Title: Instanton Counting on ALE Spaces

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Abstract:

Instanton Counting on ALE spaces.

Satoshi Minabe
(Nagoya University)

Jan. 18. 2005 , Perimeter

based on joint work with
Shigeyuki Fujii

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Instanton Counting

In 4d $N=2$ SUSY YM theory,
one counts instantons.
($\rightsquigarrow$ Donaldson invariants)

cf. curve counting
in topological string theory.
($\rightsquigarrow$ Gromov–Witten invariants)

Integrations over moduli spaces
— possible to define,
— but difficult to compute.

Nekrasov's partition function

(analogs of Donaldson inv.)

— possible to compute exactly.

— leads to the Seiberg-Witten prepotential.

— equal to topological string partition function for certain local CY 3-fold.

Interesting to study more on Nekrasov's partition function!

— from various point of view

What happens if we consider the partition function over more general 4-manifolds?

Today:

Extension to ALE spaces

Plan:

1. Introduction (done)
2. Nekrasov's partition function & Nekrasov's conjecture
3. ALE case

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Nekrasov's instanton partition function

$$Z^{inst}_{4d} = \text{"} \sum_{n=0}^{\infty} q^n \oint_{M(r,n)} 1 \text{"}$$

Problem $M(r,n)$ is non-compact

$\rightarrow \int$ is not defined

How to cure the non-compactness

Use equivariant integration & localization w.r.t. a torus action on $M(r,n)$

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framed moduli spaces of instantons on $\mathbb{R}^4$

$n \in \mathbb{Z}_{\geq 0}$, $r \in \mathbb{Z}_{>0}$

$M(n,r)$, $M_0(n,r)$

: Gieseker / Uhlenbeck (partial) compactification of framed moduli space of $SU(r)$-instantons with $c_2 = n$

$M(n,r) =$ framed moduli space of torsion-free sheaves E on $\mathbb{CP}^2 = \mathbb{C}^2 \cup \ell_\infty$ of rank $E = r$, $c_2(E) = n$.

$\bigcup \quad = \{ (E,\varphi) \mid \varphi : E|_{\ell_\infty} \xrightarrow{\simeq} \mathcal{O}_{\ell_\infty}^{\oplus r} \} / \text{isom}$

$M_0^{\text{reg}}(n,r) = \{ (E,\varphi) \mid E : \text{locally free} \}$

$\bigcap \longleftrightarrow$ framed moduli space of instantons on $\mathbb{R}^4$

$$M_0(r,n) = \coprod_{0 \leq k \leq n} M_0^{\text{reg}}(r, n-k) \times \underline{S^k \mathbb{C}^2}$$

location of the bubble

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location of the bubble

Explicit description via linear data
(ADHM construction)

$B_1, B_2 \in \mathrm{End}(\mathbb{C}^n)$

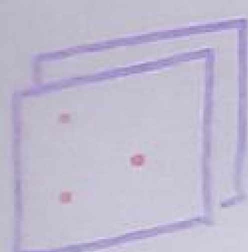
$i \in \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n), \; j \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r)$

$$\left\{ (B_1, B_2, i, j) \;\middle|\; \begin{array}{l} \cdot\ [B_1, B_2] + ij = 0 \\ \cdot\ [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + ii^\dagger - j^\dagger j \\ \qquad = \zeta\, \mathrm{id} \end{array} \right\} \Big/ U(n)$$

$$\cong \qquad (\zeta \neq 0)$$

$$M(n, r)$$

D-brane interpretation



n D0 $\rightsquigarrow$ BPS condition

$\leftrightarrow$ ADHM eq.

0–0 string $\leftrightarrow B_1, B_2$

0–4 string $\leftrightarrow i, j$

r D4 $\rightsquigarrow$ BPS condition

$\leftrightarrow$ (A)SD eq.

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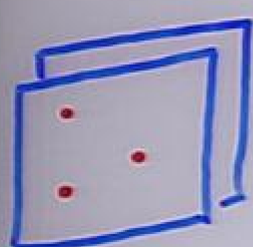

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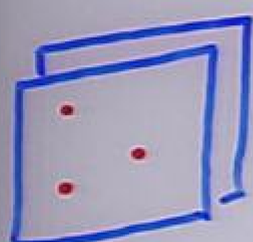

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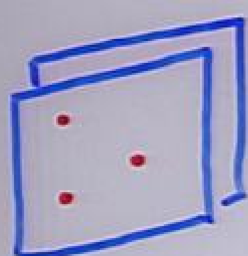

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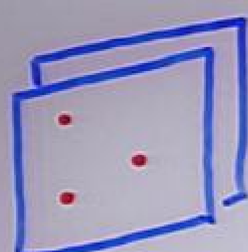

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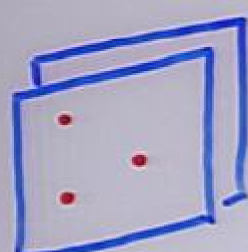

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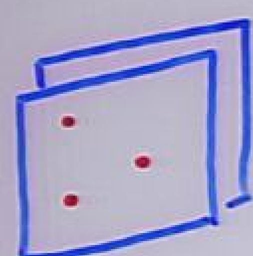

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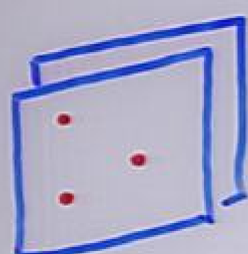

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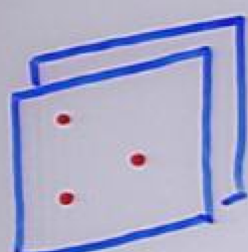

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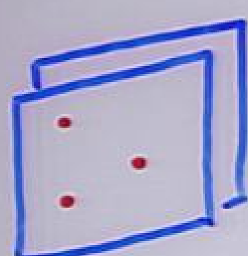

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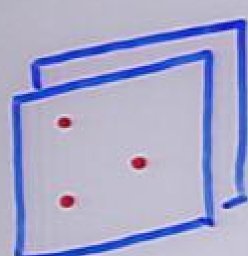

n D0 $\rightsquigarrow$ BPS condition
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0–0 string $\leftrightarrow B_1, B_2$
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Explicit description via linear data (ADHM construction)

$B_1, B_2 \in \mathrm{End}(\mathbb{C}^n)$

$i \in \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$, $j \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r)$

$$\left\{ (B_1, B_2, i, j) \;\middle|\; \begin{array}{l} \cdot\ [B_1, B_2] + ij = 0 \\ \cdot\ [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + ii^\dagger - jj^\dagger \\ \qquad = \zeta\, \mathrm{id} \end{array} \right\} \Big/ U(n)$$

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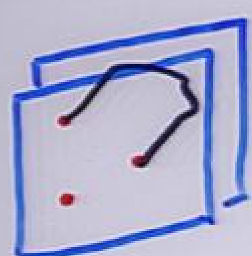

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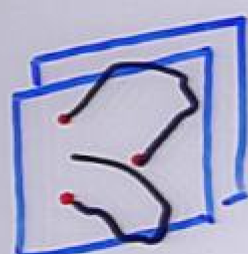

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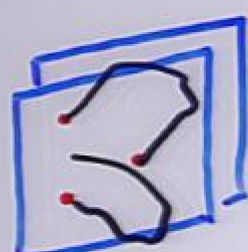

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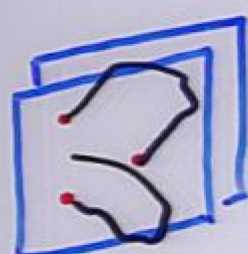

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Facts.

1) $M(n,r)$ is a smooth hyperKäler manifold of $\dim_{\mathbb{C}} = 2nr$.

2) $M_0(n,r)$ is an affine variety (with orbifold singularities).

3) $\exists$ projective morphism

$$\pi : M(n,r) \longrightarrow M_0(n,r).$$

(resolution of singularities)

Example.

$$M(n,1) = \mathrm{Hilb}^n(\mathbb{C}^2)$$

Hilbert scheme of n points on $\mathbb{C}^2$

$$= \left\{ I \subset \mathbb{C}[x,y] \;\middle|\; \dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n \right\}$$

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$\tilde{T} = (\mathbb{C}^*)^2 \times T \curvearrowright M(n, r),\ M_0(n, r)$

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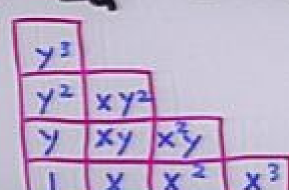

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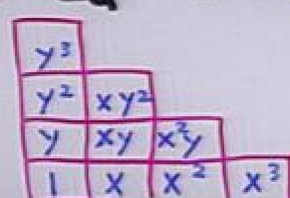

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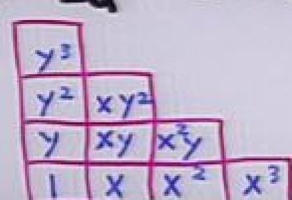

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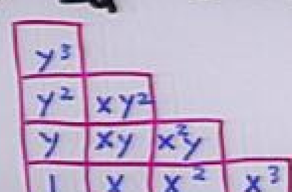

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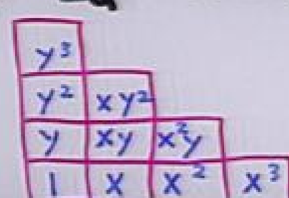

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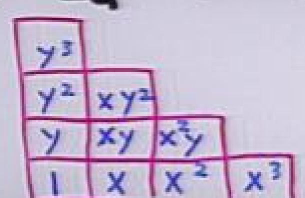

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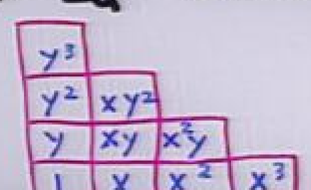

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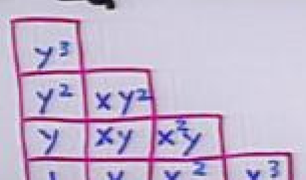

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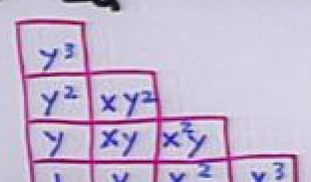

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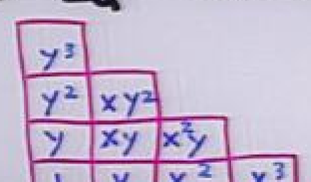

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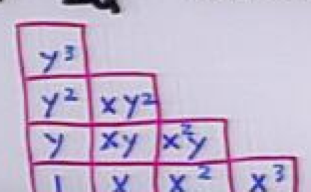

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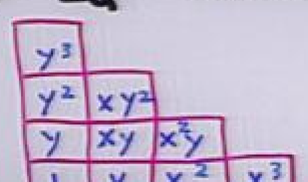

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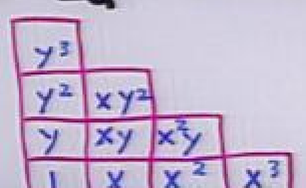
  $\underset{\text{iff}}{\iff}$ I_α is generated by monomials
  $\iff$ I_α corresponds to Young diagram Y_α.



Torus action on moduli spaces

$T = (\mathbb{C}^*)^{r-1}$: maximal torus in $SL(r,\mathbb{C})$

$$\tilde{T} = (\mathbb{C}^*)^2 \times T \curvearrowright M(n,r),\ M_0(n,r)$$

$$(B_1, B_2, i, j) \mapsto (t_1 B_1,\ t_2 B_2,\ i e^{-1},\ t_1 t_2 e j)$$

$$(t_1, t_2) \in (\mathbb{C}^*)^2,\quad e \in T$$

Fixed point set $M(n,r)^{\tilde{T}}$

- $[B_1, B_2, i, j] \in M(n,r)$ is fixed by $\tilde{T}$

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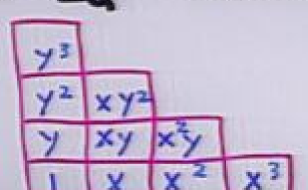

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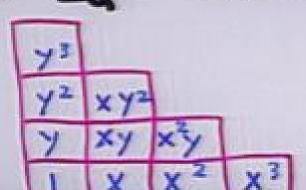

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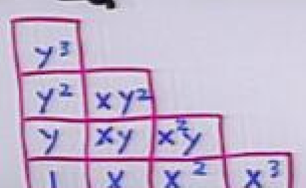

• $M(n.r)^{\widetilde{T}}$

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finite set!

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$$\int_{M(n.r)} 1 \equiv \widetilde{T}\text{- equivariant push forward to a point}$$

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$$e(T_{\vec{Y}} M(n.r)) = \prod_{\alpha,\beta} n_{\alpha.\beta}(\varepsilon_1, \varepsilon_2, \vec{a})$$

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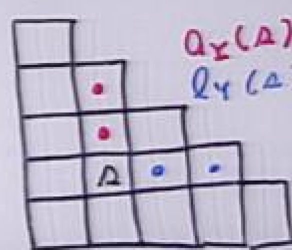

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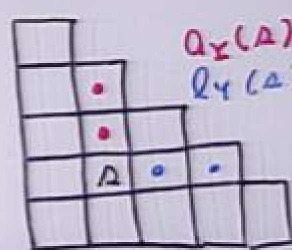

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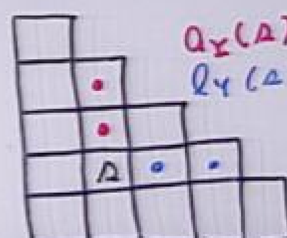

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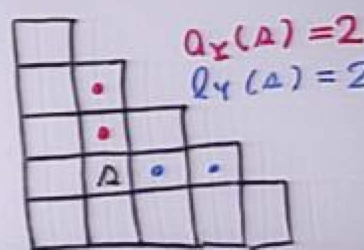

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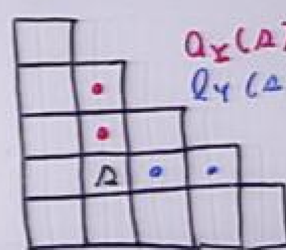

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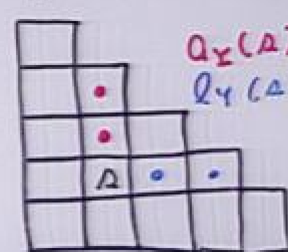

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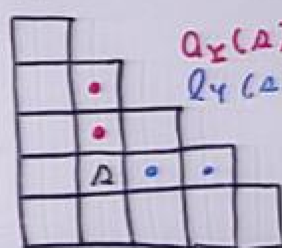

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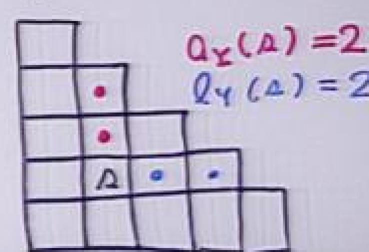

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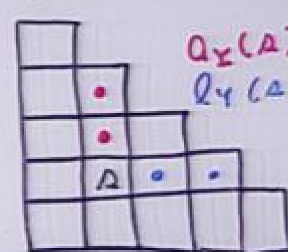

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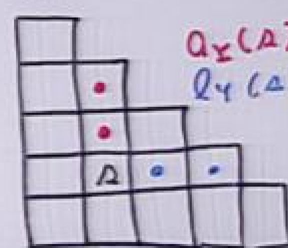

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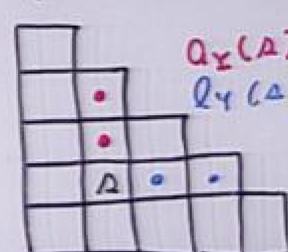

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9.

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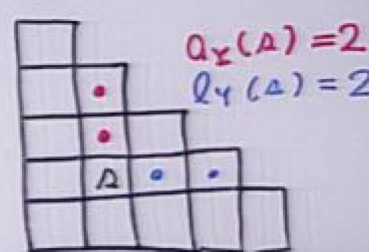

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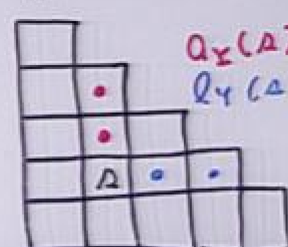

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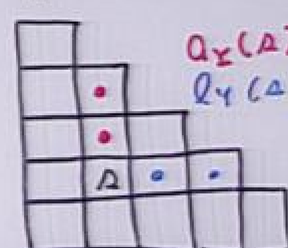

$a_Y(\Delta) = 2$

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- $M(n,r)^{\widetilde{T}}$

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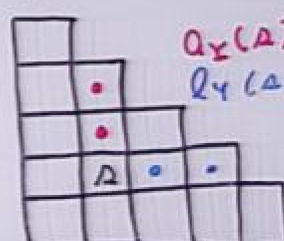

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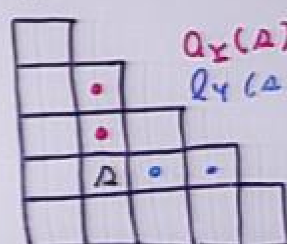

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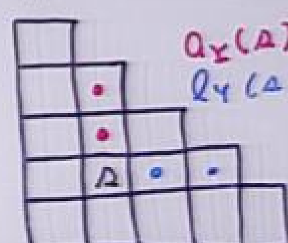

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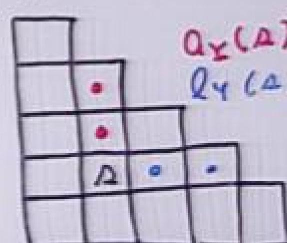

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Nekrasov's partition function

$$Z^{inst}_{4d}(\varepsilon_1 . \varepsilon_2 . \vec{a} : q) = \sum_{n \geq 0} q^n \int_{M(n.r)} 1$$

Sums over all Young diagrams & all boxes are taken into account.

Rmk.

$$\begin{array}{ccc} \{\vec{Y}\} = M(n.r)^{\tilde{T}} & \xrightarrow{\tilde{\iota}} & M(n.r) \\ \downarrow \pi' & & \downarrow \pi \\ M_0(n.r)^{\tilde{T}} & \xrightarrow{\iota} & M_0(n.r) \\ \| & & \\ \{pt\} & & \end{array}$$

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· <u>geometric engeneering</u>
gauge theory $\longleftarrow$ string theory

Set $\varepsilon_1 = -\varepsilon_2 = \hbar$
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(1). (2) — Nakajima – Yoshioka.
Nekrasov – Okounkov.

(3) — Iqbal + Kashani-Poor,
Zhou.

• Two aspects of study

① The structure of the generating function.

$\longleftrightarrow$ (1). (2)

② The meaning of the indivisual coefficients.

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also interesting from combinatorics,
Rep. theory view point

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• Two aspects of study

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Nekrasov's Conjecture.

(1) $\varepsilon_1\varepsilon_2 \log Z^{inst}_{4d}(\varepsilon_1.\varepsilon_2.\vec{a}:g)$
is regular at $\varepsilon_1=\varepsilon_2=0$

(2) $\varepsilon_1\varepsilon_2 \log Z^{inst}_{4d}\Big|_{\varepsilon_1=\varepsilon_2=0} = \mathcal{F}_0^{inst}$
is the instanton part of the SW prepotential.

· geometric engeneering
gauge theory ⟵ string theory

Set $\varepsilon_1 = -\varepsilon_2 = \hbar$

$$-\log Z^{inst}_{(5d)}(\hbar,-\hbar,\vec{a}.g)$$
$$= \frac{1}{\hbar^2}\mathcal{F}_0 + \mathcal{F}_1 + \hbar^2\mathcal{F}_2 + \cdots\cdots + \hbar^{2g-2}\mathcal{F}_g + \cdots$$

(3) $\mathcal{F}_g$ = genus g GW invariants of certain local CY 3 fold
[e.g. $K_{\mathbb{P}^1\times\mathbb{P}^1}$ for SU(2) case]

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ALE spaces.

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$c_1(E)$, $ch_2(E)$,

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• ∃ ADHM construction on X!

[Kronheimer – Nakajima
Douglas – Moore (D-brane)]

Philosophically, consider everything Γ-equivariantly.

V, W : unitary Γ-modules
with $\dim V = n$, $\dim W = r$

• $\mathrm{Hom}_\Gamma(V, V \otimes Q) \oplus \mathrm{Hom}_\Gamma(W, V) \oplus \mathrm{Hom}_\Gamma(V, W)$

[Q : 2-dim rep of Γ
defined by $\Gamma \subset SL(2, \mathbb{C})$]

+ same ADHM equation

⇝ the situation is essentially
the same with $\mathbb{C}^2$!!
We can apply same method
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- $\exists$ singularities on instanton moduli

$\rightsquigarrow$ certain moduli sp of sheaves gives a nice smooth (partial) compactification of instanton moduli.

Assume $\begin{cases} \cdot\ c_1 = 0 \\ \cdot\ \rho = \text{trivial} \end{cases}$ for simplicity

<u>Prop.</u> A component of $M(n,r)^{\Gamma}$ gives $\widetilde{T}$-equivariant resolution of singularities of $\mathcal{M}(c_1, ch_2, \rho)$.

$M(n,r)^{\Gamma}$ = Γ-fixed point set of $M(n,r)$

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<u>Rmk.</u>
Here we need theory of Quiver varieties.

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$$M^{\Gamma}(n.r) = \coprod_{\vec{v}} M(\vec{v})$$

$$\vec{v} = (v_0 \cdots v_\ell) \longleftrightarrow \oplus R_i^{\oplus v_i}$$

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$\Gamma \curvearrowright \mathbb{C}[x.y]$

$x \longmapsto \zeta x, \quad y \longmapsto \zeta^{-1} y$

- monomials in $x.y$

  $\rightsquigarrow$ 1-dim rep. of Γ

- Young diagram Y

  $\rightarrow$ $|Y|$-dim rep of Γ

e.g.

$\Gamma = \mathbb{Z}_3$

$$\begin{array}{|c|c|c|}\hline y^2 & & \\ \hline y & xy & \\ \hline 1 & x & x^2 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & & \\ \hline 2 & 0 & \\ \hline 0 & 1 & 2 \\ \hline \end{array} \qquad R_0^{\oplus 2} \oplus R_1^{\oplus 2} \oplus R_2^{\oplus 2}$$

- $\vec{Y} \in M(\vec{v})$

  $\Leftrightarrow \bigoplus Y_\alpha \simeq \bigoplus R_i^{\oplus v_i}$

  as a rep of Γ.

- Partition function on ALE spaces

  — sums over collored partition

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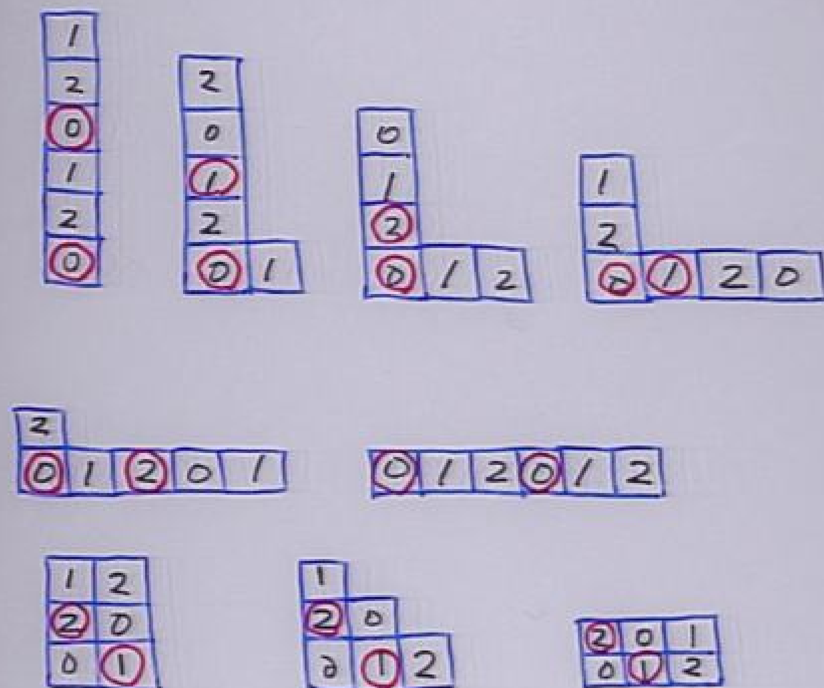

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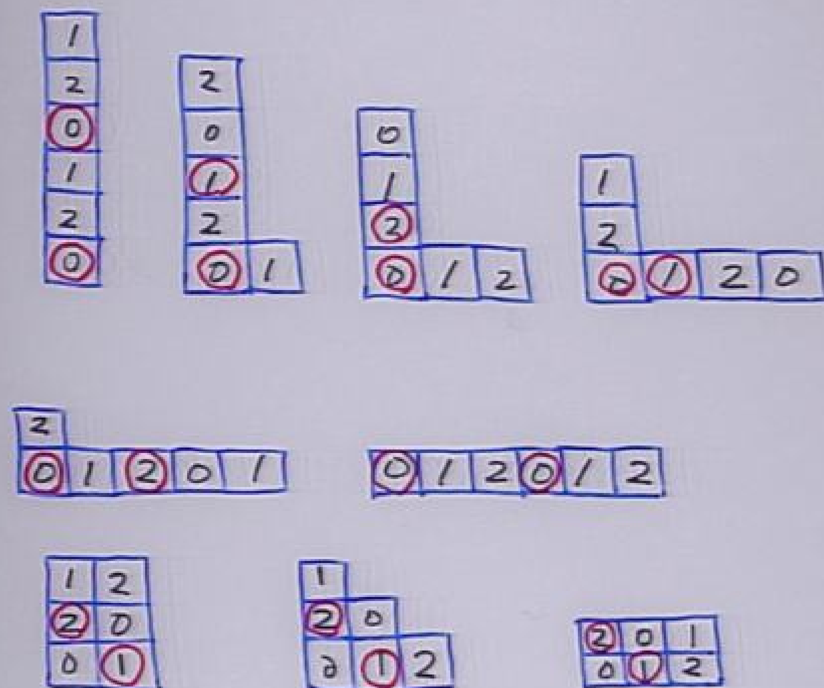

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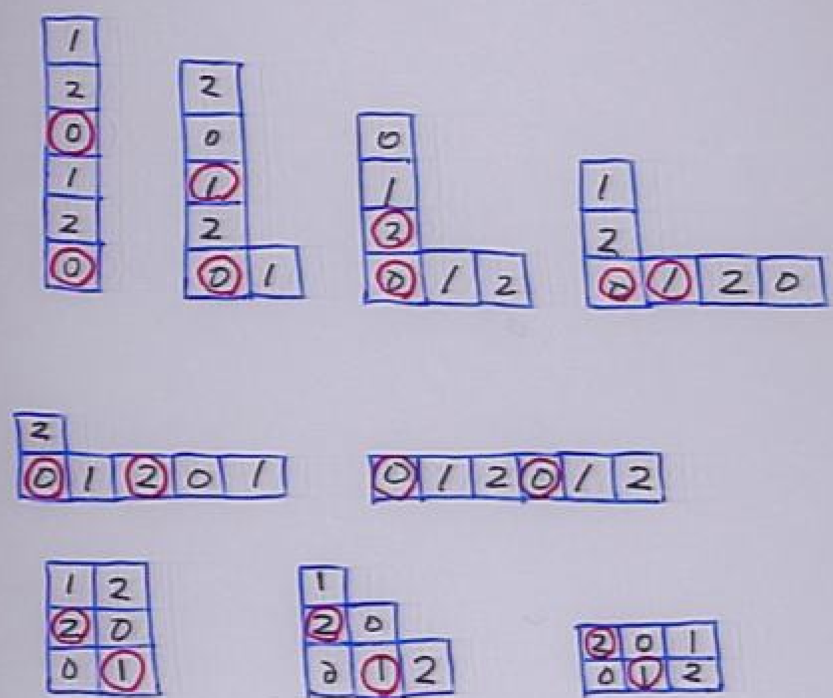

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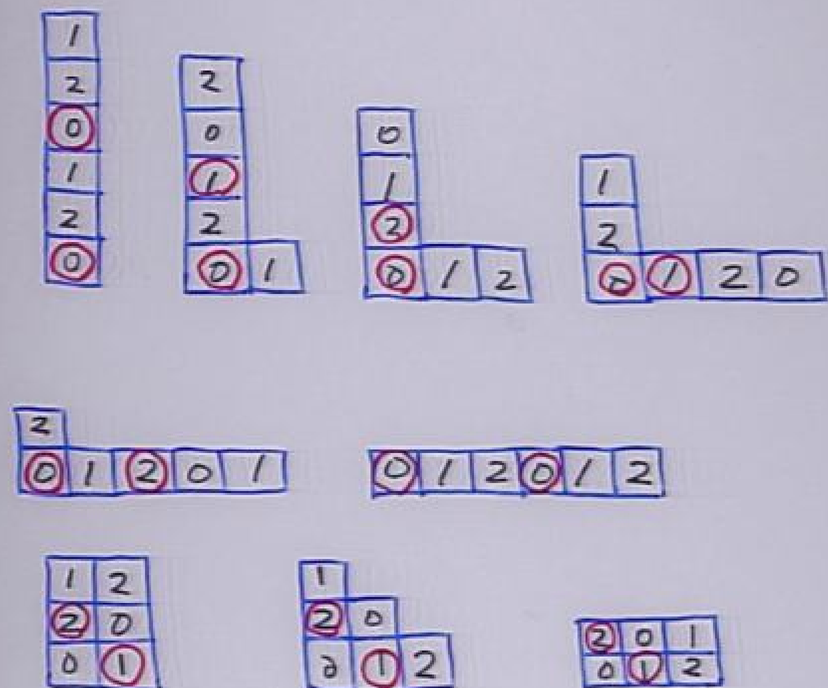

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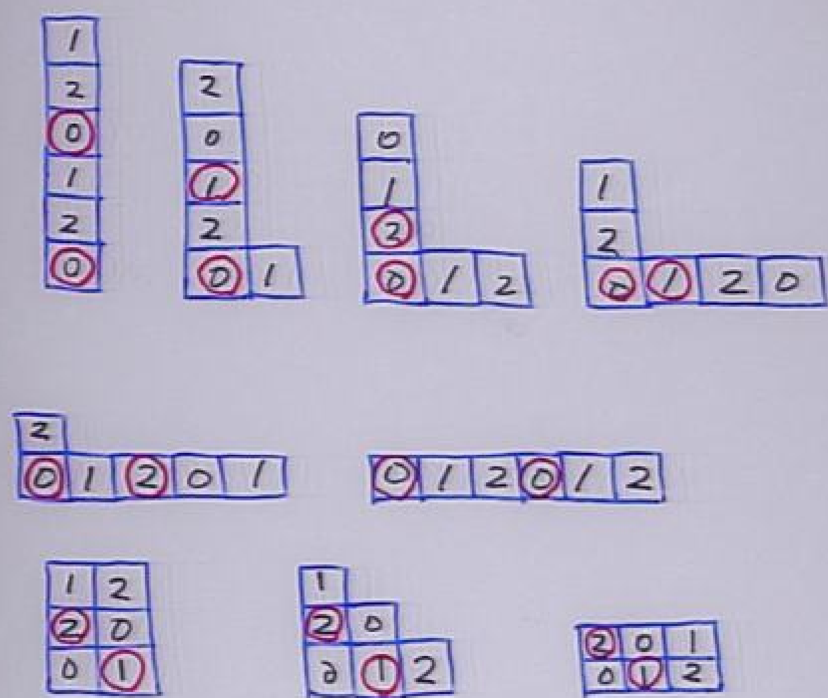

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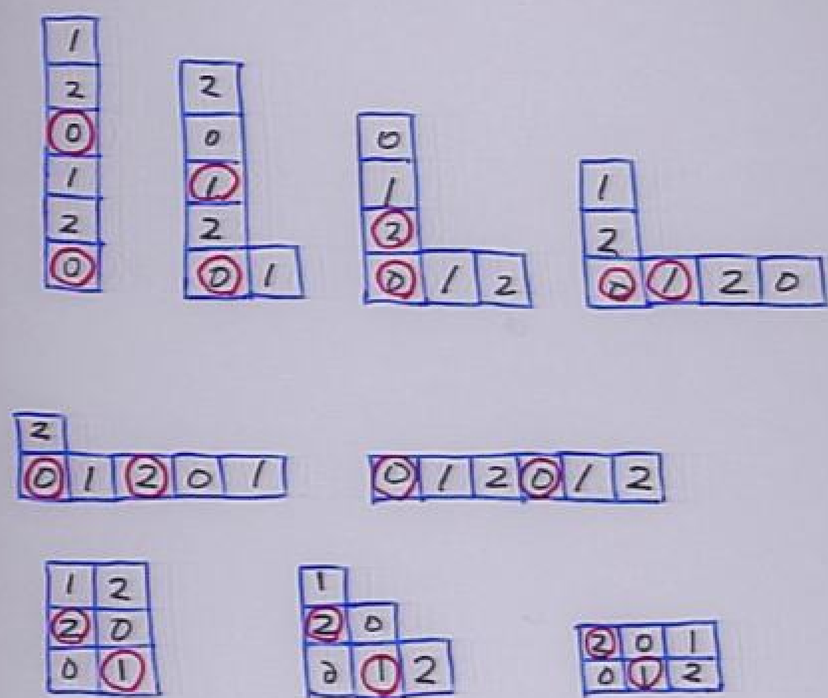

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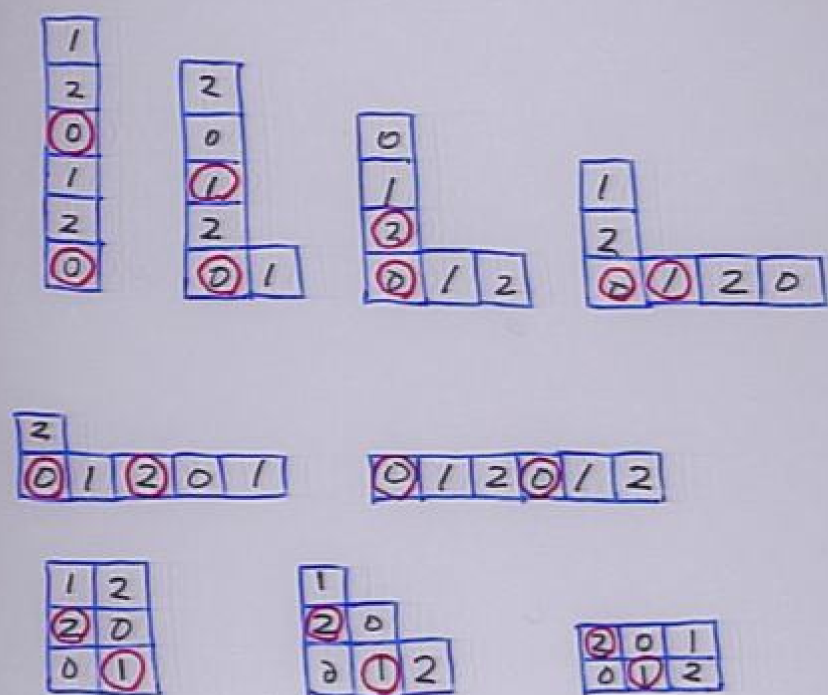

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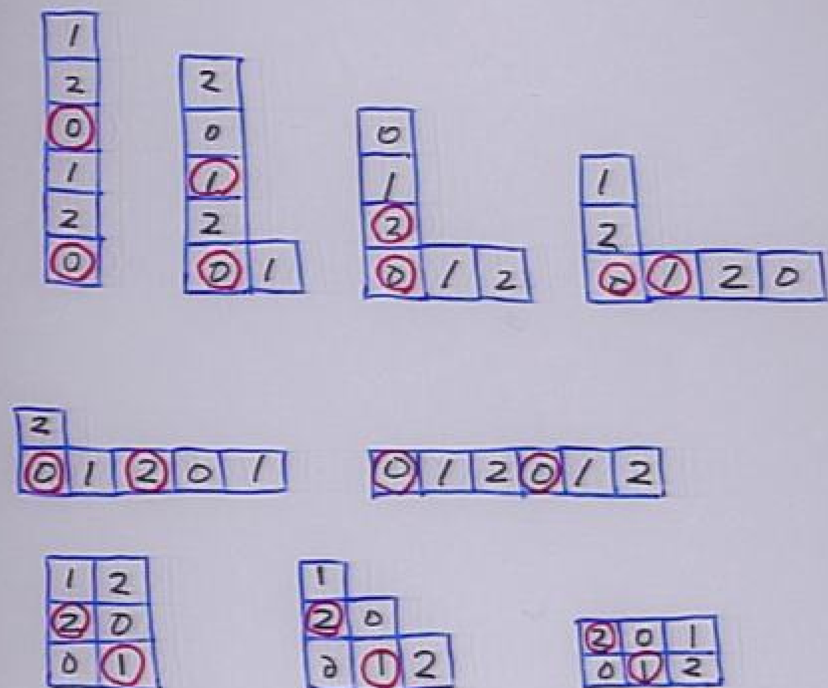

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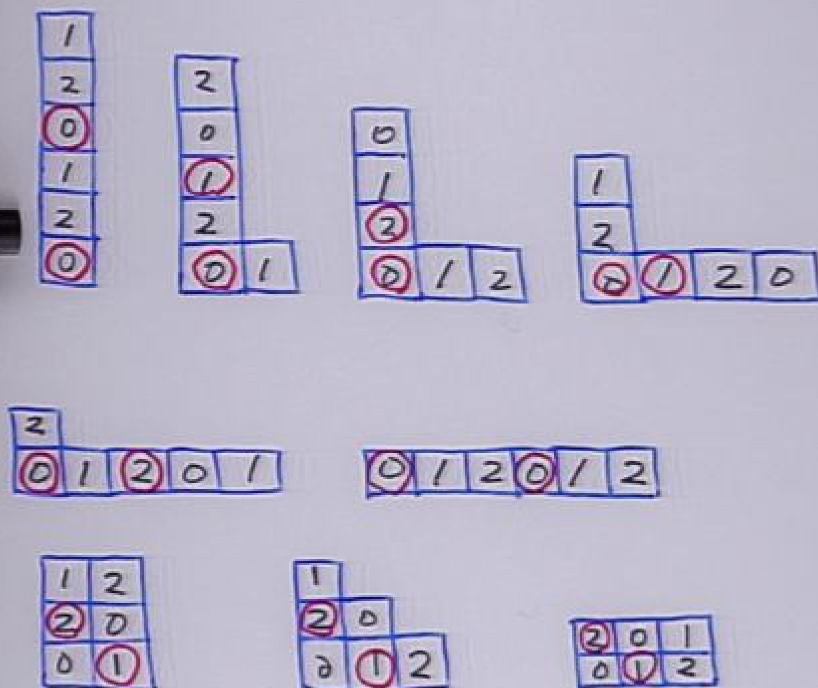

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