod_i \left(\mathrm{Tr}\, Z^{n_i}\right)^{r_i}$

Only the lowest KK mode of Z on S^3 is relevant: $Z = Z(t)$

This mode has a harmonic oscillator potential (conf. coupling to S^3)

$\Longrightarrow$ "Matrix quantum mechanics in a harmonic oscillator potential"

(7)

Gauge invariant states of this system are described by N free fermions ~~in~~ ~~a~~ subject to a single-particle hamiltonian

$$h = \frac{\hat{p}^2 + \hat{q}^2}{2} \qquad [\hat{q}, \hat{p}] = i\hbar$$

Semiclassical configurations are described by an incompressible fluid in phase space:

$$u(p,q) = 0 \quad \text{or} \quad 1$$

$$\int u \frac{dp\,dq}{2\pi\hbar} = N$$

Wigner phase space distribution

$$u(p,q,t) = \int d\eta\ \psi^+(q+\tfrac{\eta}{2},t)\,\psi(q-\tfrac{\eta}{2},t)$$
$$\exp[\ \ i\eta p]$$

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Wigner phase space distribution

$$u(p,q,t) = \int d\eta\, \psi^{+}(q + \tfrac{\eta}{2}, t)\, \psi(q - \tfrac{\eta}{2}, t)\, \exp[\,i\eta p\,]$$

Questions:

1. Clearly for AdS/CFT to work

$$u(x_1, x_2)\Big|_{SUGRA} = u(p, q)\Big|_{SYM}$$

But how does the (x_1, x_2) plane metamorphose into a phase space (p, q) ?

2. It is known that $u(p,q)$, being a function on a noncommutative plane is "fuzzy" e.g. $u(p,q) = \delta(p-p_0)\,\delta(q-q_0)$ is disallowed because of uncertainty principle. How

(9')

does such a structure arise in $u(x_1, x_2)$ which is, after all, a supergravity mode?

3. Can one quantitatively arrive at ~~derive~~ the fermion theory starting from the supergravity theory?

Plan:

1. Collective coordinate quantization of $\frac{1}{2}$ BPS supergravity

2. Giant graviton: $\frac{1}{2}$ BPS dynamics

3. D3 brane in arbitrary LLM geometry: $\frac{1}{2}$ BPS dynamics

4. Collective coordinate action $S[u(x_1, x_2$ and measure $\mathcal{D}u[x_1, x_2]$ ($x_1, x_2 \to$ non-commutative)

5. Review phase space density formalism for non-interacting fermions: action $S[u(p,q,t)]$ and measure $\mathcal{D}u[p,q]$

6. $\int \mathcal{D}u[x_1,x_2]\, e^{iS[u(x_1,x_2,t)]/\hbar}$
$= \int \mathcal{D}u[p,q]\, e^{iS[u(p,q,t)]/\hbar}$ as $\hbar \to$

(11)

7. Interesting open questions when $\hbar$ = finite

(W_∞ vs w_∞, fuzzy geometries in supergravity, ...)

droplet splitting

For $c=1$ it took 10 years to resolve the difference in terms of unstable $D0$ branes

Collective coordinate quantization of $\frac{1}{2}$ BPS supergravity

(12

Space of classical solutions coordinatized by the function
$u(x_1, x_2, 0) \equiv u(x_1, x_2)$

$u(x_1, x_2)$ obeys the constraints

ⓐ $u^2 = u$ (equivalent to $u = 0$ or 1)

ⓑ $\int \frac{dx_1 dx_2}{2\pi\hbar} u(x_1, x_2) = N$

It is clear that the space of solutions is preserved by $SDiff^{=W_\infty}$, the group of area preserving diffeomorphisms

$x_1, x_2 \xrightarrow{g} x_1', x_2'$

(12)

Collective coordinate quantization of $\frac{1}{2}$ BPS supergravity

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ⓐ $u^2 = u$ (equivalent to $u = 0$ or 1)

ⓑ $\displaystyle\int \frac{dx_1 dx_2}{2\pi k} u(x_1, x_2) = N$

It is clear that the space of solutions is preserved by $SDiff^{=W_\infty}$, the group of area preserving diffeomorphisms

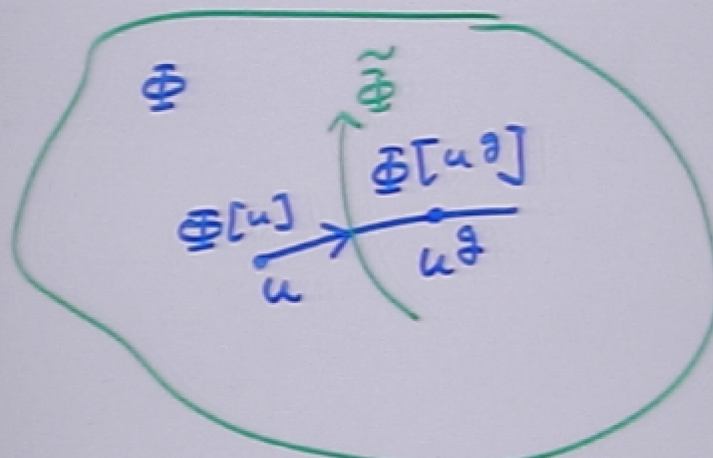
$x_1, x_2 \xrightarrow{g} x_1', x_2'$ $\quad dx_1 \wedge dx_2 = dx_1' \wedge dx_2'$

$$u(x_1,x_2) \longrightarrow u(x_1',x_2') \equiv u^g(x_1,x_2)$$

clearly

$$(u^g)^2 = u^g$$

$$\int \frac{dx_1\,dx_2}{2\pi\hbar}\, u^g(x_1,x_2) = N$$



$$\Phi(t) \longrightarrow \{\underbrace{g(t)}_{\Phi[u^{g(t)}]},\ \tilde{\Phi}(t)\}$$

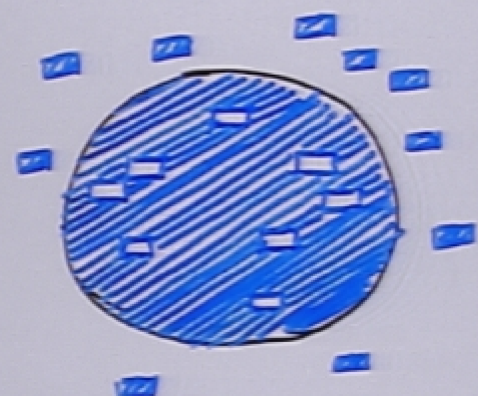
$$\int \mathcal{D}\Phi\, e^{iS[\Phi]/\hbar} \to \int \mathcal{D}g(t)\, e^{iS[g(t)]/\hbar}$$

$$S[g(t)] \longrightarrow \dot{g}^2$$

Impose BPS condition. cf. $F^2(-1)^F$?

We will take a different approach.

Use checkerboard parametrization of $u(x_1, x_2)$



With sufficient number of "particles" and "holes" we can describe all u-configurations.

We will <u>assume</u>, after LLM, any single cell (centre x_1^0, x_2^0) describes a giant graviton, dual giant graviton or in general a D3 brane in an arbitrary LLM

geometry described by the rest of the u-configuration.
The strategy for finding the collective coordinate action for u will be as follows:

(1) Consider $u = u_0 \pm \delta u$

where δu represents a single cell (particle or hole).

(2) Find $S^{(BPS)}_{brane}$ which is the ~~D3~~ action of the BPS D3-brane that corresponds to the little cell δu

(3) Find if $\exists$ an action $S[u]$

s.t. $\boxed{S[u_0 \pm \delta u] - S[u_0] = S^{(BPS)}_{brane}}$ for all $\delta u, u_0$

(16)

2. Giant gravitons

Consider

$$u = u_0 - \delta u$$

This corresponds to a giant graviton in $\underbrace{AdS_5}_{t,\rho,\Omega_3} \times \underbrace{S^5}_{\theta,\tilde{\Phi},\tilde{\Omega}_3}$ embedded as follows:

$t = \tau$

$\tilde{\Omega}_i = \sigma_i$

$\theta = \theta(\tau)$

$\tilde{\Phi} = \tilde{\Phi}(\tau)$

$\rho = 0$

LLM coordinates

$t = \tau$

$\tilde{\Omega}_i = \sigma_i$

$r = r_0(\tau) = r_0 \cos\theta(\tau)$

$\phi = \phi(\tau) = \tilde{\Phi}(\tau) + \tau$

$y = 0$

(4a)

Relation to global coordinates:

global $AdS_5 \times S^5$

$$ds^2 = R^2\Big[-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2 + \cos^2\theta\, d\tilde{\phi}^2 + d\theta^2 + \sin^2\theta\, d\tilde{\Omega}_3^2\Big] \quad (1)$$

LLM:

$$ds^2 = -\frac{y}{\sqrt{u(1-u)}}(dt + V_i dx_i)^2 + \frac{\sqrt{u(1-u)}}{y}(dy^2 + dx_i dx_i) + y\sqrt{\frac{1-u}{u}}\, d\Omega_3^2 + y\sqrt{\frac{u}{1-u}}\, d\tilde{\Omega}_3^2 \quad (2)$$

(1) = (2) if $u(x_1, x_2, 0) = \theta(r_0 - r)$

$r_0 = R^2$

and

$$\left.\begin{aligned} y &= r_0 \sinh\rho \sin\theta \\ r &= r_0 \cosh\rho \cos\theta \end{aligned}\right\} \; r \pm iy = r_0 \cos(\theta \pm i\rho)$$

$$\phi = \tilde{\phi} + t$$

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$\theta = \theta(\tau)$	$r = r_0(\tau) = r_0 \cos\theta(\tau)$
$\tilde{\Phi} = \tilde{\Phi}(\tau)$	$\phi = \phi(\tau) = \tilde{\Phi}(\tau) + \tau$
$\rho = 0$	$y = 0$

(17)

$$S = S_{DBI} + S_{CS}$$

$$= N \int d\tau \left[-\sin^3\theta \sqrt{1 - \cos^2\theta\, \dot{\tilde{\phi}}^2 - \dot{\theta}^2} - \sin^4\theta\, \dot{\tilde{\phi}} \right]$$

BPS Condition :

Das, Jevicki, Mathur 00 08 088

$$\begin{cases} (\hat{\Gamma} - 1)\,\epsilon = 0 \\ \text{where } \epsilon = AdS_5 \times S^5 \text{ Killing spinor} \end{cases}$$

$$\Rightarrow \quad \dot{\tilde{\phi}} = -1, \quad \dot{\theta} = 0$$

$$\Rightarrow \quad p_\theta = 0, \quad p_{\tilde{\phi}} = -N \sin^2\theta$$

(BPS condition could also be obtained by imposing

$$H = -p_{\tilde{\phi}}$$)

(18)

Impose the BPS constraints as Dirac constraints on phase space.

Result:

- 4 dim. phase space $(\theta, p_\theta, \tilde{\Phi}, p_{\tilde{\Phi}})$ gets reduced to a 2 dim. phase space
- ~~And~~ The reduced phase space can be coordinatized by θ, $\tilde{\Phi}$ which have a Dirac bracket

$$\{\theta, \tilde{\Phi}\}_{DB} = \frac{1}{2N \sin\theta \cos\theta}$$

$$\Rightarrow \{\sin^2\theta, \tilde{\Phi}\}_{DB} = \frac{1}{N}$$

In LLM coordinates

$$\{\tfrac{r^2}{2}, \phi\}_{DB} = \frac{r_0^2/2}{N} = 2\pi l_p^4 = \hbar_{LLM}$$

(19)

In terms of path integrals

$$= \int \mathcal{D}\theta\, \mathcal{D}\tilde{\Phi} \exp i S[\theta, \tilde{\Phi}]$$

$$Z_{full} = \int \mathcal{D}p_\theta\, \mathcal{D}\theta\, \mathcal{D}p_{\tilde{\Phi}}\, \mathcal{D}\tilde{\Phi} \exp\left[i \int d\tau \left(\dot{\tilde{\Phi}} p_{\tilde{\Phi}} + \dot{\theta} p_\theta - H \right)\right]$$

$$\Big\downarrow \text{BPS}$$

$$= \int \mathcal{D}[\sin^2\theta]\, \mathcal{D}\tilde{\Phi}\, e^{i \underbrace{\int d\tau \left(-N \sin^2\theta\, \dot{\tilde{\Phi}} - \tilde{H} \right)}_{S_{BPS}[\theta, \tilde{\Phi}]}}$$

$$\tilde{H} = N \sin^2\theta$$

- Note that $S_{BPS}[\theta, \tilde{\Phi}]$ is first order in time derivative, consistent with $\theta, \tilde{\Phi}$ being a phase space.

first order.

Thus the action $S[g]$ should be first-order in time. There is only one such action consistent with the symmetry group G known, which is obtained by Kirillov's coadjoint orbit method.

Dhar, G.M., Wadia
4204028
4207011
4309028

$$S_{BPS}[g] = -\hbar \int dt \langle g^{-1}\dot{g}, u_0 \rangle + \int dt \langle g^{-1} h g, u_0 \rangle$$

Using $g \leftrightarrow u_0^g = u$ (upto isotropy group of u_0) the above action can be rewritten as

$$S_{BPS}[u] = -\int \frac{dx_1\, dx_2}{2\pi\hbar}\, \hbar^2 \int dt\, ds\; u\{\partial_t u, \partial_s u\}/6 + \int \frac{dx_1\, dx_2}{2\pi\hbar}\, u(x_1, x_2, t)\, \frac{x_1^2+x_2^2}{2}$$

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(21)

D3 brane in arbitrary LLM geometry

Consider $u = u_0 - \delta u$ (hole)

$$S_{brane} = \frac{1}{\hbar}\int d\tau \left[-\frac{\sqrt{(1+v_r\dot{r}+v_\phi\dot{\phi})^2 + f(\dot{r}^2+r^2\dot{\phi}^2)}}{f} + r^2\dot{\phi} + \frac{1+v_r\dot{r}+v_\phi\dot{\phi}}{f} \right]$$

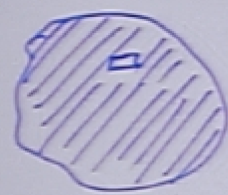
$$= \frac{1}{\hbar}\int d\tau \left[-\frac{\sqrt{(1+v_r\dot{r}+v_\phi(\dot{\tilde{\phi}}+1))^2 + f(\dot{r}^2+r^2(\dot{\tilde{\phi}}+1)^2)}}{f} + r^2(\dot{\tilde{\phi}}+1) + \frac{1+v_r\dot{r}+v_\phi(\dot{\tilde{\phi}}+1)}{f} \right]$$

The BPS conditions can once again be read off from $H = -p_{\tilde{\phi}}$ which amounts to

$$p_r = 0 \qquad p_{\tilde{\phi}} = r^2$$

$$S_{BPS} = \frac{1}{\hbar}\int d\tau \, r^2(\dot{\tilde{\phi}}+1)$$

(21a)



$$u_0 = 1 - y^2 f(x_1, x_2) + O(y^4)$$

$$-g_{tt} = \frac{1}{g_{yy}} \approx \frac{1}{\sqrt{f}} \quad \text{upto } (1+O(y^2))$$

$$g_{\Omega\Omega} \approx y^2\sqrt{f}\,, \quad g_{\tilde\Omega\tilde\Omega} \approx \frac{1}{\sqrt{f}}$$

Metric:

$$ds^2 = \frac{1}{\sqrt{f}}\Big[-(dt + V_i\,dx^i)^2 + f\big(dx_i\,dx^i + \underbrace{dy^2}\big) + d\tilde\Omega_3^2 + f\,\underbrace{y^2\,d\Omega_3^2}\Big]$$

RR-field:

$$C^{(4)} = \tilde B_t\,(dt + V_i\,dx^i) + \hat B$$

$$= -\frac{1}{4}\Big[\frac{1}{f}(dt + V_i\,dx^i) + r^2\,d\phi\Big]$$

$$\tilde B_t = -\frac{1}{4f} \qquad d\hat B = -\frac{1}{4}\,y^3 *_3 d\Big(\frac{u_0}{y^2}\Big)$$

$$= -\frac{1}{4}\,d(x^1\,dx^2 - x^2\,dx^1)$$

D3 brane in arbitrary LLM geometry

(21)

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$$= \frac{1}{\hbar}\int d\tau\left[-\frac{\sqrt{(1+v_r\dot{r}+v_\phi(\dot{\tilde{\phi}}+1))^2+f(\dot{r}^2+r^2(\dot{\tilde{\phi}}+1)^2)}}{f} + r^2(\dot{\tilde{\phi}}+1) + \frac{1+v_r\dot{r}+v_\phi(\dot{\tilde{\phi}}+1)}{f}\right]$$

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$$S_{BPS} = \frac{1}{\hbar}\int d\tau\, r^2(\dot{\tilde{\phi}}+1)$$

It can be shown after a series of manipulations with delta-functions and step-functions that

$$S_{BPS}[u_0 \pm \delta u] - S_{BPS}[u_0]$$

$$= S_{BPS}^{Brane}$$

in all the instances.

$$\delta u = [\ \theta(x_1^0 + \tfrac{\varepsilon}{2} - x_1) \times$$
$$\times \theta(-x_1^0 + \tfrac{\varepsilon}{2} + x_1) \times$$
$$\times \theta(x_2^0 + \tfrac{\varepsilon}{2} - x_2) \times$$
$$\times \theta(-x_2^0 + \tfrac{\varepsilon}{2} + x_2)\ ]$$

x_1^0, x_2^0

area $= \varepsilon^2$

$$\boxed{\frac{\varepsilon^2}{2\pi\hbar} = 1}$$

$$\delta u(t,s) \Rightarrow x_i^0(t,s)$$

$$\int \pm \delta u \{\delta \dot{u}, \delta u'\}_{PB} \pm \int \delta u\, h = S_{BPS}^{Brane}$$

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$$\iint \delta u \, \hbar \, dt = \int \frac{r^2}{2\hbar} \, dt$$

$$\int \delta u \{\delta \dot{u}, \delta u'\}_{PB} = A \int \frac{r^2}{2\hbar} \dot{\Phi} \, dt$$

$$A = \frac{3\hbar}{\pi} \delta_{x_1}(0) \, \delta_{x_2}(0)$$

$$\delta_{x_1}(0) = \delta_{x_2}(0) = \frac{\sqrt{\pi/3}}{\sqrt{\hbar}}$$

However no such subtlety in the equation of motion.

$$\partial_t \delta u - \left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right) \delta u = 0$$

~~or~~ iff

$$\frac{d}{dt} x_1^0 = x_2^0, \quad \frac{d}{dt} x_2^0 = -x_1^0$$

(upto leading order in $\hbar$)

Fermion theory in terms of Wigner function

$$Z_W = \int \mathcal{D}u(p,q)\, \exp\Big[i \int \frac{dp\, dq}{2\pi\hbar}\, \hbar^2 \int d\tau\, ds\; L_{kin}$$

$$- i \int \frac{dp\, dq}{2\pi\hbar} \int d\tau\; u(x_1, x_2, \tau) * h(x_1, x_2) \Big]$$

$$h(x_1, x_2) = \frac{x_1^2 + x_2^2}{2}$$

$$L_{kin} = u(x_1, x_2, \tau, s) * \{ \partial_\tau u, \partial_s u \}_{MB}$$

$$(a * b)(p, q) = e^{i\frac{\hbar}{2}(\partial_q \partial_{p'} - \partial_p \partial_{q'})} a(p,q)\, b(p',q')$$

$p', q' \to p, q$

$$\{a, b\}_{MB} = a * b - b * a$$

$$Z_W = Z_F = \int [\mathcal{D}\psi] \exp i \Big[\int dt\, dx\; \psi^\dagger(x,t) \big(i\partial_t - h(x,\partial) \big) \psi(x,t) \Big]$$

$$h(x, \partial) = \frac{x^2}{2} - \frac{\partial^2}{2}$$

Matching

Clearly

$\mathcal{D}u(x_1, x_2)$ and $S_{BPS}[u(x_1, x_2)]$

obtained from SUGRA reproduces ~~the~~ the $\hbar \to 0$ limits of

$\mathcal{D}u(p,q)$ and $S[u(p,q)]$

obtained from the matrix quantum mechanics.

$\hbar$ = finite

1. The group involved in Z_W or Z_F is W_∞

   It describes $U(p,q)$'s which are fuzzy and can describe splitting of droplets easily.

2. The group obtained in the case of Z_{BPS} in supergravity, viz. w_∞ (= SDiff) cannot describe splitting of droplets.

   Hence SDiff cannot ~~span the~~ cover the entire moduli space of classical $\frac{1}{2}$ BPS supergravity which are ~~different nu~~ various numbers of disconnected components.

3. Can one predict loop/nonperturbative physics (esp. NC) from the exact fermions

Conclusion

- We showed that spacetime coordinates can become noncommutative under BPS conditions.
- Supergravity modes become non-commutative
$$\int \mathcal{D}g = \int \mathcal{D}g_{\text{coord}} \, \mathcal{D}g_{\cancel{\text{phase}}\ \text{momentum}}$$
- The non-commutative action (and measure) agrees with the $\hbar \to 0$ limit of free fermions in a harmonic oscillator potential.
- Collective coordinate method subject to BPS constraint can lead to counting of 5D "black hole" and other configurations
- Near-extremal = approximate NC ? (S_{cs} + S_{ym})
- Lessons from finite $\hbar$

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