Title: Twisted Generalized Calabi-Yau Manifolds and Topological Sigma Models with Flux (Part 2)

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Abstract: In these lectures, we examine how twisted generalized Calabi-Yau (GCY) manifolds arise in the construction of a general class of topological sigma models with non-trivial three-form flux. The topological sigma model defined on a twisted GCY can be regarded as a simultaneous generalization of the more familiar A-model and B-model. Emphasis will be given to the relation between topological observables of the sigma model and a Lie algebroid cohomology intrinsically associated with the twisted GCY. If time permits, we shall also discuss topological D-branes in this more general setting, and explain how the viewpoint from the Lie algebroid helps to elucidate certain subtleties even for the conventional A-branes and B-branes. The lectures will be physically motivated, although I will try to make the presentation self-contained for both mathematicians and physicists.

$(2,2)$ σ-models.

$(g, I_+, I_-, H) \Longleftrightarrow$ TGK Geom.

$\underline{U(1)_A} \Longrightarrow$ G. B-model. $\Longrightarrow G(\ldots) = 0$

$U(1)_V \Longrightarrow$ G. A-model. $\Longrightarrow G(\ldots) = 0$

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$\underline{U(1)_A} \Rightarrow$ G. B-model $\Rightarrow c_1(E_1) = 0$

$U(1)_V \Rightarrow$ G. A-model $\Rightarrow c_1(E_2) = 0$

$(g, I_+, I_-, H) \Longrightarrow$ GK Geom.

- $\underline{U(1)_A}$ $\Rightarrow$ G. B-model $\Rightarrow G(E_1) = 0$
- $U(1)_V$ $\Rightarrow$ G. A-model $\Rightarrow G(E_2) = 0$

Gen. B-model

Before twisting, $Q_\pm$, $\tilde{Q}_\pm$ $\leftarrow$ spin-1/2 fields

After twist : $Q_+ + i\tilde{Q}_+$, $Q_- + i\tilde{Q}_-$: scalars

$U(1)_A$

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- $\underline{U(1)_A} \Rightarrow$ G. B-model $\Rightarrow G(E_1) = 0$

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Gen. B-model

Before twisting, $Q_\pm, \tilde{Q}_\pm$ ← spin-½ fields

After twist, $U(1)_A$: $Q_+ + i\tilde{Q}_+ , \; Q_- + i\tilde{Q}_-$: scalars

$Q_+ - i\tilde{Q}_+ , \; Q_- - i\tilde{Q}_-$: spin-1

$Q_+ - i\tilde{Q}_+ \,,\ Q_+ - i\tilde{Q}_- \quad : \quad$ spin-1

Def. $Q_B = \frac{1}{2}(Q_+ + i\tilde{Q}_+) + \frac{1}{2}(Q_- + i\tilde{Q}_-)\,, \quad Q_B^2 = 0$

$Q_+ - i\tilde{Q}_+ \,,\; Q_+ - i\tilde{Q}_-$: spin

Def. $Q_B = \frac{1}{2}(Q_+ + i\tilde{Q}_+) + \frac{1}{2}(Q_- + i\tilde{Q}_-)$, $Q_B^2 = 0$

$O = \{Q_B, \lambda\}$

$\Rightarrow$ Top. obs. $\Longleftrightarrow$ Q_B-cohomology pre-observables: func. of spinor fields.

spin-0 : ϕ , $\chi = \frac{1}{2}(1+i\Gamma_+)\psi_+$, $\lambda = \frac{1}{2}(1+i\Gamma_-)\psi_-$

$\psi_- \quad , \quad \lambda = \frac{1}{2}(1+iJ_+)\psi_+ \quad , \quad \lambda = \frac{1}{2}(1+iJ_-)$

$$\psi = \frac{1}{\sqrt{2}}(\psi_+ + i\psi_-) \qquad \rho = \frac{1}{\sqrt{2}}\, g(\psi_+ - i\psi_-)$$

$$\psi \in \Gamma(TM) \qquad \rho \in \Gamma(TM^*),$$

$$\underline{\Psi} = \begin{pmatrix} \psi \\ \rho \end{pmatrix} \in \Gamma(TM \oplus TM^*).$$

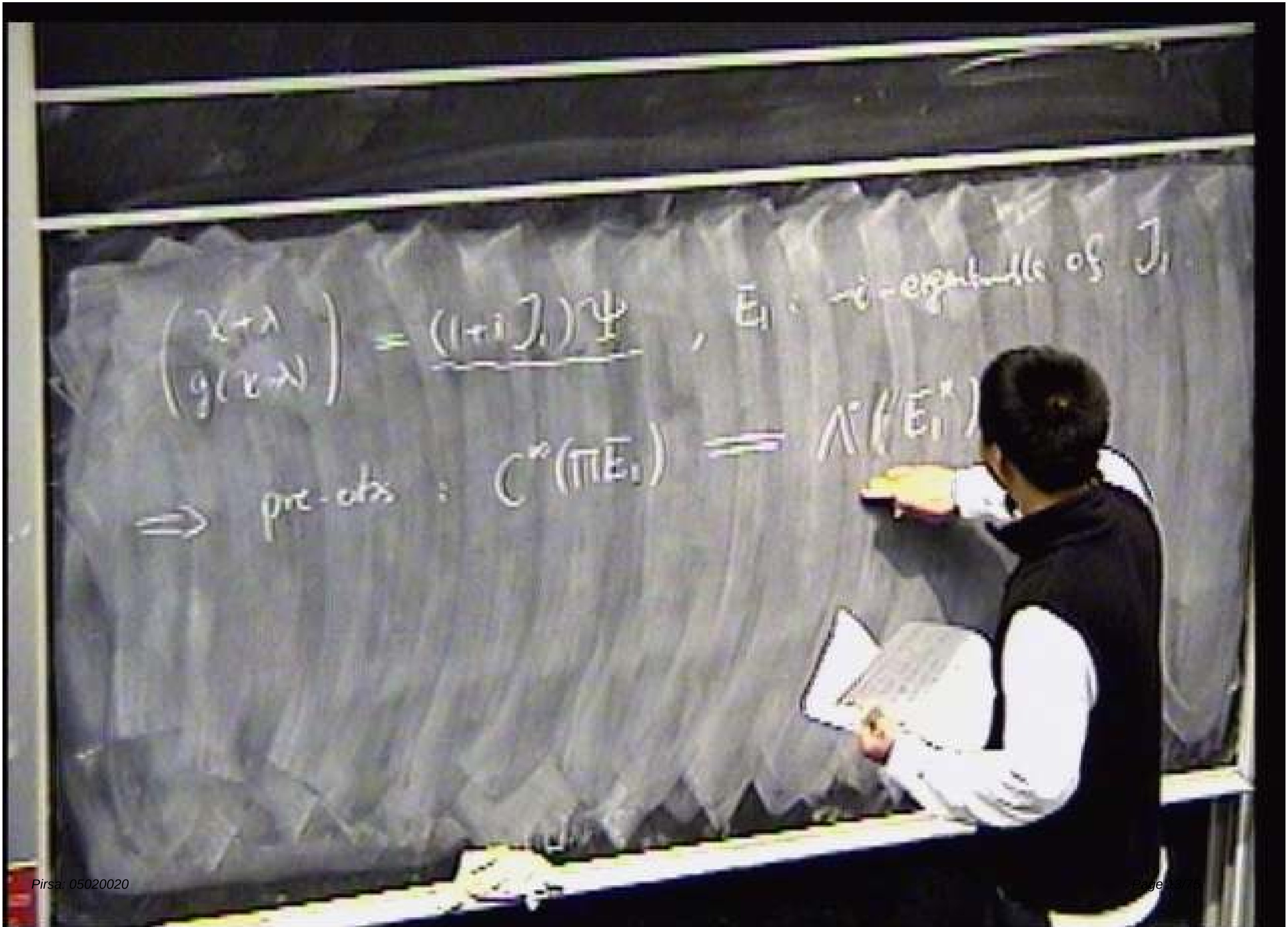
$$\begin{pmatrix} \chi+\lambda \\ g(\chi-\lambda) \end{pmatrix} = (1 + i J_1)\Psi$$

$$\begin{pmatrix} x+\lambda \\ g(x-\lambda) \end{pmatrix} = \underline{(1+iJ_1)\Psi} \quad , \quad E_1 : -i\text{-eigenbundle of } J_1$$

$$\Longrightarrow \quad \text{pre-obs} \; : \; C^\infty(E_1)$$

$$\begin{pmatrix} x+\lambda \\ g(x-\lambda) \end{pmatrix} = \underline{(1+iJ_1)\psi}, \quad \bar{E}_1 : -i\text{-eigenbundle of } J_1.$$

$$\Longrightarrow \text{pre-obs} : C^\infty(\Pi E_1)$$



(x+λ, g(x+λ)) = ((1+iJ_1)Ψ , E_1 : -i-eigenbundle of J_1
⟹ pre-obs : C^∞(ΠE_1) ⟶ Λ^•(E_1^*)

$) = \underline{(1+iJ_1)\Psi}$, E_1 : $-i$-eigenb...

-obs : $C^\infty(T\!\!\cdot E_1) = \underline{\cancel{\Gamma}(\Lambda^\bullet E_1^*)}$

$$\left(\begin{array}{c} \cdots + \lambda \\ (\xi - \lambda) \end{array} \right) = \underline{(1 + i\,\mathcal{J}_1)\Psi} \;, \quad E_1 : -i\text{-eigenbundle o}$$

pre-obs : $C^\infty(\Pi E_1) = \underline{\Gamma(\Lambda^\bullet E_1^*)}$

E_1 : Lie-algebroid $\Longrightarrow$ $(\Lambda^\bullet E_1^*, d_{E_1})$

E_1 : Lie-algebroid $\Longrightarrow$ (

Q_B : odd homological vector field on ΠE_1

$\Longrightarrow$ pre-... : (...

E_1 : Lie-algebroid $\Longrightarrow$ $(\Lambda^\bullet E_1^*, d_{E_1})$

Q_{E_1} : odd homological vector field on ΠE_1

$Q_{E_1}^2 = 0$

$$C^\infty(\Pi E_1) = \Gamma(\Lambda^\bullet E_1^*)$$

$$(\Lambda^\bullet E_1^*, d_{E_1})$$

Lie-algebroid $\Longrightarrow$

odd homological vector field on ΠE_1 $\Longleftrightarrow$ Q_b differential on $\Lambda^\bullet E_1^*$

$Q_b^2 = 0$

Claim: $\mathcal{D}_{B} = d_{E_1}$

Claim: $\mathcal{O}_B = d_{E_1}$

TFT: Ops $\overset{\text{iso}}{\longleftrightarrow}$ States

Claim: $\mathcal{O}_B = \mathcal{L}_{E_1}$

TFT: Ops $\xleftrightarrow{\text{iso}}$ states

$\uparrow$ twist $\qquad \uparrow \qquad \uparrow$

N=2 SCFT chiral ring R-R ground states

Claim: $\mathcal{O}_B = \mathcal{A}_{E_1}$

$$\begin{array}{lccc}
\text{TFT:} & \text{Ops} & \xleftrightarrow{\text{iso}} & \text{States} \\
\uparrow \text{twist} & \uparrow & & \uparrow \\
N=2\ \text{SCFT} & \text{chiral ring} & \rightleftarrows & \text{R-R ground states}
\end{array}$$

Claim: $\Omega_{B} = d_{E_1}$

TFT: Ops $\overset{\text{iso}}{\longleftrightarrow}$ States

$\uparrow$ twist

$\mathcal{N}=2$ SCFT chiral ring $\underset{\longleftarrow}{\overset{\text{spec flow}}{\longrightarrow}}$ R-R ground states

Claim: $\mathcal{O}_{B} = d_{B1}$

$$
\begin{tikzcd}
\text{TFT:} & \text{Ops} \arrow[r, leftrightarrow, "\text{iso}"] & \text{States} \\
\mathcal{N}=2\ \text{SCFT} \arrow[u, "\text{twist}"'] & \text{chiral ring} \arrow[u] \arrow[r, shift left, "\text{spec. flow}"] & (\text{R-R ground states}) \arrow[u] \arrow[l, shift left] \\
& & (\Omega^{\bullet}(M), \bar{\partial}_H) \arrow[u, "\text{will show}"']
\end{tikzcd}
$$

Claim: $Q_B = d_{E_1}$

$$\begin{array}{ccccc}
\text{TFT:} & & \text{Ops} & \xleftrightarrow{\text{iso}} & \text{States} \\
\uparrow \text{twist} & & \uparrow & & \uparrow \\
N=2 \text{ SCFT} & & \text{chiral ring} & \overset{\text{spec flow}}{\rightleftarrows} & (\text{R-R ground states}) \\
 & & & & \uparrow \text{will show} \\
(\Lambda^\bullet E_1^*, d_{E_1}) & & & & (\Omega^\bullet(M), \bar\partial_H)
\end{array}$$

Claim: $Q_B = d_{E_1}$

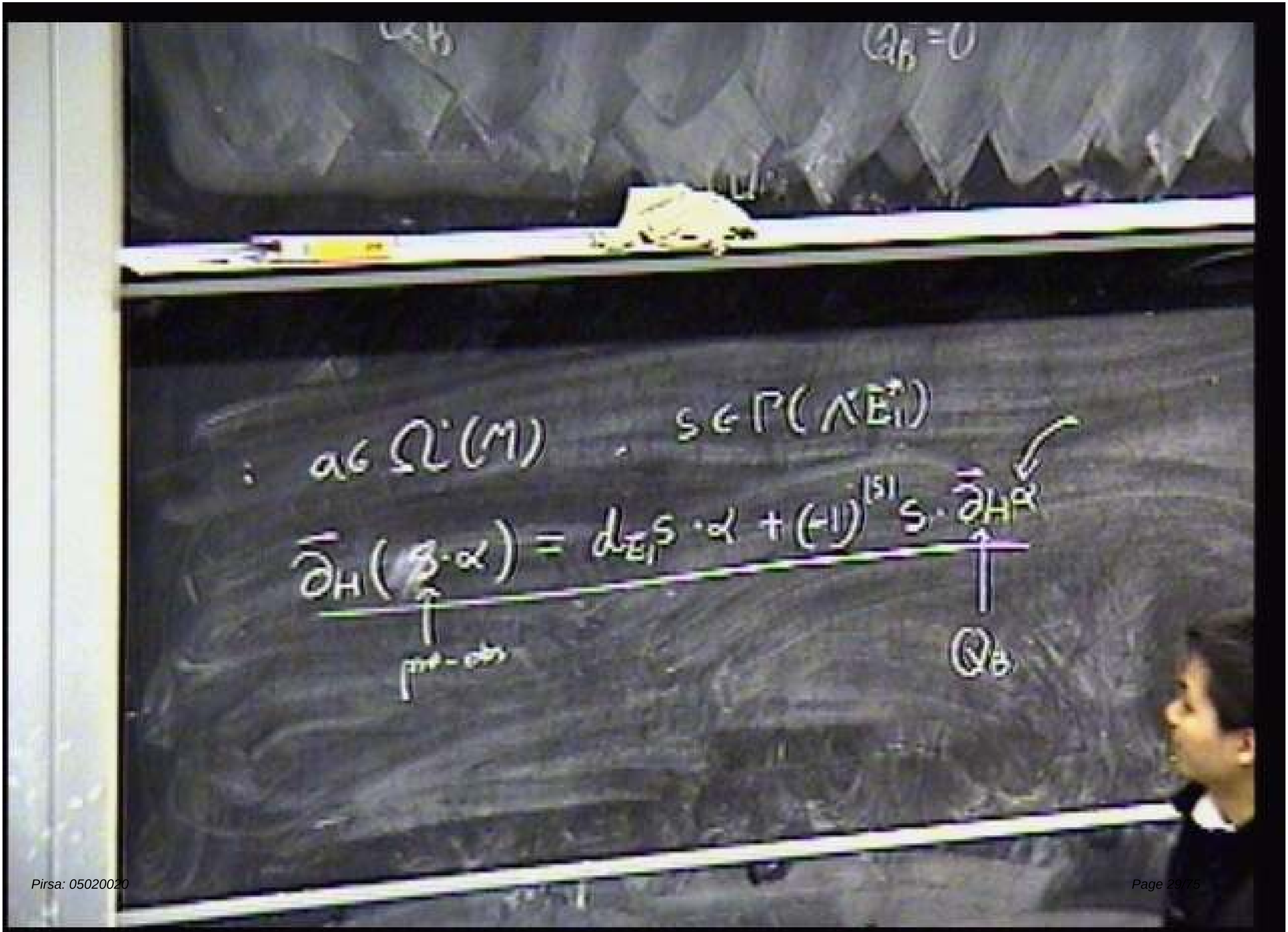
$$
\begin{tikzcd}
\text{TFT}: & \text{Ops} \arrow[rr, leftrightarrow, "\text{iso}"] & & \text{States} \\
\mathcal{N}=2\ \text{SCFT} \arrow[u, "\text{twist}"] & \text{chiral ring} \arrow[u] \arrow[rr, shift left, "\text{spec. flow}"] & & (\text{R-R ground states}) \arrow[u] \arrow[ll, shift left] \\
(\Lambda^{\bullet} E_1^{*}, d_{E_1}) \arrow[rrr, bend left, "\text{DG-module}"'] & & & (\Omega^{\bullet}(M), \bar{\partial}_H) \arrow[u, "\text{will show}"]
\end{tikzcd}
$$

$\alpha \in \Omega^{\cdot}(M)$, $s \in \Gamma(\Lambda^{\cdot} E_1^{*})$

$$\bar{\partial}_H (s \cdot \alpha) = d_{E_1} s \cdot \alpha + (-1)^{|s|} s \cdot \bar{\partial}_H$$

$a \in \Omega^{\cdot}(M)$, $s \in \Gamma(\Lambda^{\cdot}\vec{E}_1)$

$$\bar{\partial}_H(s \cdot \alpha) = d_{E_1}s \cdot \alpha + (-1)^{|s|}\, s \cdot \bar{\partial}_H \alpha$$


Q_h
$Q_h = 0$
$\alpha \in \Omega^{\cdot}(M)$, $s \in \Gamma(\wedge^{\cdot} E_1)$
$\bar{\partial}_H(s \cdot \alpha) = d_{E_1} s \cdot \alpha + (-1)^{|s|} s \cdot \bar{\partial}_H \alpha$
Q_0

$Q_B = 0$

$\alpha \in \Omega^{\cdot}(M)$, $s \in \Gamma(\Lambda^{\cdot} E_1^{*})$

$$\bar{\partial}_H(s \cdot \alpha) = d_{E_1} s \cdot \alpha + (-1)^{|s|} s \cdot \bar{\partial}_H \alpha$$

Q_B pnt-chrg Q_B

$$\Rightarrow \{Q_B, s\} = d_{E_1} s$$

Q_B


R-R ground state
restricting to const modes.

R-R ground state

x^+, x_L^+

restricting to const modes

$$\{\psi_+, \psi_+\} = g'' = \{\psi_-, \psi_-\}, \quad \{\psi_+, \psi_-\} = 0$$

R-R ground state : Restricting to constant modes.

x^0, ξ^0

$$\{\psi_+, \psi_+\} = g' = \{\psi_-, \psi_-\} \;,\; \{\psi_+, \psi_-\} = 0$$

$$\{\psi, \psi\} = \{\rho, \rho\} = 0 \;,\; \{\psi^a, \rho_b\} = \delta^a_b$$

State..

$$\{\psi_+, \psi_+\} = g^{-1} = \{\psi_-, \psi_-\} \quad , \quad \{\psi_+, \psi_-\} = 0$$

$$\{\psi, \psi\} = \{\rho, \rho\} = 0 \quad , \quad \{\psi^a, \rho_b\} = \delta^a_b$$

States: $\Omega^\bullet(M)$

$\psi^a \cdot dx^a, \quad \rho_a = \iota_{\partial_a}$

Q

$$Q = Q_+ + iQ_- \,, \quad Q^\dagger = Q_+ - iQ_- .$$

$$Q^2 = 0, \quad Q^{\dagger 2} = 0, \quad \{Q, Q^\dagger\} \propto \mathcal{H}$$

$$Q = Q_+ + iQ_- \,, \quad Q^* = Q_+ - iQ_-$$

$$Q^2 = 0, \quad Q^{*2} = 0, \quad \{Q, Q^*\} \propto \mathcal{H}$$

$$Q \longleftrightarrow d_H$$

$$Q = Q_+ + iQ_- \,, \quad Q^* = Q_+ - iQ_- .$$

$$Q^2 = 0, \quad Q^{*2} = 0, \quad \{Q, Q^*\} \propto \mathcal{H}$$

$$Q \Longleftrightarrow d_H = d - H_\wedge .$$

$\ldots Q \ldots = 0, \quad \{Q, Q^* \ldots$

$\Longleftrightarrow \quad d_H = d - H\wedge.$

R-R ground states $\xleftrightarrow{1:1}$ Q-coh.

$\{Q, Q^*\} \propto \mathcal{H}$

$\Rightarrow$ Q-coh. $=$ d_M-coh.

$$Q = Q_+ + iQ_- \,, \quad Q^\dagger = Q_+ - iQ_-.$$

$$Q^2 = 0, \quad Q^{\dagger 2} = 0, \quad \underline{\{Q, Q^\dagger\} \propto \mathcal{H}}.$$

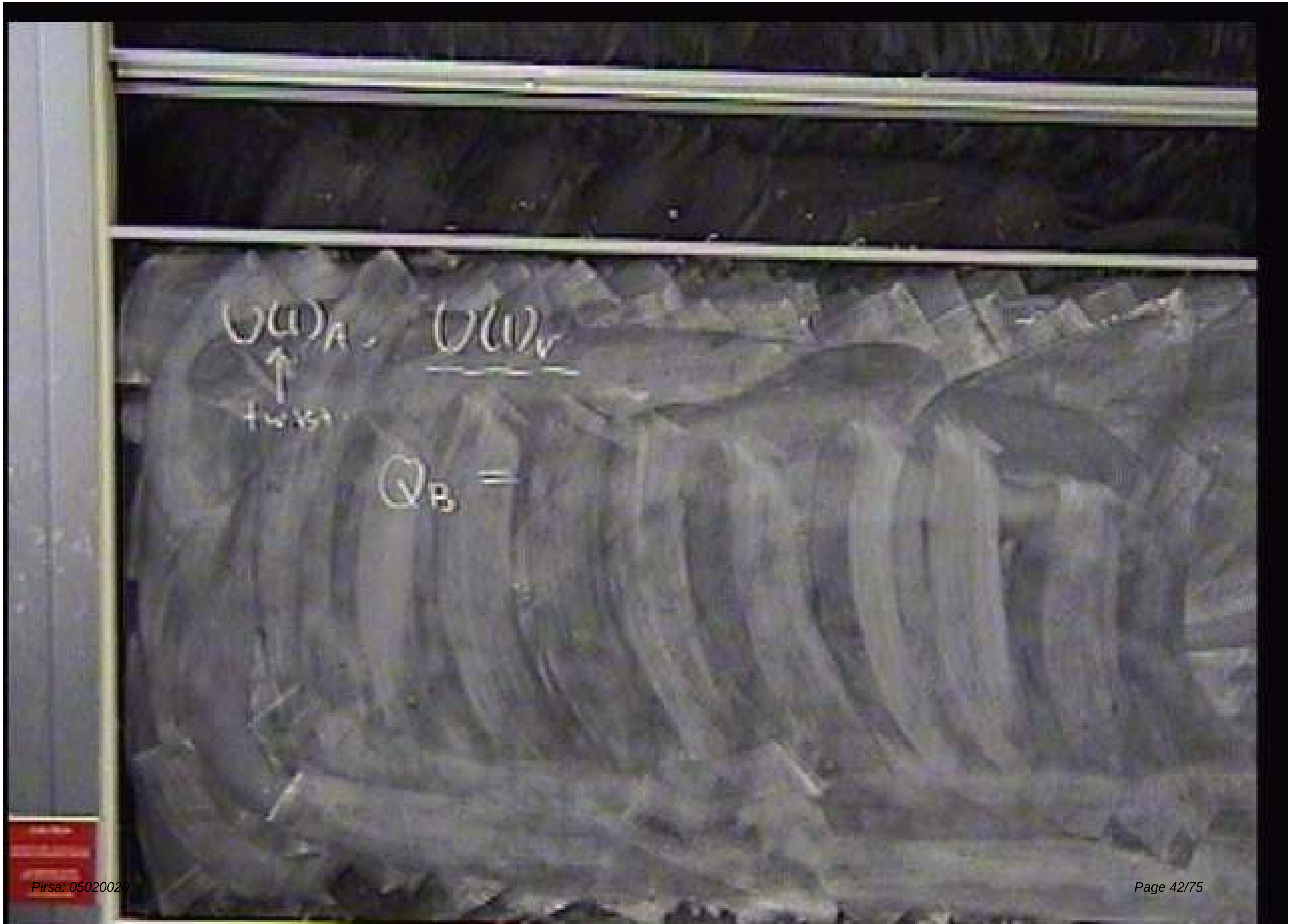
$$Q \longleftrightarrow d_H = d - H\wedge$$

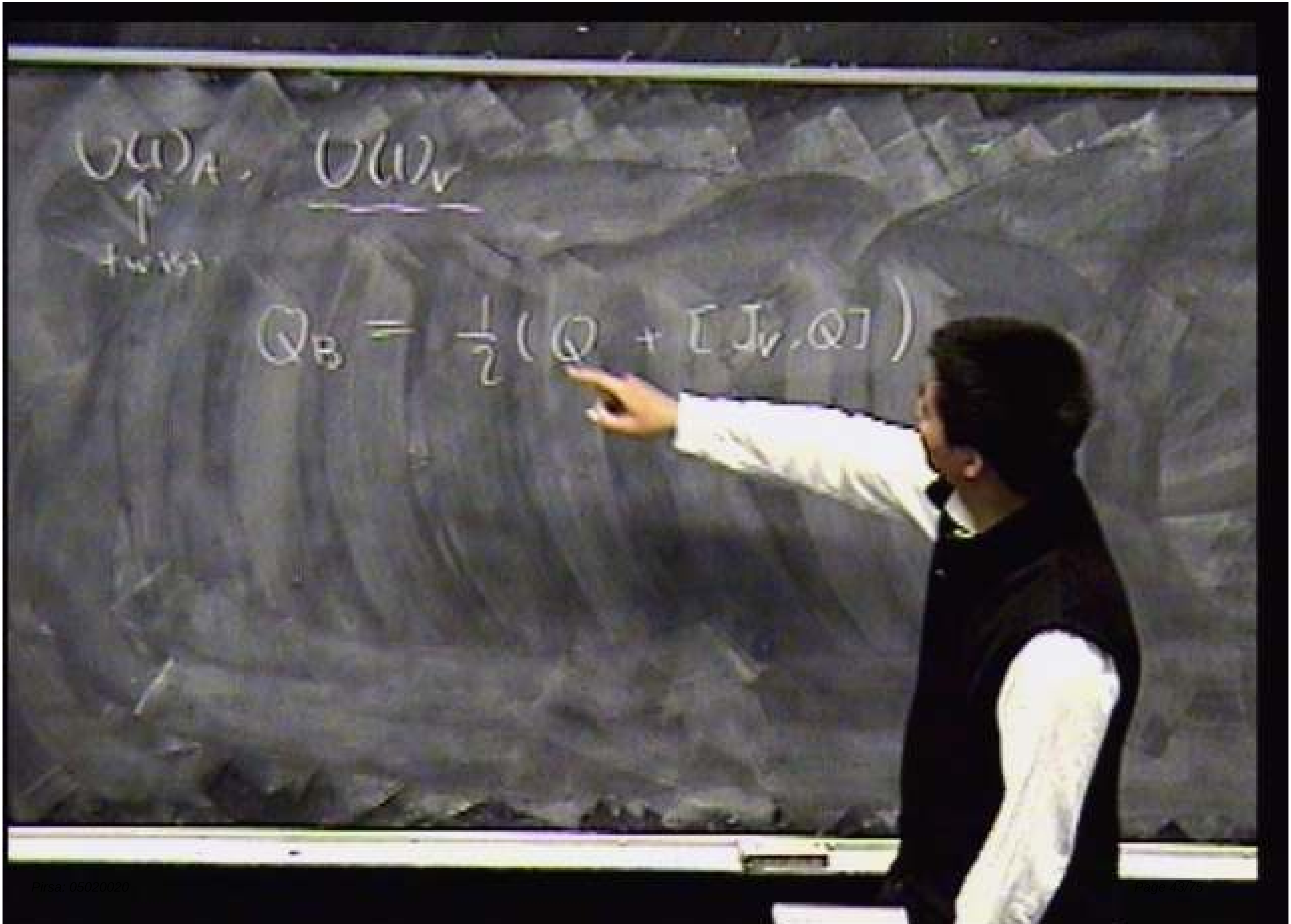
R-R ground states $\xleftrightarrow{1:1}$ Q-coh. = d_H-coh.

Q_3 ?

$U(1)_A$, $\underline{U(1)_V}$

twist.


U(1)_A , U(1)_V
twist
Q_B =


U(1)_A , U(1)_V
twist
Q_B = 1/2 (Q + [J_V, Q])

$$Q_B = \frac{1}{2}\left(Q + [J_V, Q]\right)$$

$\underline{U(1)_V}$

$$Q_B = \frac{1}{2}\left(Q + [J_V, Q]\right)$$

By Noether

$$J_V = -\frac{i}{2}\left(\omega_+(\psi_+, \psi_+) + \omega_-(\psi_-, \psi_-)\right)$$

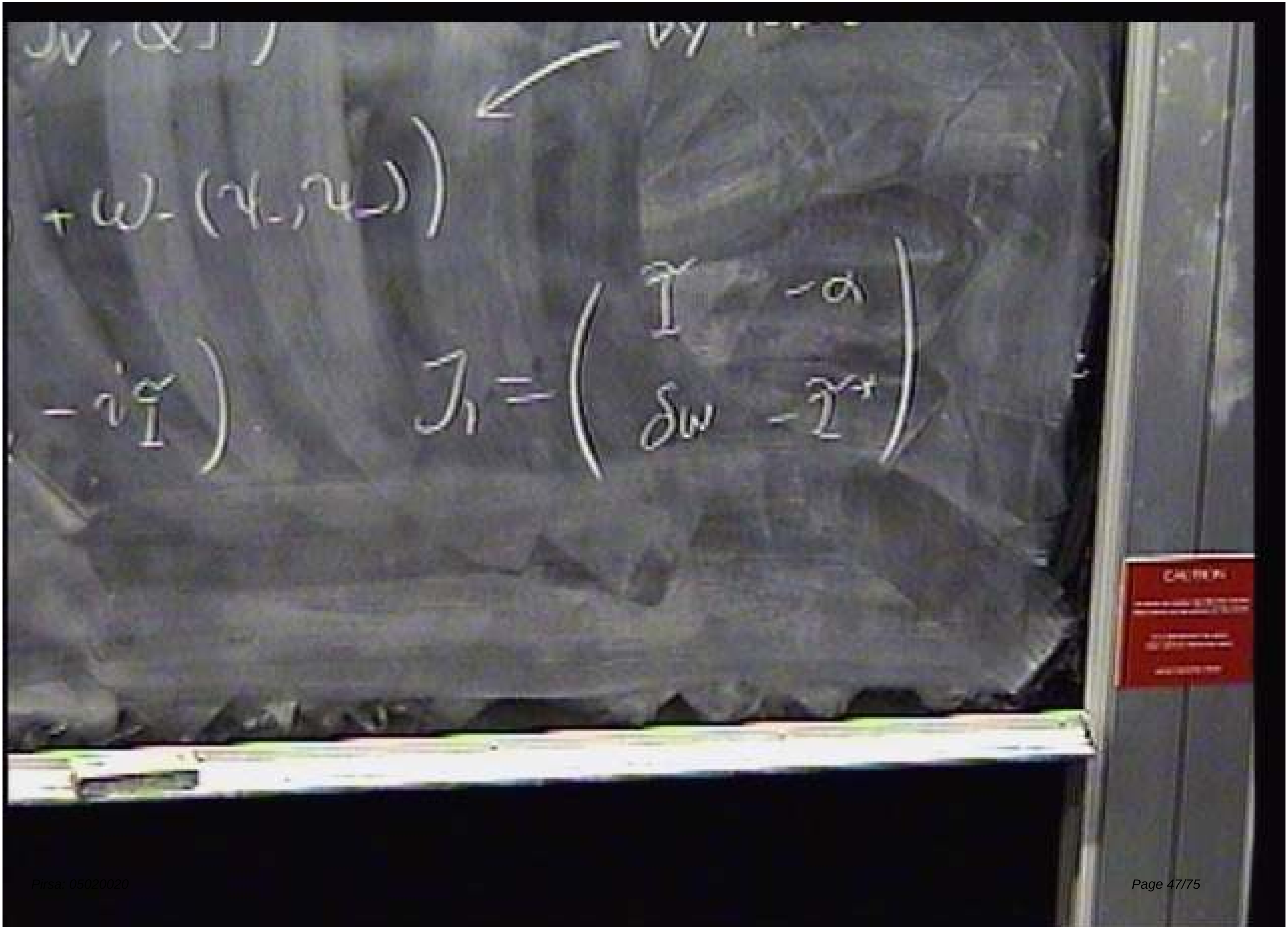
$$Q_B = \frac{1}{2}\left(Q + [J_V, Q]\right)$$

B,

$$J_V = -\frac{i}{2}\left(\omega_+(\psi_+, \psi_+) + \omega_-(\psi_-, \psi_-)\right)$$

$$\| $$

$$-i\left(\delta\omega \wedge - i_\alpha - i\tilde{I}\right)$$


$+\omega_-(\psi_-,\psi_-))$
$-i\tilde{I})$
$J_1 = \begin{pmatrix} \tilde{I} & -\alpha \\ \delta\omega & -\tilde{I}^t \end{pmatrix}$

$\frac{1}{2}\left(Q + [J_{v}, Q] \right)$

By Noether

$\frac{i}{2}\left(\omega_+(\psi_+, \psi_+) + \omega_-(\psi_-, \psi_-) \right)$

$\|$

$-i\left(\delta\omega \wedge - i_{\alpha} - i_{\tilde{I}} \right)$

$$\mathcal{J}_1 = \begin{pmatrix} \tilde{I} & -\alpha \\ \delta\omega & -\tilde{I}^{*} \end{pmatrix}$$

$i_{\tilde{I}} = \tilde{I}^{a}{}_{b}\, dx^{b}\, i_{\partial_a}$

$\Omega^{\cdot}(M), \ \bar{\partial}_H \qquad U_0$ ann. by $\overline{E}_1$

$$\Omega^{\cdot}(M) \simeq U_0 \oplus U_1 \oplus \cdots \oplus U_n$$

$$U_k = \Lambda^k \overline{E}_1 \, U_0$$

$$\Omega^{\cdot}(M) \cong V_0 \oplus V_1 \oplus \cdots \oplus V_n$$

$$V_k = \Lambda^k \bar{E}_1 \cdot V_0$$

$$\bar{\partial}_H : \pi_{k+1} d_H : \Gamma(V_k) \to \Gamma(V_{k+1})$$

$\Omega^{\cdot}(M),\ \bar{\partial}_H$ .. U_0 ann. by $\bar{E}_1$

$$\Omega^{\cdot}(M) \simeq U_0 \oplus U_1 \oplus \cdots \oplus U_n$$

$$U_k = \Lambda^k \bar{E}_1 \cdot U_0$$

$$\bar{\partial}_H : \pi_{k+1}\, d_H : \Gamma(U_k) \to \Gamma(U_{k+1}) \quad , \ \partial_H .$$

J_1 is integ. $\Longleftrightarrow$ $d_H = \bar{\partial}_H + \partial_H$.

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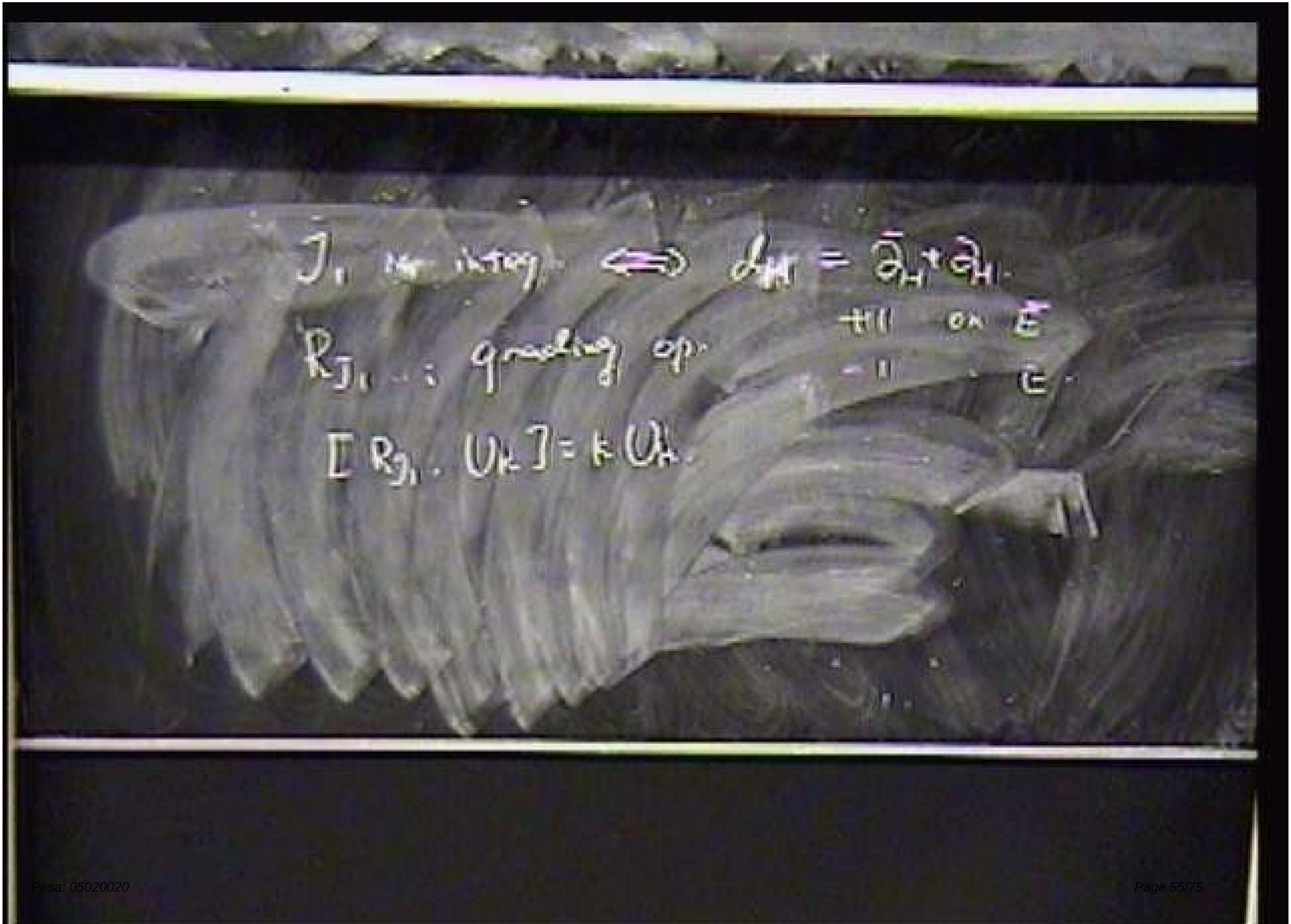
R_{J_1} : grading op.

J_1 is integ. $\Longleftrightarrow$ $d_H = \bar{\partial}_H + \partial_H$

R_{J_1} : grading op.

$[R_{J_1}, U_k] = k\, U_k$


J₁ is integr. ⟺ d_H = ∂̄_H + ∂_H
+1 on E
R_{J₁} : grading op.
−1 on Ē
[R_{J₁}, O_k] = k O_k

$\mathcal{J}_1$ is integ. $\Longleftrightarrow$ $d_H = \bar{\partial}_H + \partial_H$.

$R_{\mathcal{J}_1}$: grading op. $\quad +i$ on $\bar{E}$
$\qquad\qquad\qquad\qquad\qquad\quad -i$ on E.

$$[R_{\mathcal{J}_1}, U_k] = k\, U_k$$

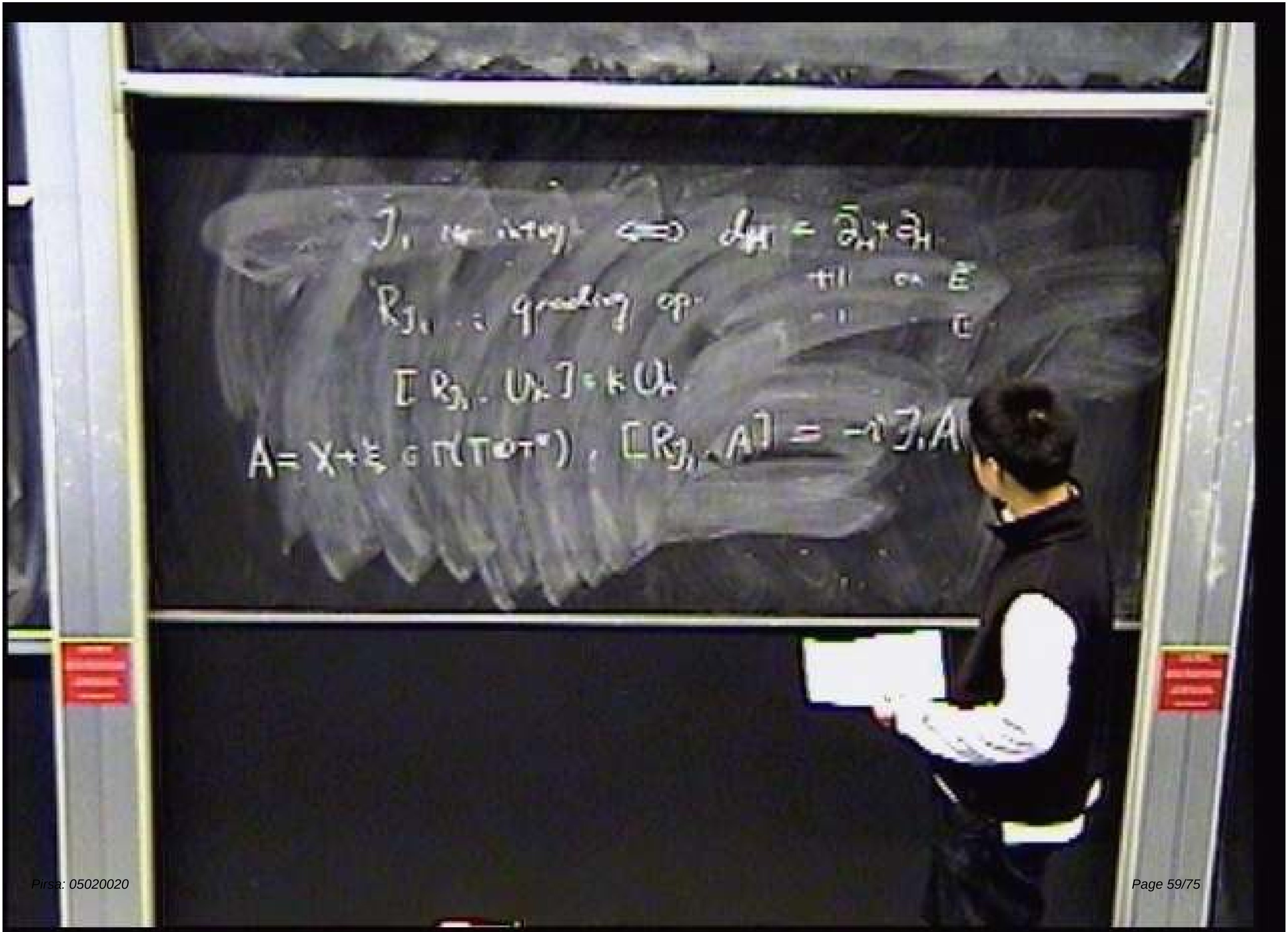
$$A = X + \xi \in \Gamma(T \oplus T^*)$$

$$[R_j, U_k] = k\, U_k$$

$$A = X + \xi \in \Gamma(T \oplus T^*) \,, \quad [R_j, A]$$

$$[R_{J_1}, U_k] = k\, U_k$$

$$A = X + \xi \in \Gamma(T \oplus T^*) \,, \quad [R_{J_1}, A] = -i J_1 A$$


J₁ integrable ⟺ d_H = ∂̄_H + ∂_H
R_{J₁} : grading op.
[R_{J₁}, U_k] = k U_k
A = X + ξ ∈ Γ(T⊕T*) , [R_{J₁}, A] = −i J₁.A

$$[R_\xi, A] = -i\,[J, A]$$

$$= i\,\xi^x + L_{\alpha(\xi)} - L_x g_\omega - I_\xi^\omega \Lambda$$

$$A] = -i \, J_1 A$$

$$= i_{\xi} X + L_{\alpha(\xi)} - L_X \delta\omega - I^{*}_{\xi} \wedge$$

$$R_{J_1} = -i\left(\delta\omega\lambda - (\omega - \omega_{\frac{\gamma}{1}} \quad)\right)$$

$$R_{J_1} = -i\left(\delta\omega_{r} - (L - L_{I}\;\;)\right)$$

$$\Rightarrow \quad R_{J_1} \Longleftrightarrow J_V$$

$$\Rightarrow R_{J_1} \Leftrightarrow J_V$$

$$Q_B = \frac{1}{2}\left(Q + [J_V, Q]\right)$$

$$= \frac{1}{2}\left(d_H\right.$$

Q

$$R_{J_1} \Leftrightarrow J_V$$

$$Q_B = \frac{1}{2}\left(Q + [J_V, Q]\right)$$
$$= \frac{1}{2}\left(d_H + [R_{J_1}, d_H]\right)$$
$$= \bar{\partial}_H$$

Q


$Q_B = d_{E_1}$
A:

$\Omega^{\cdot}(M),\ \bar{\partial}_H$ $\underline{V_0}$ gen. by E_1 $\Omega \in \Gamma(V_0)$

$d_H \Omega$

$\Omega^{\cdot}(M) \simeq V_0 \oplus V_1 \oplus \cdots \oplus V_n$

$V_k = \Lambda^k E_1 V_0$

$\bar{\partial}_H = \pi_{k+1} d_H : \Gamma(V_k) \to \Gamma(V_{k+1})$, ∂_H.

$\Omega^{\cdot}(M)$, $\bar{\partial}_H$

U_0 ann by E_1

$\Omega \in \Gamma(U_0)$

$d_H \Omega = \bar{\partial}_H \Omega$

$\Omega^{\cdot}(M) \simeq U_0 \oplus U_1 \oplus \cdots \oplus U_n$

$U_k = \Lambda^k \bar{E}_1 \cdot U_0$

$\bar{\partial}_H = \pi_{k+1} \circ d_H : \Gamma(U_k) \to \Gamma(U_{k+1})$, ∂_H.

$\Omega^{\cdot}(M)$, $\bar{\partial}_H$ $\underline{V_0}$ gen. by E_1 $\Omega \in \Gamma(V_0)$

$$d_H \Omega = \bar{\partial}_H \Omega = 0$$

$$\Omega^{\cdot}(M) \simeq V_0 \oplus V_1 \oplus \cdots \oplus V_n$$

$$V_k = \Lambda^k E_1 \cdot V_0$$

$$\bar{\partial}_H : \pi_{k+1} d_H : \Gamma(V_k) \to \Gamma(V_{k+1}) \quad , \partial_H$$

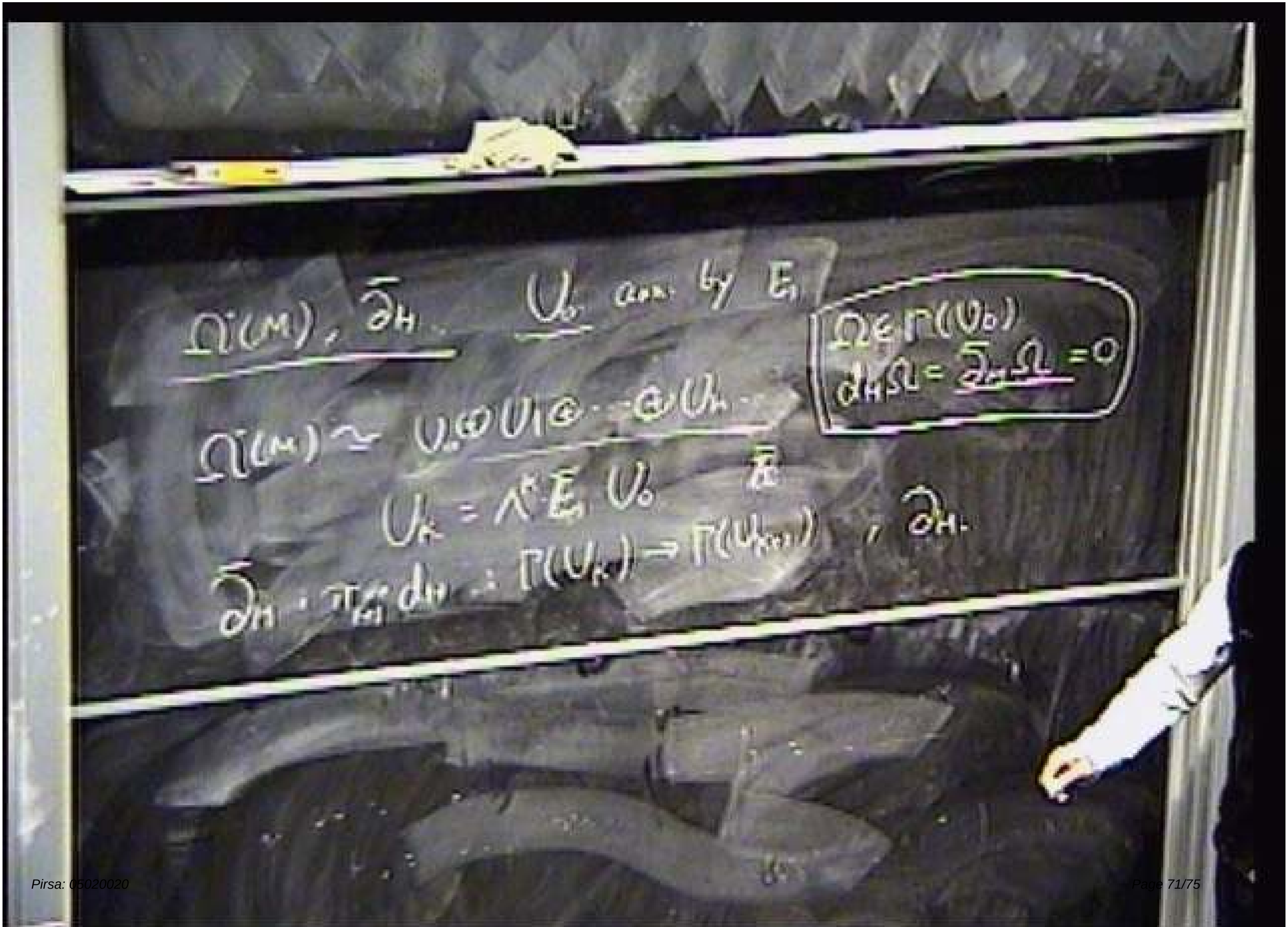
$\Omega^{\cdot}(M)$, $\bar{\partial}_H$ .. $\underline{V_0}$ gen. by E_1

$$\boxed{\begin{array}{l} \Omega \in \Gamma(V_0) \\ d_H \Omega = \underline{\bar{\partial}_H \Omega} = 0 \end{array}}$$

$\Omega^{\cdot}(M) \simeq V_0 \oplus V_1 \oplus \cdots \oplus V_n$

$V_k = \Lambda^k E_1 V_0$

$\bar{\partial}_H = \pi_{k+1} d_H : \Gamma(V_k) \to \Gamma(V_{k+1})$ // ∂_H


$\Omega(M)$, $\bar\partial_H$ .. U_0 gen by E_1
$\Omega \in \Gamma(U_0)$
$d_H \Omega = \bar\partial_H \Omega = 0$
$\Omega(M) \simeq U_0 \oplus U_1 \oplus \cdots \oplus U_n$ ..
$U_k = \Lambda^k E_1 \, U_0$ $\bar{E}$
$\bar\partial_H : \pi_{k+1} d_H : \Gamma(U_k) \to \Gamma(U_{k+1})$, ∂_H

$), \bar{\partial}_H$ .. $\underline{U_0}$ ann. by E_1

$\Omega \in \Gamma(U_0)$
$d_H \Omega = \underline{\bar{\partial}_H}$

$) \simeq U_0 \oplus U_1 \oplus \cdots \oplus U_n$

$U_k = \Lambda^k \bar{E}_1 U_0$ E

$\Gamma(U_k) \to \Gamma(U_{k+1})$, ∂_H.

$\pi_{k+1} d_H$

$U(1)_A$, $\underline{U(1)_V}$

$\uparrow$

twist

$c_1(E_1) = 0$

$$Q_B = \frac{1}{2}\left(Q + [J_V, Q]\right)$$

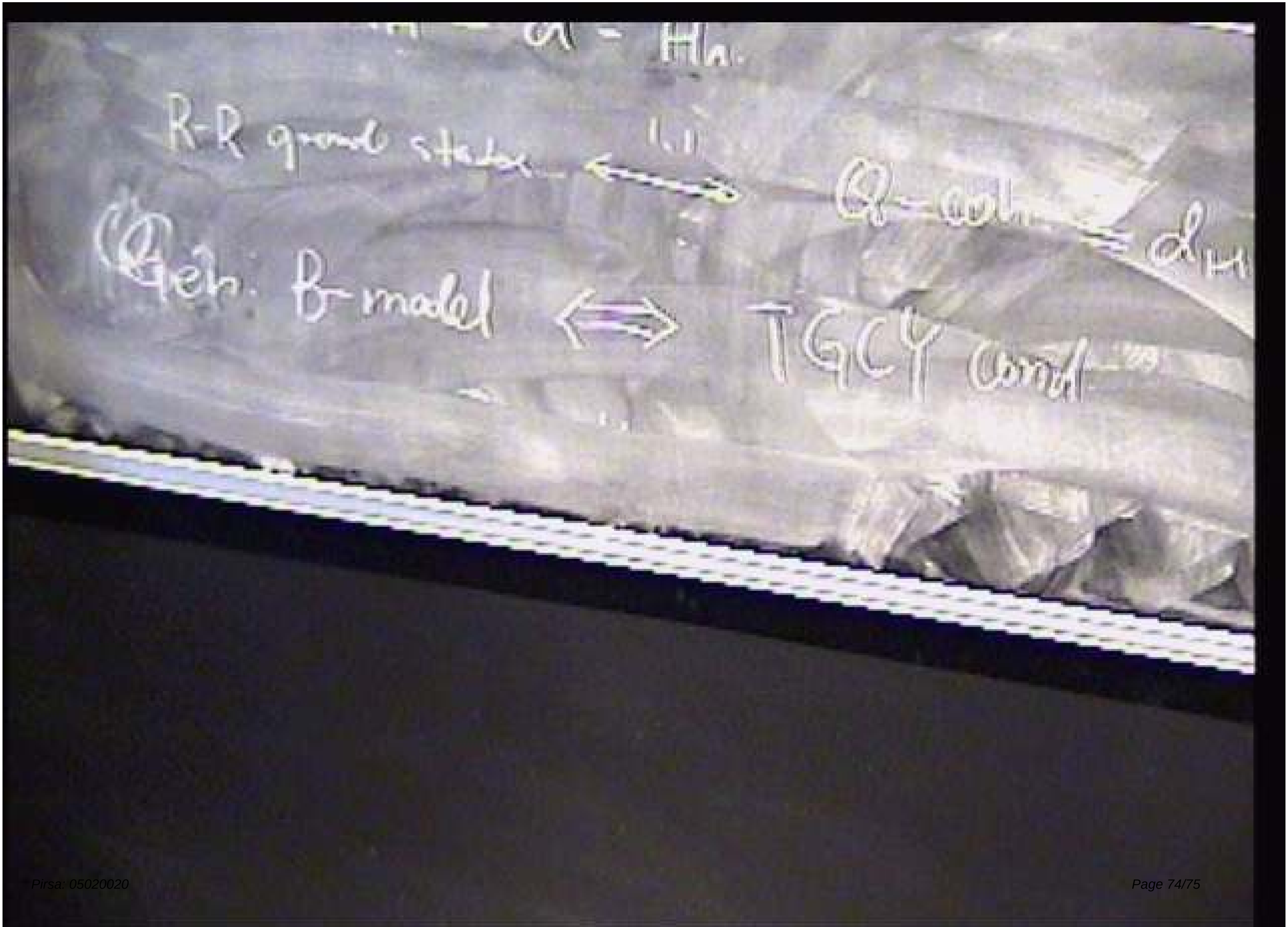
by Noether

$$J_V = -\frac{1}{2}\left(\omega_+(\nu_+, \nu_+) + \omega_-(\nu_-, \nu_-)\right)$$

$$\| $$

$$-i\left(\delta\omega \wedge - i_\alpha - i_I\right) \qquad J_1 = \begin{pmatrix} I & -\alpha \\ \delta\omega & -I^* \end{pmatrix}$$

$$i_I = I^a{}_b \, dz^b \, i_{a}$$


R-R ground states
B-model
TGCY cond