Title: Generalized Geometric Structures (Part 2)

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Abstract:

H odd degree closed form

curvature of a $(o-3)$-gerbe

close

ature of a $(n-3)$-gerbe

$$f^* H^\bullet = dF^\bullet$$

odd

even

$$d + H \wedge \; : C^\infty(\Lambda^{ev} T^*)$$

$$T^*) \rightleftarrows C^\infty(\Lambda^{0,d} T^*)$$

$d_H = d + H \wedge \; : C^\infty(\Lambda^{ev} T^*) \rightleftarrows$

$\Rightarrow H^{\bullet}_{d_H}(N)$ twisted coh.

$\rho \in \Omega^{\bullet}(N)$ st

$\Lambda^{ev} T^*) \rightleftarrows C^\infty(\Lambda^{od} T^*)$

ed coh.

$)$ st $d_H \rho = 0.$

$\Omega^{\cdot}(N)$ st c

$d(e^{F^{\cdot}} f^{*} \rho) = dF^{\cdot}$

$+$

st $\quad d_H \rho = 0$

$$) = \quad dF \wedge e^F f^* \rho$$

$$+ e^F f^* \left(-H|_M \wedge \rho\right)$$

$$-dF \wedge \rho$$

$$d\left(e^{F} f^{*} \rho\right) = \quad dF$$

$$+$$

$$e^{F} f^{*} : H^{\cdot}_{H}(N) \longrightarrow H^{\cdot}(M)$$

$$H^{\bullet}(M) \longrightarrow H_H^{\bullet + \mathrm{codim}(M)}(N$$

$$
\begin{tikzcd}
d_H : \mathcal{S} \arrow[r] \arrow[d, phantom, "\shortparallel"] & \mathcal{S} \arrow[d, phantom, "\shortparallel"] \\
C^\infty(\Lambda^1 T^*) \arrow[r] & C^\infty(\Lambda^1 T^*)
\end{tikzcd}
$$

$$C^\infty(\Lambda^i T^*) \longrightarrow C^\infty(\Lambda^j T^*)$$

generates a bracket operation on $CL(T \oplus T^*)$

$$[A, B]_H \cdot \rho = [[A, d_H], B] \cdot \rho$$

$$\mathbb{H}^{\cdot} \to d_H : S \to S$$

$$S \subset C^\infty(\Lambda T^*), \quad S = C^\infty(\Lambda T^*), \quad C^\infty(\Lambda T^*) \to C^\infty(\Lambda T^*)$$

⇒ derived bracket construction generates a bracket operation on $CL(T \oplus T^*)$

$$[A, B]_H \cdot \rho = [[A, d_H], B] \cdot \rho$$

if $A, B \in T \oplus \Lambda^{\cdot} T^*$, then $[A, B]_H$ is a section of $T \oplus \Lambda^{\cdot} T^*$.

$$H^\cdot \to d_H : S \to S$$

$$C^\infty(\Lambda^\cdot T^*) \to C^\infty(\Lambda^\cdot T^*)$$

$\Rightarrow$ derived bracket construction generates a bracket operation on

$$CL(T \oplus T^*) \cong \Lambda^\cdot(T \oplus T^*)$$

$$[A, B]_H \cdot \rho = [[A, d_H], B] \cdot \rho$$

$A, B \in \Gamma$

is a section of

$$[X+\xi,\; Y+\eta]_H = [$$

$T \quad \Lambda^\cdot T^* \qquad \Lambda^\cdot T^*$

$\oplus \Lambda^{\cdot} T^{*}$, then $[A, B]_{H}$

$T \oplus \Lambda^{\cdot} T^{*}$ ..

$[X, Y] + L_{X}\eta - L_{Y}\xi - \frac{1}{2} d(i_{X}\eta - i_{Y}\xi)$

$\oplus \Lambda^{\bullet} T^{*}$, then $[A,B]_H$

$T \oplus \Lambda^{\bullet} T^{*}$ ..

$x,y] + L_x \eta - L_y \xi - \frac{1}{2} d(i_x \eta - i_y \xi)$

$+ i_x i_y H$

$$T \oplus T^{*} \longleftrightarrow H^{(3)}$$

Enlarged symmetry gr

$$\mathcal{D}iff_H \subset Sym([\ ,\ ]_H)$$

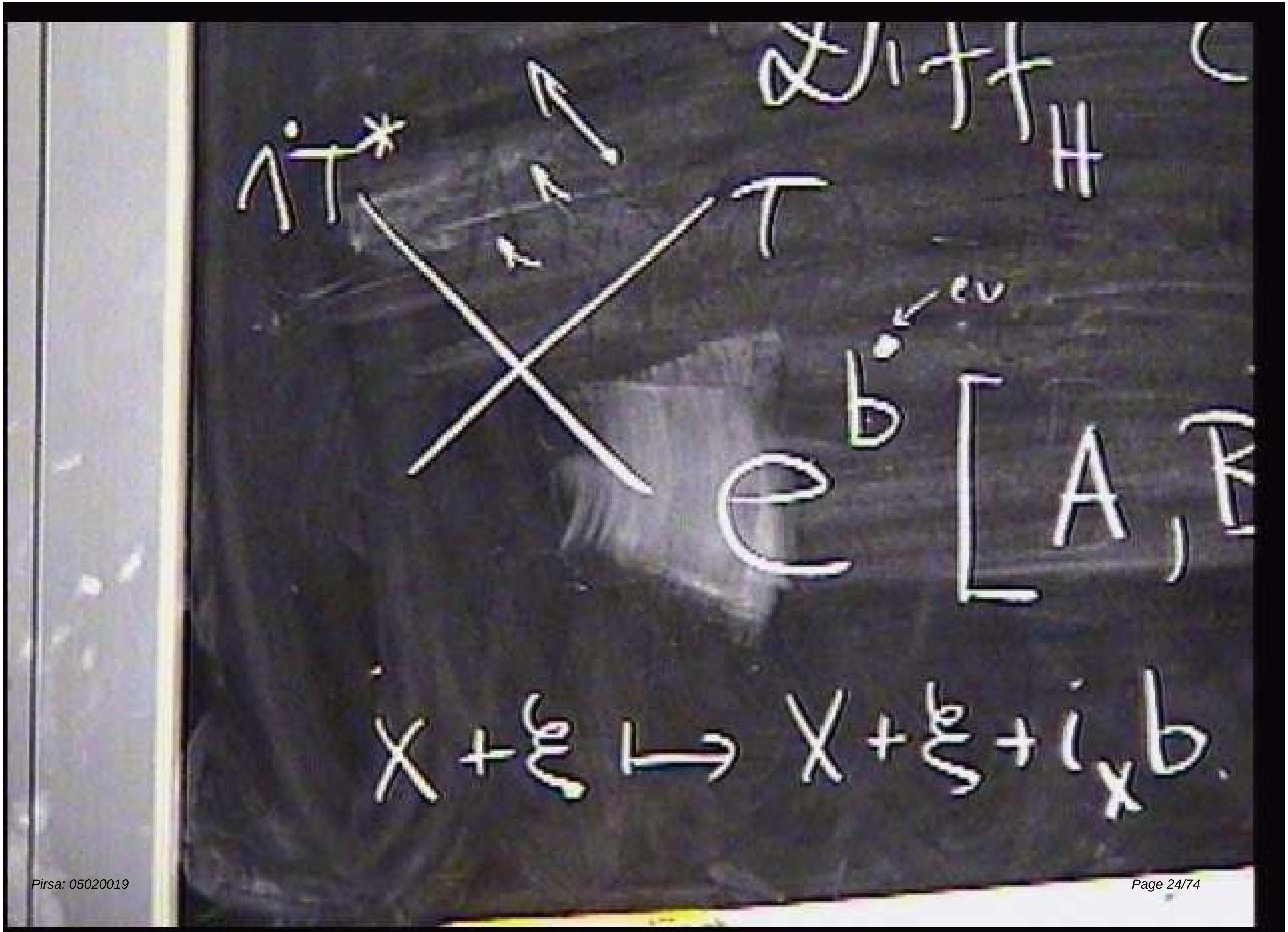
$$e^b [A,B]_H = [e^b A, e^b B]_{H+db}$$

group

if $db = 0$

then e^{b} is

a symmetry.


T^*
T
H
ev
$e^{b}[A,B$
$X+\xi \mapsto X+\xi+i_X b$

$$\begin{tikzcd} T_1 \oplus T_1^* \arrow[r, "F"] \arrow[d] & T_2 \oplus T_2^* \arrow[d] \\ N_1 \arrow[r, "f"'] & N_2 \end{tikzcd}$$

nmetry group for [

ourant bracket $\rightarrow T \oplus T^*$ is <u>not</u> a Lie brack

$$\mathrm{Jac}(A,B,C) = [[A,B],C] + \text{c.p.}$$

$$= d\, \mathrm{Nij}(A,B,C)$$

$$\mathrm{Nij}(A,B,C) = \frac{1}{3}\left(\langle [A,B],C\rangle + i_z i_y i_x\, dH + \text{c.p.}\right)$$

$L \subset T \oplus T^*$ subbundle

1. L Lagrangian (max. isotropic)
2. $C^\infty(L)$ closed under Courant bracket

ourant & Weinstein

f we find $L \subset T \oplus T^*$ subbundle

st 1. L Lagrangian (max. isotropic)

2. $C^\infty(L)$ closed under Courant bracket

hen $N_{ij}|_L = 0 \Rightarrow [\,,\,]$ is a Lie bracket on $C^\infty(L)$

if we find $L \subset$

Dirac Structure $\begin{cases} 1.\ L \ \ldots \\ 2.\ C(L) \ \ldots \end{cases}$

then $N_{ij}\big|_L = 0 \Rightarrow [$

T closed if $H=0$

$$\underset{\underset{T}{\cap}}{[X,Y]_H} = [X,Y] + i_X i_Y H$$

$$[X,Y]_H = [X,Y] +$$

$$e^b T = \{X + i_X b\} = Gr(b)$$

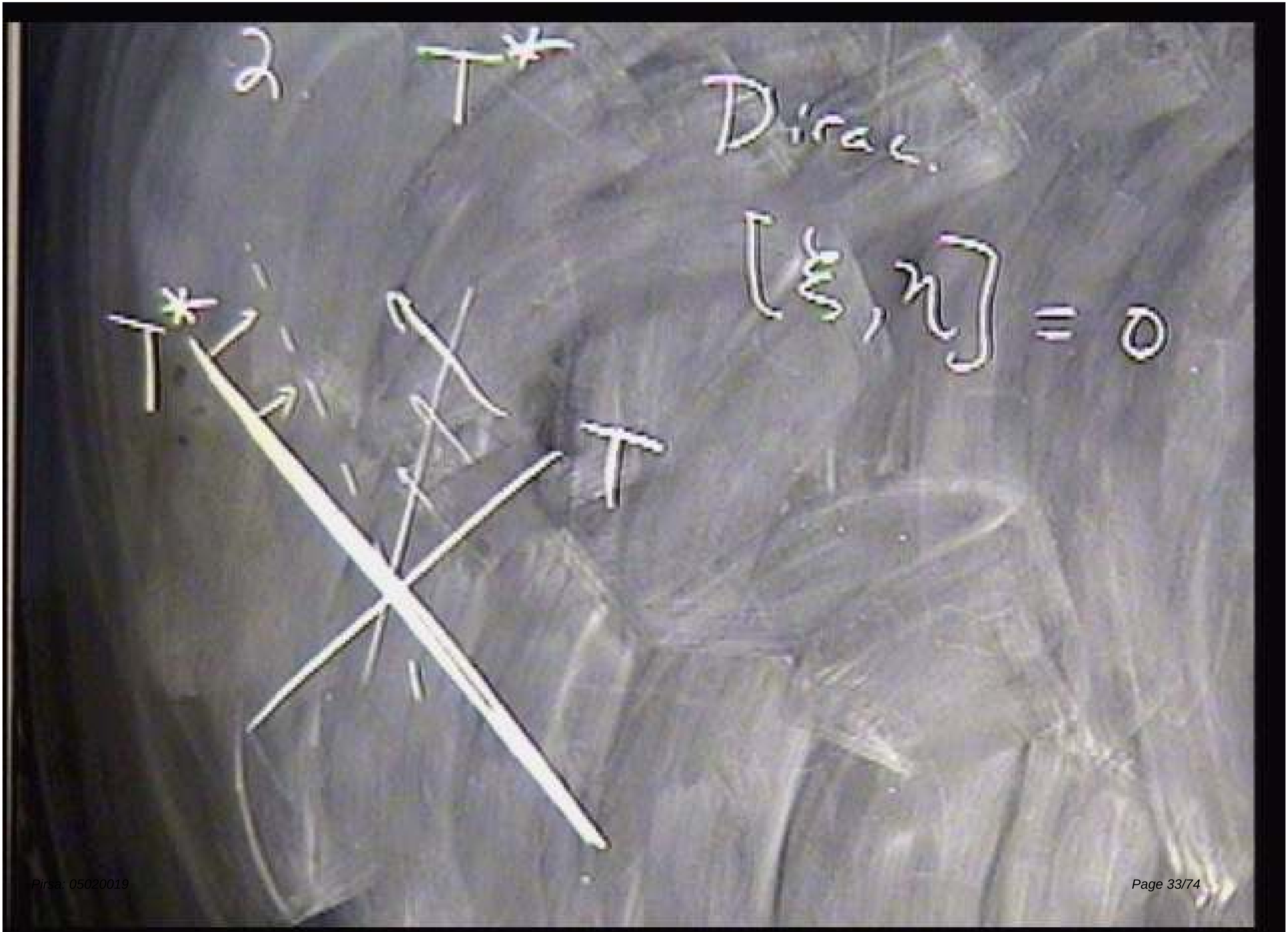
this is Dirac $\iff$ $db = H$.

$$[X,Y]_H = [X,Y] + i_X i_Y H$$

$$e^b T = \{X + i_X b\} = Gr(b)$$

this is Dirac $\Leftrightarrow$ $db = H$.

$H=0$ then closed 2-forms are Dirac


2. T^*
Dirac.
$[\xi, \eta] = 0$
T^*
T


] = 0
β ∈ C(Λ² T)
β, ∈ so(T⊕T*)

irac.

$\beta^* H \in \Lambda^5 T$

$\xi, \eta] = 0$

$H \in \Lambda^3 T^*$

$\beta \in C^\infty(\Lambda^2 T)$

$\beta \in \Gamma O(T \oplus$

$e^\beta T^*$

$\beta : T^* \to T$

Dirac $\Leftrightarrow$ $[\beta, \beta] = \beta^* H$

$H =$

$(C^\infty(L), [\ ,\ ])$ Lie algebr

$$C^\infty(\Lambda^k L^*) \xrightarrow{d_L} C^\infty(\Lambda^{k+1} L^*)$$

$(C^\infty(L), [\ ,\ ])$ Lie algebroid

$$C^\infty(\Lambda^k L^*) \xrightarrow{d_L} C^\infty(\Lambda^{k+1} L^*)$$

$d_L^2 = 0$ diff complex

$H^\bullet_{d_L}(L)$ Lie algebroid cohomol

$C^\infty(\Lambda^\bullet T^*) \circlearrowleft d$

$C^\infty(\Lambda^{0,\bullet} T^*) \circlearrowleft \bar\partial$

Poincaré lemma
Dolbeault lemma) exact as complexes of sheaves

$C^\infty(L^*) \circlearrowleft d_L$ <u>not</u> exact as cx of sheaves

$\mathcal{H}^\bullet(L)$ cohomology sheaves of $C^\infty(\Lambda^\bullet L^*)$

$$E_2^{p,q} = H^p(N, \mathcal{H}^{\bullet}(L))$$

$$\Downarrow \quad \longleftarrow \text{Hypercoh. s.s.}$$

$$H^{\bullet}_{dR}(L).$$

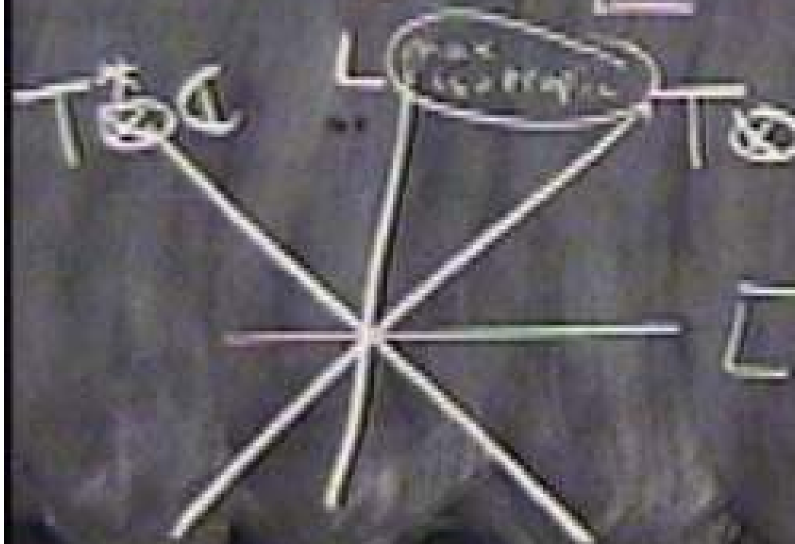
$\mathbb{J} : T \oplus T^* \circlearrowleft \quad \mathbb{J}^2 = -1$

$T \oplus T^*$

$\mathbb{J}$ orthog in $\langle \, , \, \rangle$

$O(2n, 2n)$
$\downarrow$
(M even) $U(n, n)$

$\mathbb{J}$ integ $\quad L = +i$ eigenbundle $\quad$ closed under Cou



$l_1, l_2 \in L$

$$\langle l_1, l_2 \rangle = -\langle i l_1, i l_2 \rangle$$
$$= -\langle \mathbb{J} l_1, \mathbb{J} l_2 \rangle$$
$$= -\langle l_1, l_2 \rangle$$
$$= 0$$

$\boxed{\text{Gen Cx Str}}$

$\updownarrow$

$\boxed{\begin{array}{l}\text{Cx. Dirac Structure} \\ L \text{ st } L \cap \bar{L} = \{0\}\end{array}}$

ex: $\mathcal{J}_f = \begin{pmatrix} -J & \\ & J^* \end{pmatrix}$

$\Downarrow$

integ $\Rightarrow$ integ (x

and $\mathbb{J}_w = \begin{pmatrix} & -\tilde{w}' \\ w & \end{pmatrix}$

$\downarrow$

$d_w = 0$

ex: $\mathcal{J} = \begin{pmatrix} -J & \\ & -J^* \end{pmatrix}$ and $\mathcal{J}_\omega = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}$

integ $\Rightarrow$ integ cx $\qquad d\omega = 0$

$C^\infty(\wedge^\bullet L^*)$ is complex of sheaves

$\left(\quad + J^* \right)$ and $\mathbb{J}_w = \left(\, w \right.$

<u>Integ</u> $\Rightarrow$ integ cv

$\downarrow \; dw = 0$

$C^\infty(\Lambda^\bullet L^*)$ is <u>elliptic</u> <u>complex</u> of sheaves.

governs deformation theory of $\mathbb{J}$.

cx mfld

$$L = T_{0,1} \oplus T^*_{1,0}$$

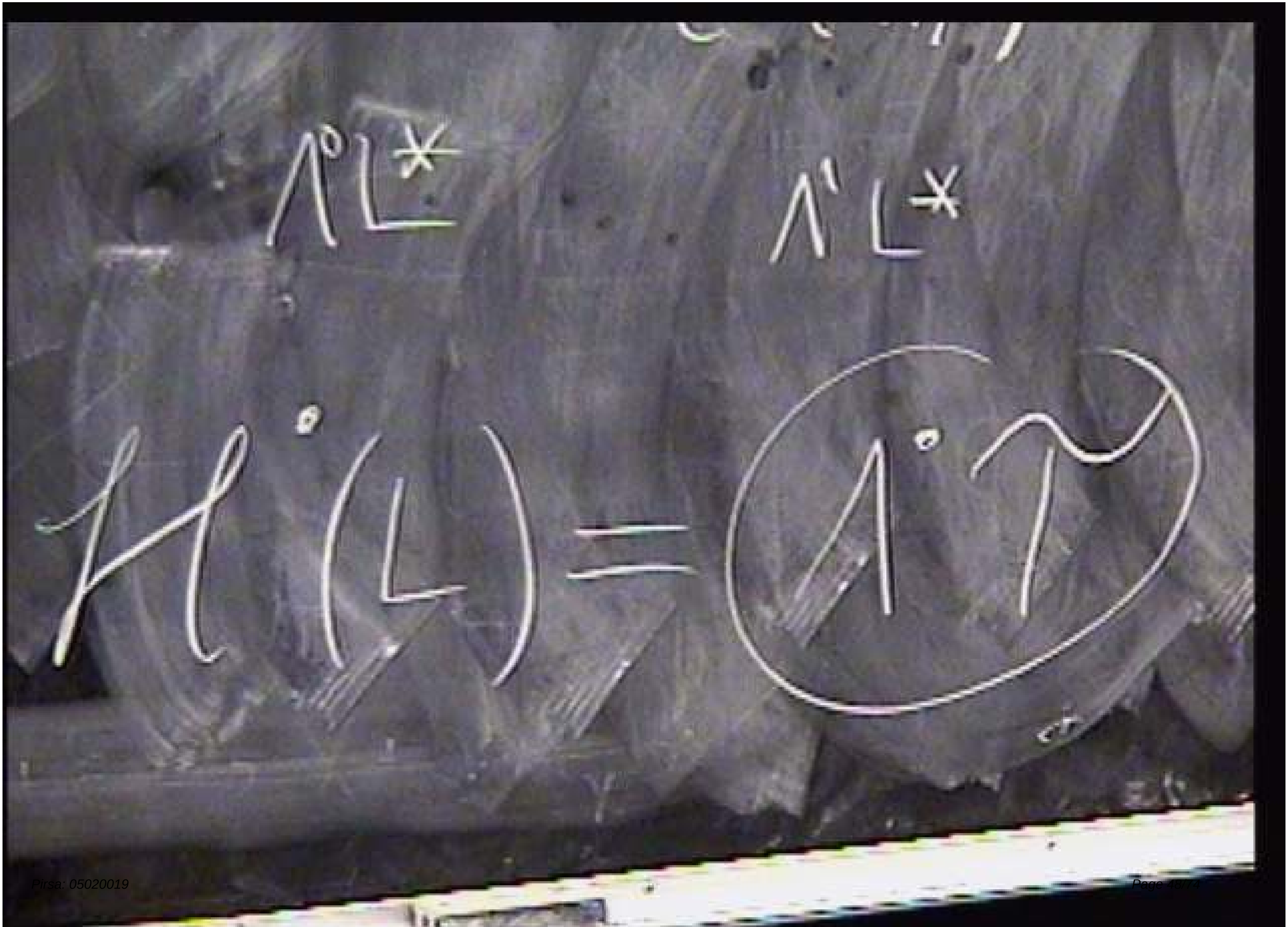
$$C^\infty(M,\mathbb{C}) \qquad C^\infty(T^*_{0,1}) \oplus C^\infty(T_{1,0}) \qquad C^\infty(\Lambda^2 T^*_{0,1}) \oplus C^\infty(T_{1,0} \otimes T^*_{0,1}) \oplus C^\infty(\Lambda^2 T_{1,0}).$$

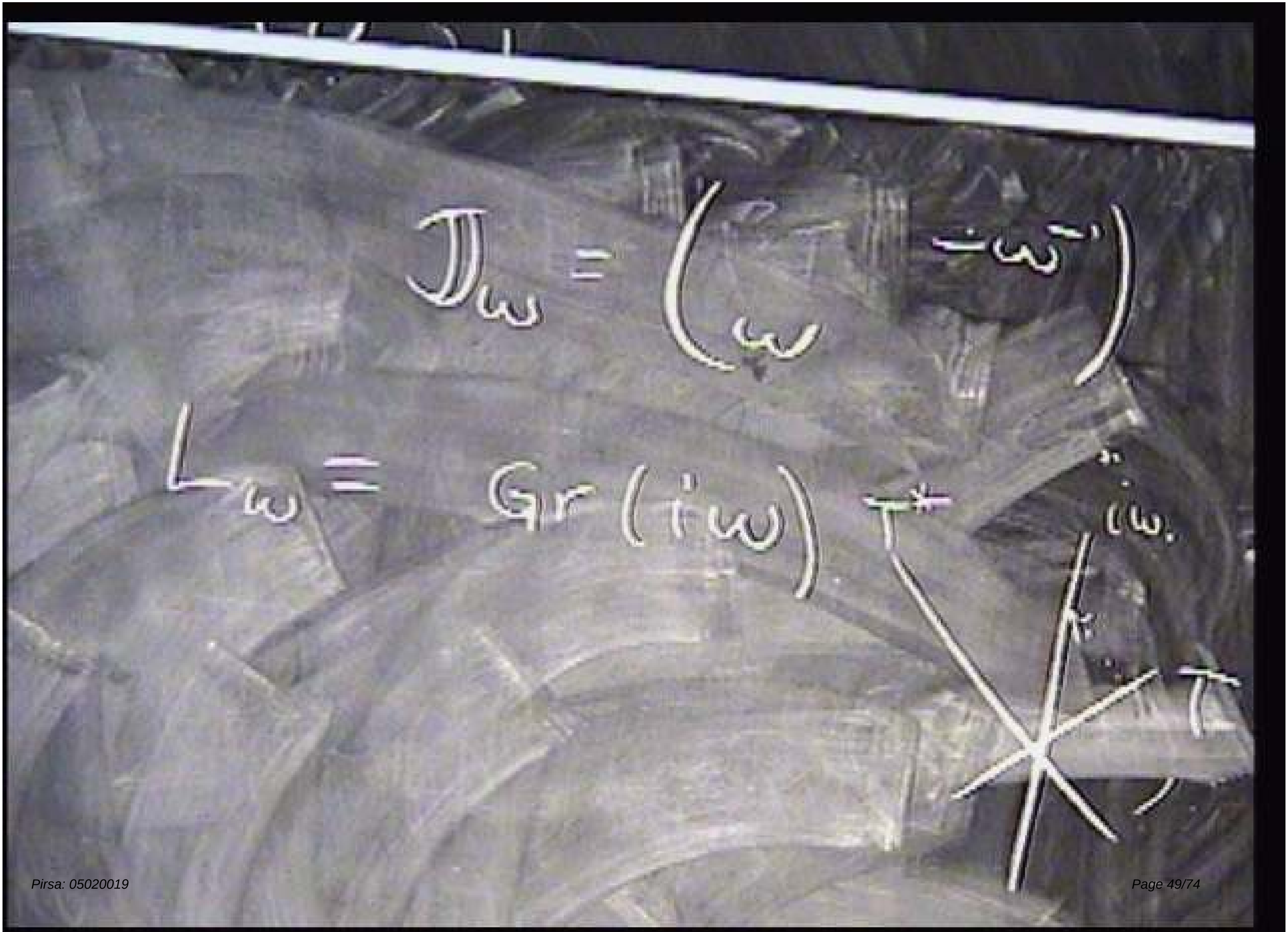
$$\Lambda^0 L^* \qquad \Lambda^1 L^*$$

mfld $\quad L = T_{0,1} \oplus T^*_{1,0}$

$$C^\infty(M,\mathbb{C}) \xrightarrow{\bar\partial} \begin{matrix} C^\infty(T^*_{0,1}) \\ \oplus \\ C^\infty(T_{1,0}) \end{matrix} \xrightarrow{\bar\partial} \begin{matrix} C^\infty(\Lambda^2 T^*_{0,1}) \\ \oplus \\ C^\infty(T_{1,0} \otimes T^*_{0,1}) \\ \oplus \\ C^\infty(\Lambda^2 T_{1,0}) \end{matrix} \xrightarrow{\bar\partial}$$

$$\Lambda^0 L^* \qquad \Lambda^1 L^* \qquad \Lambda^2 L^*$$


$\Lambda^0 L^*$
$\Lambda^1 L^*$
$H^0(L) = \Lambda^0 \mathcal{T}$



$\mathbb{J}_\omega = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}$
$L_\omega = Gr(i\omega)\ T^*$
T

$$\mathbb{J}_\omega = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}$$

$$L_\omega = Gr(i\omega) \quad T^*$$

$$\Lambda^i L^*_\omega \cong \Lambda^i T^* \otimes \mathbb{C}$$

$$d) = \begin{cases} 0 & \text{for all } k > 0 \\ \mathbb{C} & \text{for } k = 0 \end{cases}$$

$$\mathbb{J}_\omega = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}$$

$$L_\omega = \mathrm{Gr}(i\omega) \quad T^*$$

$$\Lambda^\cdot L_\omega^* \cong \Lambda^\cdot T^* \otimes \mathbb{C}$$

$$\Lambda^\cdot T^* \otimes \mathbb{C}, d) = \begin{cases} 0 & \text{for all } k > 0 \\ \mathbb{C} & \text{for } k = 0. \end{cases}$$

L Lag subbundle $\subset (T \oplus T^*) \otimes \mathbb{C}$

$\updownarrow$

Pure spinor line $U \subset \Lambda^{\bullet} T^*$

$L = \mathrm{Ann}(U)$

Canonical Line bundle of $\mathbb{J}$

ub bundle $\subset (T \oplus T^*) \otimes \mathbb{C}$

line $U \subset \Lambda^* T^* \otimes \mathbb{C}$

$L = \mathrm{Ann}(U)$ "

Line bundle of $\mathbb{J}$

$\Rightarrow$ L Lag subbundle $\subset (T \oplus T^*) \otimes \mathbb{C}$

Pure spinor line $U \subset \Lambda^{\bullet} T^* \otimes \mathbb{C}$

$L = \mathrm{Ann}(U)$

Canonical Line bundle of $\mathbb{J}$

$\rho = c\, e^{B + i\omega}\, \theta_1 \wedge \ldots \wedge \theta_k \longrightarrow k$ called the $\underline{\underline{\text{type}}}$ of $\mathbb{J}$ at

$k \in \{0, \ldots, n\}$

L — Lag subbundle $\subset (T \oplus T^*) \otimes \mathbb{C}$

$\updownarrow$

Pure spinor lines $U \subset \wedge^\bullet T^* \otimes \mathbb{C}$

$L = \mathrm{Ann}(U)$ "

Canonical Line bundle of $\mathbb{J}$

$\rho = c e^{B+i\omega} \theta_1 \wedge \ldots \wedge \theta_k \longrightarrow k$ called the <u>type</u> of $\mathbb{J}$ at x.

$k \in \{0, \ldots, n\}$

$L_n \bar{C} = 0 \Leftrightarrow \omega^{n-k} \wedge$

cx mfld $\qquad \rho =$

$\omega^{*k} \wedge \Omega \wedge \bar{\Omega} \neq 0$ $\quad e^{B}\Omega.$

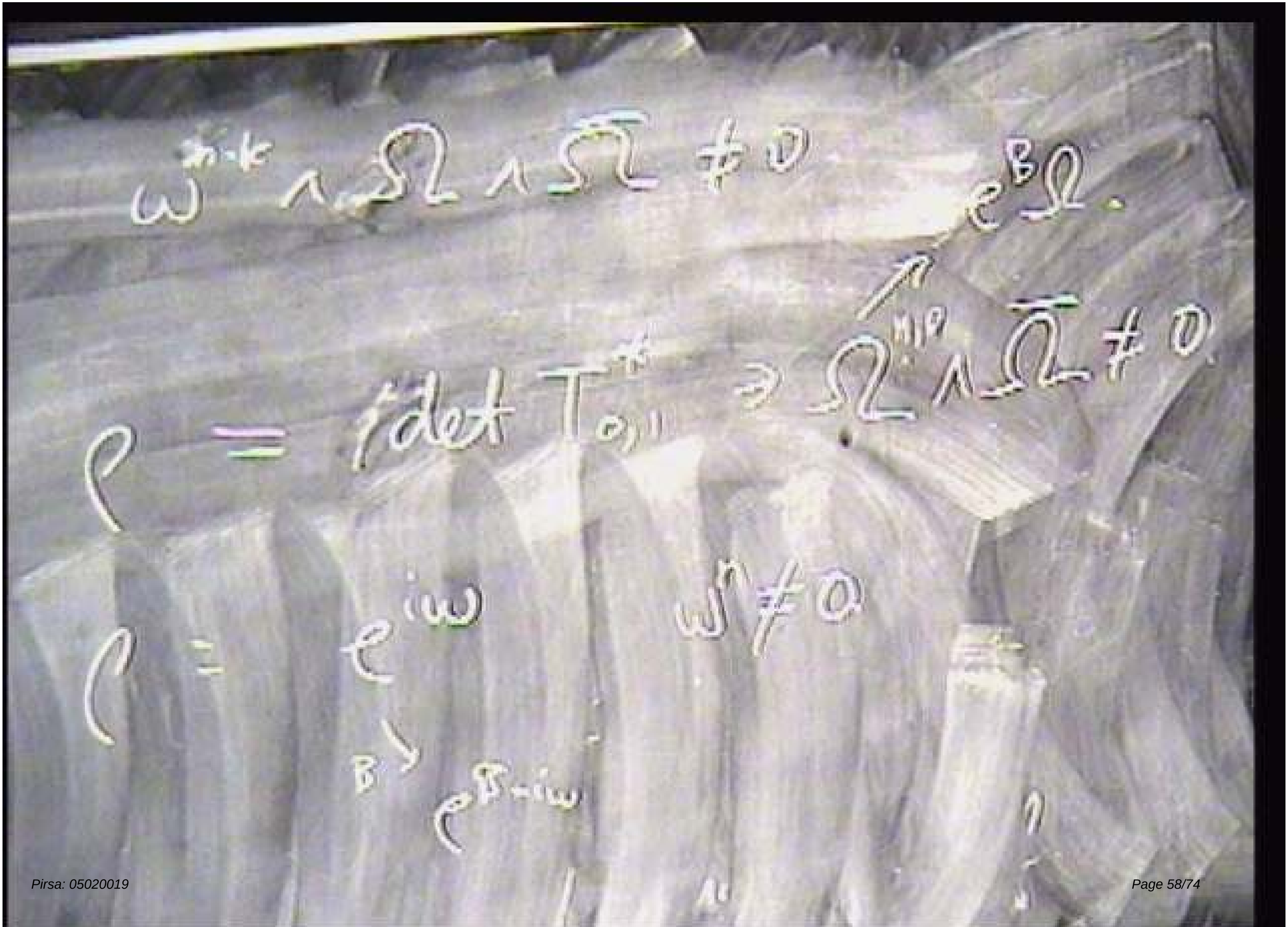
$\Omega^{n,0} \wedge \bar{\Omega} \neq$

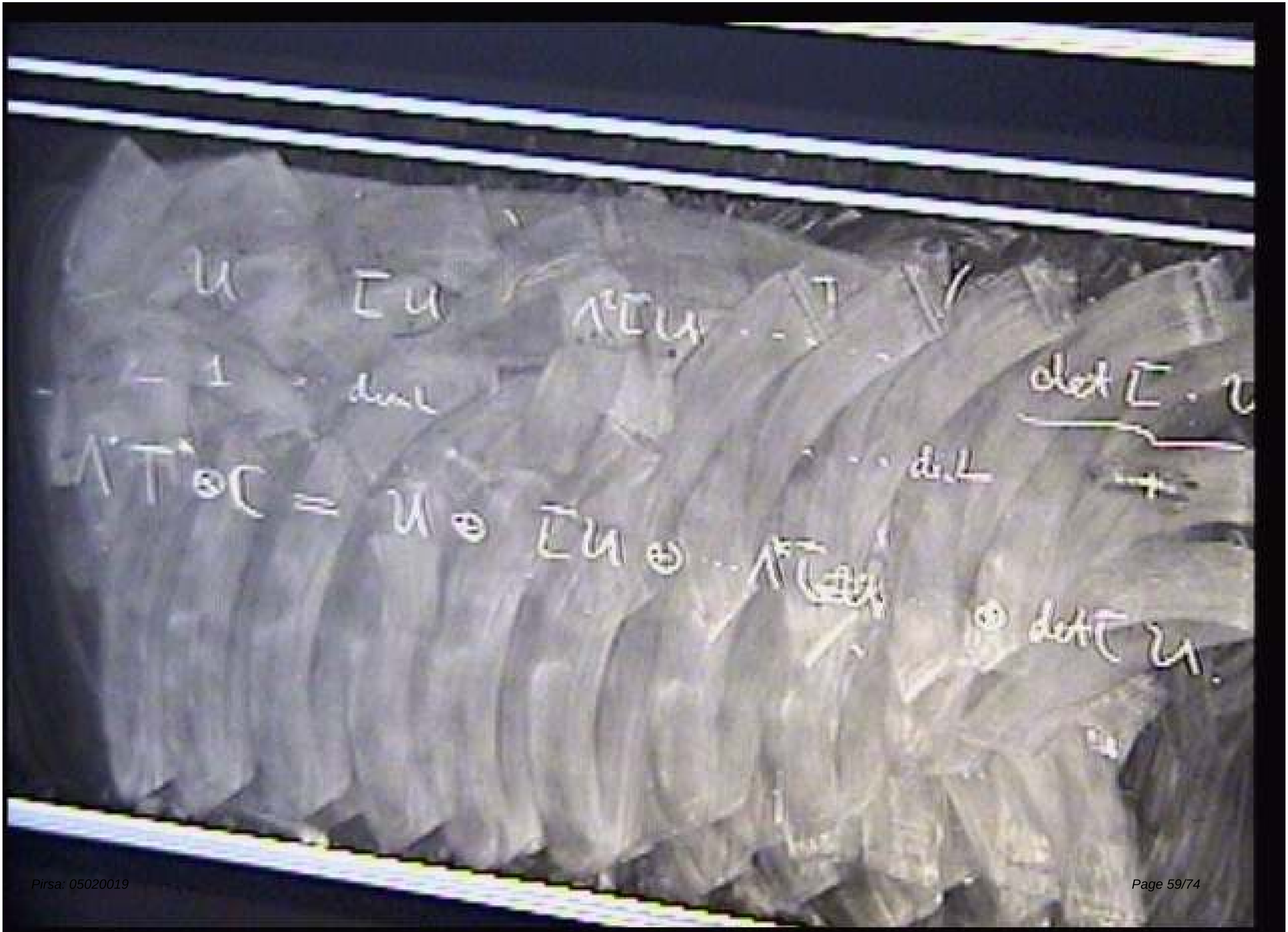
$= f\det T^{*}_{0,1}$

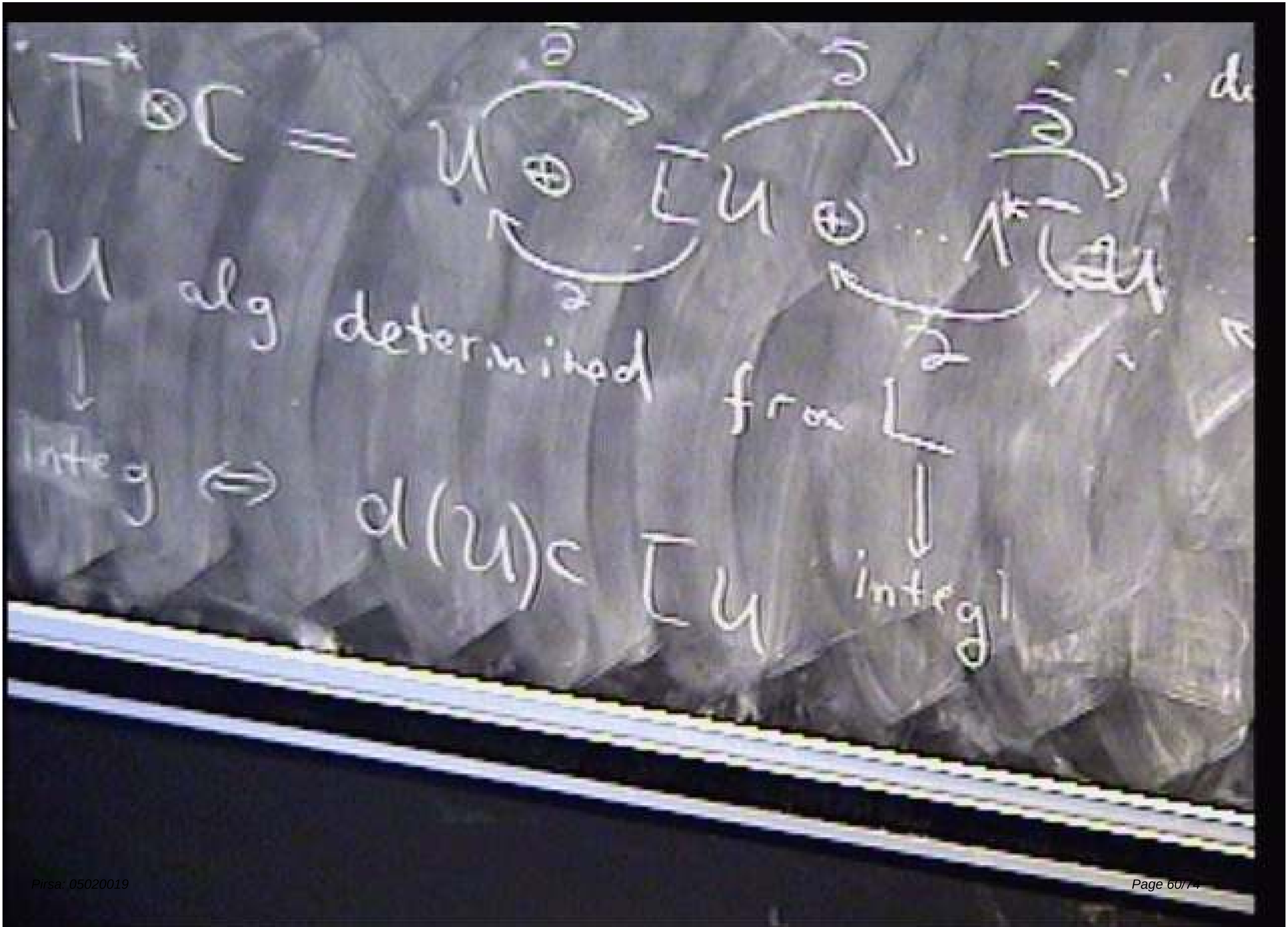
$= e^{i\omega} \quad \omega^{n} \neq 0$

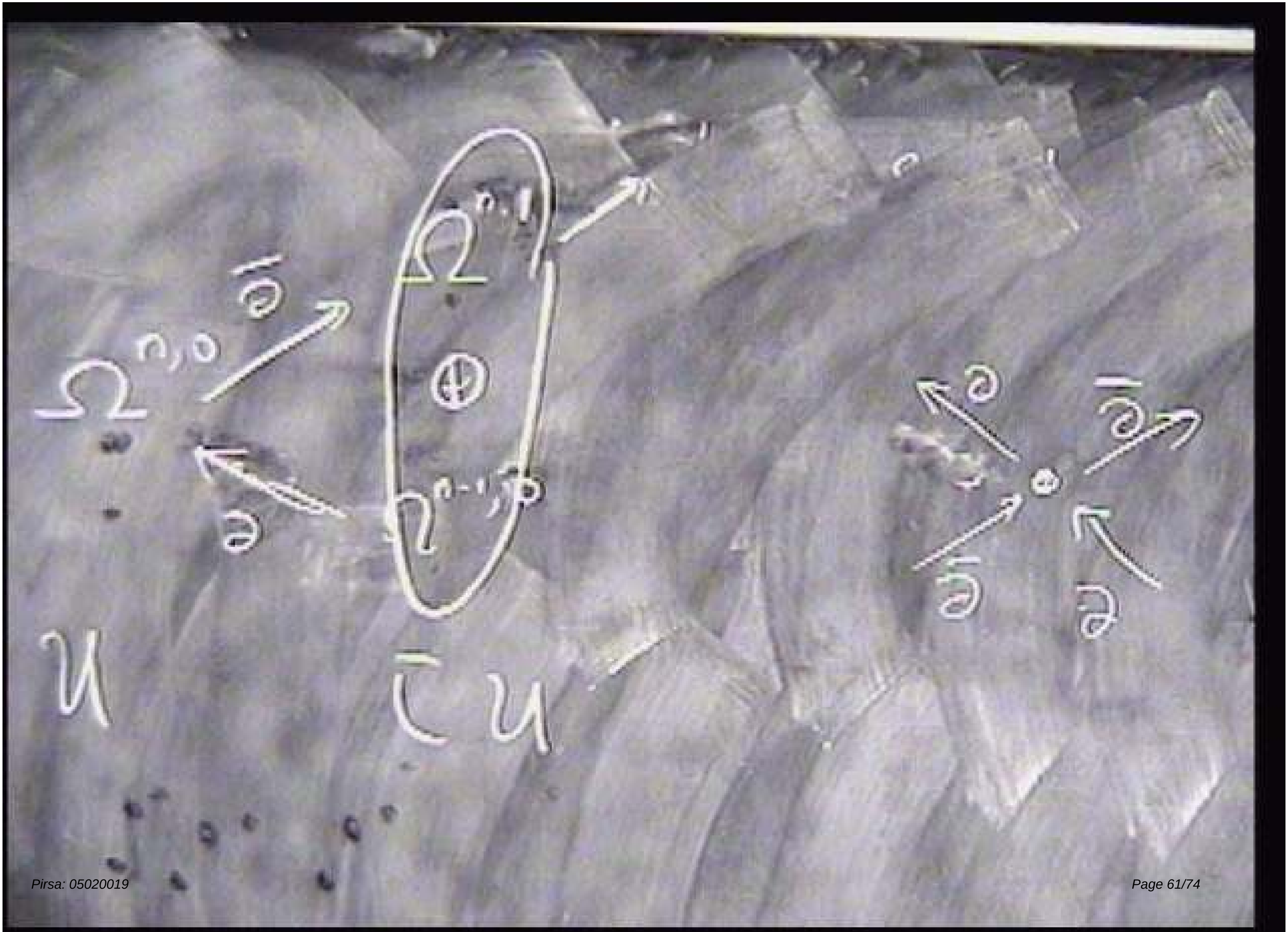
B

$e^{B+i\omega}$


$\omega^{n-k} \wedge \Omega \wedge \bar{\Omega} \neq 0$
$e^{B}\Omega$
$\rho = f \det T^{*}_{0,1} \ni \Omega^{n,0} \wedge \bar{\Omega} \neq 0$
$\rho = e^{i\omega}$
$\omega^{n} \neq 0$
B
$e^{B+i\omega}$


dual
det


T*⊗C = U ⊗ [U ⊗ ... Λ* (c,u)
U alg determined from L
integ ⟺ d(U)c [U integ]


$\Omega^{n,0}$
$\bar\partial$
∂
$\Omega^{n,1}$
$\oplus$
$\Omega^{n-1,0}$
$\mathcal{U}$
$\bar{C}\,\mathcal{U}$
∂
$\bar\partial$
$\bar\partial$
∂

CY condition

$\mathcal{U}$ canonical bundle

Suppose

$c_1\mathcal{U}=0$. Choose nonvanishing section ϕ of $\mathcal{U}$

$d\phi \in \bar{L}\,\mathcal{U}$

$\Rightarrow d\phi = \alpha\cdot\phi$, $\alpha\in \bar{L}=L^*$

$\Rightarrow d_L\alpha = 0$ in $C^\infty(\Lambda^2 L^*)$

$[\alpha] \in H^1_{d_L}(L)$

$(\alpha) = 0 \iff \exists$ section $\tilde{\phi} \in U$

$\Updownarrow$

CY condition

$c_1(K) = 0$ $\quad [K] \in H^1(M, \mathcal{O}) \longrightarrow H^1(M, \mathcal{O}^*)$

section $\tilde{\phi} \in \mathcal{U}$ st $d\tilde{\phi} = 0$

$$\ldots(\mathcal{O}) \longrightarrow \underset{K}{H^1(M, \mathcal{O}^*)} \xrightarrow{c_1} H^2(M, \mathbb{Z})$$

$$\ldots) + dz_1 \wedge dz_2 .$$

- if $z_1 = 0$ $\quad (dz_1 \wedge dz_2) \wedge (d\bar{z}_1 \wedge d\bar{z}_2) \neq 0$
- $z_1 \neq 0$

$$z_1 \left(1 + \frac{dz_1 \wedge dz_2}{z_1} \right)$$

$$= z_1 e^{\frac{dz_1 \wedge dz_2}{z_1}} \quad \text{also pure.}$$

$$d\rho = dz_1 \, \frac{\partial}{\partial z_2} \cdot \rho$$

$dz_1 \wedge dz_2$.

- if $z_1 = 0$ $\quad (dz_1 \wedge dz_2) \wedge (d\bar{z}_1 \wedge d\bar{z}_2) \neq 0$
- $z_1 \neq 0$

$$z_1 \left(1 + \frac{dz_1 \wedge dz_2}{z_1} \right)$$

$$= z_1 e^{\frac{dz_1 \wedge dz_2}{z_1}}$$

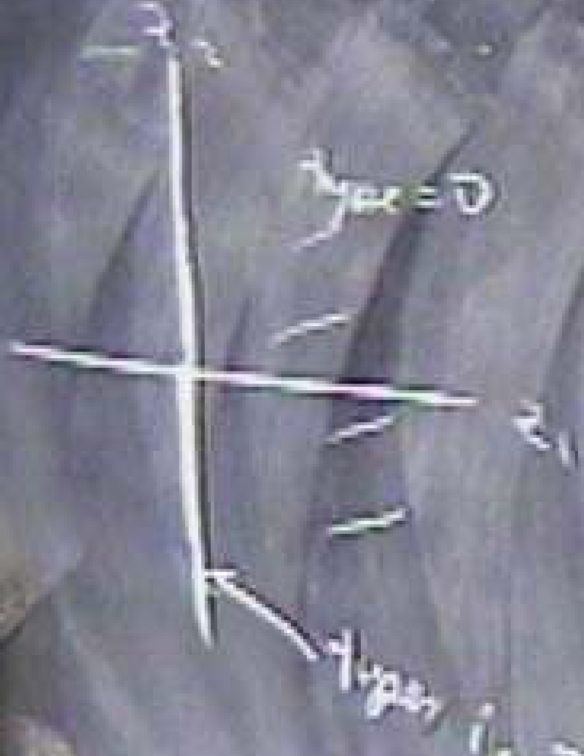
also pure.

$$d\rho = dz_1 \; \frac{\partial}{\partial z_2} \cdot \rho \quad \checkmark$$

inte

$\mathbb{C}^2$

(z_1, z_2) $\quad \rho = z_1 + dz_1 \wedge dz_2.$



- if $z_1 = 0$ $\quad (dz_1 \wedge d$

- $z_1 \neq 0$ $\quad z_1 \left(1 + \right.$

$= z_1 e^{\frac{dz_1 \wedge dz_2}{z_1}}$

In

$d\rho = dz_1 \ldots \cdot \rho$

$0 \neq [-\partial] \in H^1(M)$

$\mathbb{C}P^2$

cx structure

$$H^2_{d_L}(L) = \text{tgt space to def}^{\text{ms}}$$

$$\| $$

$$\underbrace{H^0(\Lambda^2 T)}_{\beta} \oplus \underbrace{H^1(T)}_{\substack{\text{usual} \\ \text{def}^{\text{ms}} \text{ of } J}} \oplus \underbrace{H^2(\mathcal{O})}_{B\text{-fields}}$$

$\beta \in C^\infty(\Lambda^{0,1} T_p)$ actual def^{m} iff $\bar\partial\beta + \cancel{\tfrac{1}{2}[\beta,\beta]} = 0$

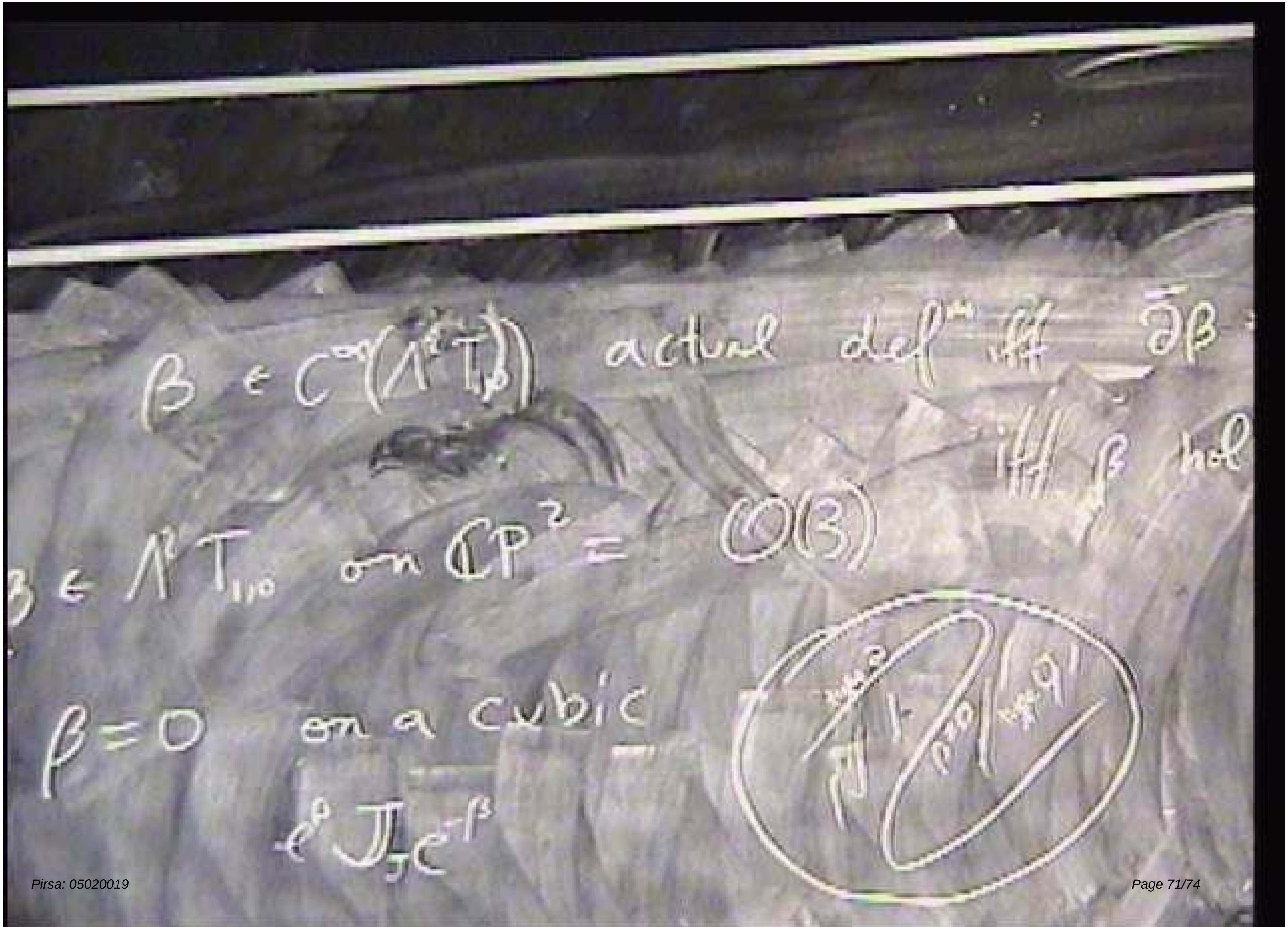
iff β hol

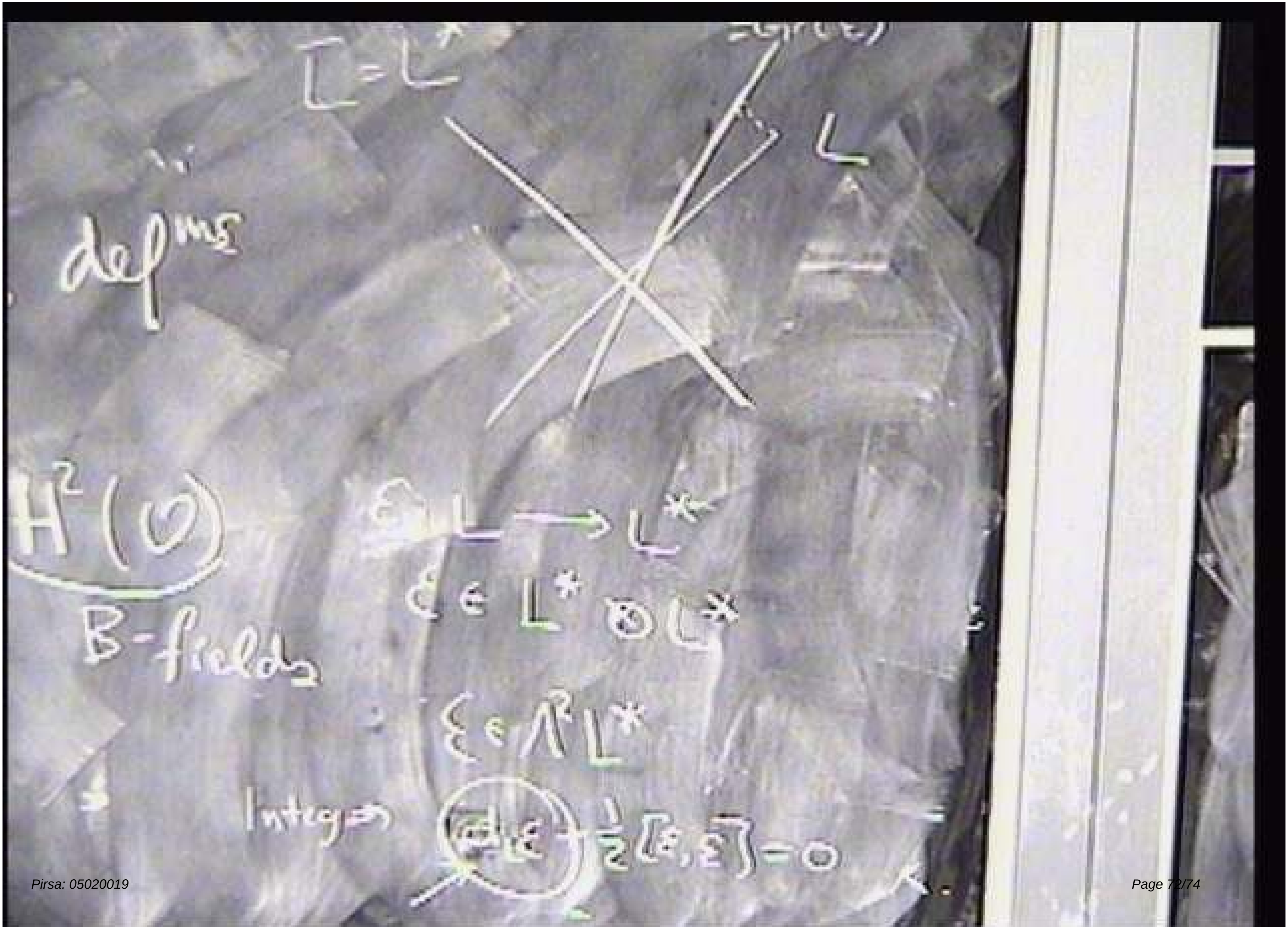
$\beta \in C^\infty(\Lambda^2 T_{1,0})$ actual def^n iff $\bar\partial\beta = 0$

iff β hol

$\beta \in \Lambda^2 T_{1,0}$ on $\mathbb{CP}^2 = \mathcal{O}(3)$

$\beta = 0$ on a cubic


$\beta \in C^\infty(\Lambda^2 T_{1,0})$ actual def^m iff $\bar\partial\beta$
iff β hol
$\beta \in \Lambda^2 T_{1,0}$ on $\mathbb{C}P^2 = \mathcal{O}(3)$
$\beta = 0$ on a cubic
$e^\beta \mathbb{J}_J e^{-\beta}$


L = L*
L
def^ms
H²(O)
B-fields
ε: L → L*
ε ∈ L* ⊗ L*
ε ∈ Λ²L*
Integrable
dε + ½[ε, ε] = 0

$$\underbrace{H(X)} \oplus \underbrace{H(F)}$$

β

usual defns of

$$\pi_{T\otimes G} L = E$$

$$E \cap \bar{E}$$

$$\mathcal{D}_{\pi} = \{ \xi + i_{\xi}\pi \} \leftarrow T \oplus T^{*}$$

$$\Lambda^{\cdot} T$$

$$e^{\pi} \in \Lambda^{\cdot} T$$