Title: Quasi-normal modes for black-holes with generic singularities

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URL: http://pirsa.org/04120007

Abstract:

Perturbations of a Black Hole

$$\boxed{\frac{d^2R(r)}{dx^2} + \left[\omega^2 - V(r)\right] \; R(r) = 0}$$

$$\omega = \Re(\omega) + i\Im(\omega)$$

$$\delta A = \delta(16\pi M^2) = 32\pi M \, \delta M \equiv 32\pi M \omega_{QNM}$$

$$\uparrow$$

$$[\; \delta A = 8\pi\gamma\sqrt{j_m(j_m+1)} \;\; \Rightarrow \gamma = \frac{4M\omega_{QNM}}{\sqrt{j_m(j_m+1)}} \leftarrow \text{Immirzi parameter}]$$

$$S = N\ln(2j_m+1) = \left[\frac{A}{\delta A}\right] \; \ln(2j_m+1) = \frac{A}{32\pi M_{QNM}} \ln(2j_m+1)$$

$$= \frac{A}{4} \, \frac{\ln(2j_m+1)}{8\pi M\omega_{QNM}} = S_{BH} \; \frac{\ln(2j_m+1)}{8\pi M\omega_{QNM}}$$

Question: $\ln(2j_m+1) = 8\pi M\omega_{QNM}$?

Yes: $\omega_{QNM} = \frac{\ln(3)}{8\pi M} \;\; \Rightarrow j_m = 1$

O. Dreyer (gr-qc/0211076)

QNM and Black Hole Area Spectrum

$$\delta A = 4\ln(3) \;\rightarrow\; A = 4n\ln(3) \;\rightarrow \Omega_{BH} = \exp(A/4) = 3^n \text{ (integer)}$$

What About Other Black Holes?

First: $\ln(3) \rightarrow \ln(k)$

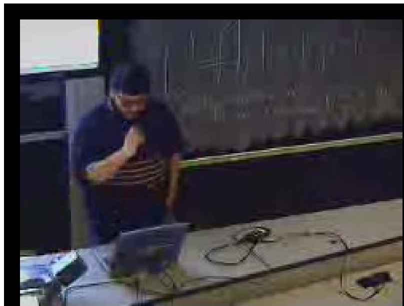
Second: $S_{BH} = \frac{A}{4} = [\ln(k)]\,\mathbf{n} \leftarrow$ Adiabatic invt (Bekenstein)?

If so, then: Periodic system: $S_{BH} \sim \int \frac{dE}{\omega(E)}$

Third: $\omega_{QNM} = \frac{\ln(3)}{8\pi M} \rightarrow T_H \ln(k)$

$\Rightarrow$ Adia Invt. $= \frac{1}{\ln(k)} \int \frac{dE}{T_H} \propto S_{BH}$

J. D. Bekenstein, Phys. Rev. **D7** (1973) 2333;
S. Hod, gr-qc/9812002; G. Kunstatter, gr-qc/0212014.



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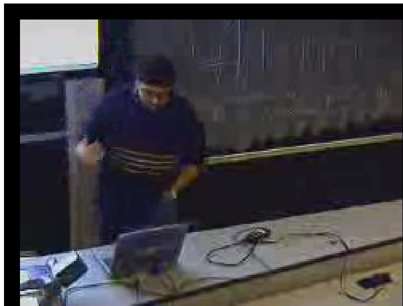


$\boxed{\text{Horizon Area = Adiabatic Invariant?}}$

Reduced action for a Sph Symm BH:

$$I_{Red} = \int dt \left(P_M \dot{M} - \underbrace{H(M)}_{\text{No } P_M} \right)$$

Can. Trans. $(M, P_M) \to (X, \Pi_X)$ to incorporate periodicity $(\sim 1/T_H)$

$$\begin{aligned} X &= \sqrt{\frac{B(M)}{\pi}} \cos\left(2\pi P_M T_H(M)\right) \\ \Pi_X &= \sqrt{\frac{B(M)}{\pi}} \cos\left(2\pi P_M T_H(M)\right) \end{aligned}$$

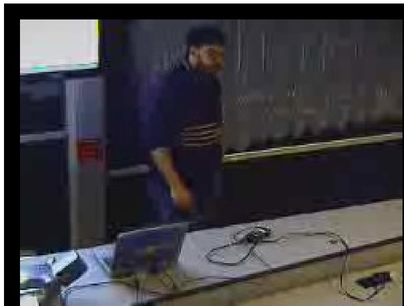
$\{X, \Pi_X\} = T_H \frac{\partial B(M)}{\partial M} \Rightarrow B(M) \equiv S_{BH}(M)$ (From First Law)

Square & add: $\hat{S}_{BH} = \pi\left(\hat{X}^2 + \hat{\Pi}_X^2\right) \Leftarrow$ SHO w/ $m = 1/2, \omega = 2$

$$\Rightarrow \boxed{S_{BH} = 2\pi\left(n + \tfrac{1}{2}\right)}$$

Adiabatic invt: $= \int \frac{dM}{T_H} = S_{BH}$ (First law)
$\propto n + \cdots$ (Bohr-Sommerfeld-Wilson)

A. Bravinsky, S. Das, G. Kunstatter, hep-th/0209039;



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A. Bravinsky, S. Das, G. Kunstatter, hep-th/0209039;
G. Gour, A. J. M. Medved, gr-qc/0211089;
S. Das, H. Mukhopadhyay, P. Ramadevi, hep-th/0407151 .

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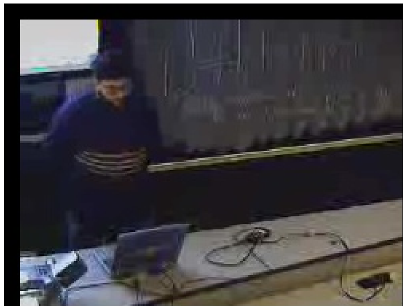

Spherically Symmetric Single Horizon Black Holes

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{dr^2}{g(r)} + \rho^2(r)d\Omega_D^2 \\ &= f(r)\left[-dt^2 + dx^2\right] + \rho^2(r)d\Omega_D^2 \end{aligned}$$

$$x = \int \frac{dr}{\sqrt{f(r)g(r)}} \quad \text{(Tortoise)}$$

Near horizon (r_h)

$$f(r) = f'(r_h)\,(r - r_h);\; g(r) = g'(r_h)\,(r - r_h)$$

$$T_H = \frac{1}{4\pi}\sqrt{g'(r_h)\,f'(r_h)}$$

$$x = c_0(r_h)\,\ln(r - r_h)$$

For example: $f(r) = g(r) = 1 - \left(\frac{r_h}{r}\right)^{D-1}$. $T_H = \frac{(D-1)}{4\pi r_h}$ (Schwarzschild)

$$f(r) = g(r) = \frac{(r^{D-1} - r_h^{D-1})^2}{r^{2(D-1)}} \quad (\textit{Extremal}\ \text{RN})$$

Near the Singularity ($y = \beta t$)

$$\begin{aligned} ds^2 &\overset{r\to 0}{\simeq} \eta r^{2p/q}dy^2 - \frac{4\eta}{q^2}r^{2(p-q+2)/q}dr^2 + r^2 d\Omega_D^2 \\ &= \eta x^p(dy^2 - dx^2) + x^q d\Omega_D^2 \quad \text{[Generic Power Law Singularity]} \end{aligned}$$

($\eta = 1, 0, -1$ Space/Null/Timelike)

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$$= f(r)\left[-dt^2 + dx^2\right] + \rho^2(r)d\Omega_D^2$$

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$$R_{abcd}R^{abcd} = \frac{cx^{-2p} + dx^{-2q+4} + ex^{-p-q+2}}{x^4}$$

P. Szekeres, V. Iyer, Phys. Rev. **D47** (1993) 4362.

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For example:

$$p = \frac{1-D}{D}; \quad q = \frac{2}{D}; \quad x = \frac{r^D}{\beta} \ \text{(Schwarzschild)}$$

$$p = -2\left(\frac{D-1}{2D-1}\right); \quad q = \frac{2}{2D-1}; \quad x = \frac{r^{2D-1}}{\beta} \ \text{(Non-Extr RN)}$$

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In general:

$$f(r) = -\eta\beta^2 r^{2p/q};\ \frac{1}{g(r)} = -\frac{4\eta}{q^2} r^{2(p-q+2)/q};\ \rho(r) = r\ ;\ x = \frac{r^{2/q}}{\beta}$$

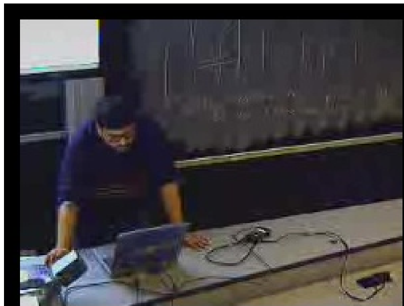
$\boxed{\text{Quasi Normal Modes}}$

$$\Box\,\Phi \equiv \frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\right) = 0\,,$$

$$\frac{d^2R(r)}{dx^2} + \left[\omega^2 - V(r)\right]R(r) = 0$$

$$\begin{aligned} V(r) \;&=\; \frac{l(l+D-1)}{r^2}\,f(r) + \left(\frac{D}{2}\right)\rho(r)^{-\frac{D}{2}}\sqrt{f(r)\,g(r)} \\ &\times\; \frac{d}{dr}\left(\rho(r)^{\frac{D-2}{2}}\frac{d\rho(r)}{dr}\sqrt{f(r)\,g(r)}\right)\ \text{[Regge-Wheeler potential]} \end{aligned}$$

$$V(r) \overset{r\to r_h}{\simeq} \left[\frac{l(l+D-1)}{\rho^2(r_h)}f'(r_h) + \frac{D}{2}f'(r_h)\,g'(r_h)\,\frac{\rho'(r_h)}{\rho(r_h)}\right](r-r_h) + \mathcal{O}[(r-r_h)^2]$$



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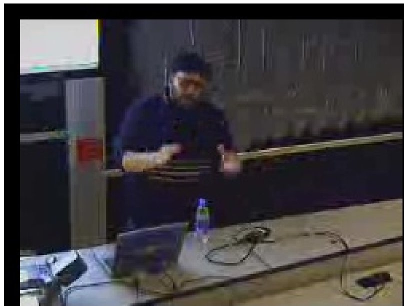
$$\boxed{\text{Solution (with } R(x)\sim e^{\pm i\omega x} \text{ as } x\to\mp\infty)}$$



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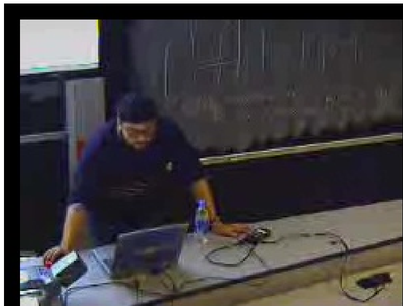

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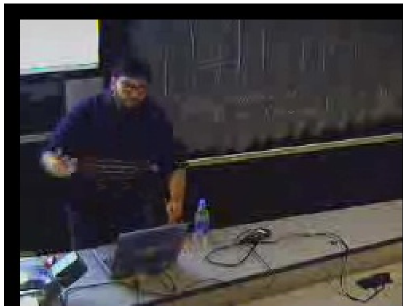

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$$A_+ e^{-i\alpha_+} + A_- e^{-i\alpha_-} = 0$$

$$R(z) \sim \left[A_+ e^{+i\alpha_+} + A_- e^{+i\alpha_-}\right] e^{-i\omega z} \quad \text{as} \quad \omega z \to \infty$$

Rotate: $\Im(\omega z) \to \infty \to \Im(\omega z) \to -\infty$
$r \to r\exp(i3\pi q/2) \ \Rightarrow \ z \to \exp(i3\pi q/2 \times 2/q) = \exp(3i\pi)z.$
$J_\nu(ze^{im\pi}) = e^{im\nu\pi} J_\nu(z)$, (as $\omega z \to -\infty$)

$$\begin{aligned} R(z) \ &\sim \ \left(A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}\right) e^{-i\omega z} \\ &+ \ \left(A_+ e^{7i\alpha_+} + A_- e^{7i\alpha_-}\right) e^{+i\omega z} \end{aligned}$$

$$\begin{aligned} \text{Monodromy} &= \frac{A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}}{A_+ e^{i\alpha_+} + A_- e^{i\alpha_-}} \\ - \ \ \frac{e^{6i\alpha_+} - e^{6i\alpha_-}}{e^{2i\alpha_+} - e^{2i\alpha_-}} &= \frac{\sin(3\pi\nu)}{\sin(\pi\nu)} = 1 + 2\cos\left(\frac{\pi}{2}[Dq-2]\right) \end{aligned}$$

<u>Near horizon ($z \to -\infty$):</u>

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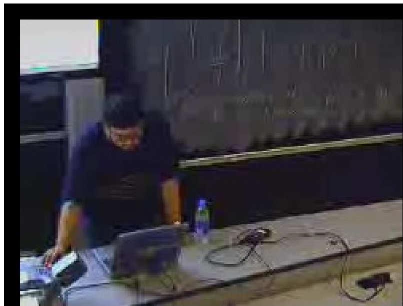

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$r \to r\exp(i3\pi q/2) \;\Rightarrow\; z \to \exp(i3\pi q/2 \times 2/q) = \exp(3i\pi)z.$
$J_\nu(ze^{im\pi}) = e^{im\nu\pi} J_\nu(z)$, (as $\omega z \to -\infty$)

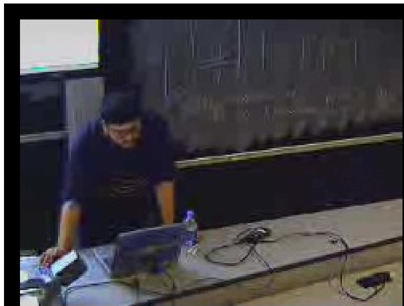
$$\begin{aligned} R(z) \;&\sim\; \left(A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}\right) e^{-i\omega z} \\ &+\; \left(A_+ e^{7i\alpha_+} + A_- e^{7i\alpha_-}\right) e^{+i\omega z} \end{aligned}$$

$$\text{Monodromy} = \frac{A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}}{A_+ e^{i\alpha_+} + A_- e^{i\alpha_-}}$$

$$- \frac{e^{6i\alpha_+} - e^{6i\alpha_-}}{e^{2i\alpha_+} - e^{2i\alpha_-}} = \frac{\sin(3\pi\nu)}{\sin(\pi\nu)} = 1 + 2\cos\left(\frac{\pi}{2}[Dq - 2]\right)$$

Near horizon ($z \to -\infty$):

$R(z) \sim e^{+i\omega z} \sim \exp[i\,c_0\,\omega \ln(x - x_h)]$, Monodromy $= \exp(4\pi\omega c_0)$



$$c_\pm\sqrt{\omega z}\,J_{\pm\nu}(\omega z) \sim 2\cos(\omega z - \alpha_\pm) \quad \text{as} \quad \omega z \to \infty\,; \quad (\alpha_\pm \equiv \tfrac{\pi}{4}[1 \pm 2\nu]\,)$$

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Equating: $$\boxed{\omega_{QNM} = \frac{i}{2c_0}\left[n + \frac{1}{2}\right] \pm \frac{1}{4\pi c_0} \log\left[1 + 2\cos\left(\frac{\pi}{2}[Dq-2]\right)\right]}$$

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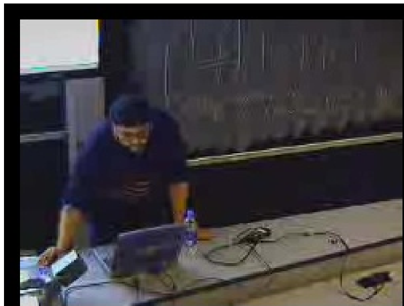


$$\boxed{\Re\left(\omega_{QNM}\right) = \pm\frac{1}{4\pi c_0}\log\left[1+2\cos\left(\frac{\pi}{2}[Dq-2]\right)\right]}$$

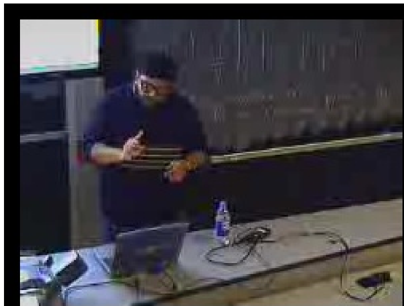
↑ Horizon ↑ Singularity (no p)

Note:

$\Re(\omega_{QNM}) =$ Integer, iff

$$(a)\quad \left(\frac{Dq-2}{2}\right)\pi = 2\,j\pi \quad \text{where} \quad j \in \mathcal{Z}$$
$$(b)\quad \left(\frac{Dq-2}{2}\right)\pi = \frac{m\,\pi}{3} \quad \text{where} \quad m \in \mathcal{Z}$$

L. Motl, A. Neitzke, hep-th/0301173;
SD, S. Shankaranarayanan, hep-th/0410209.



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$$\Re\left(\omega_{QNM}\right) = \pm\frac{1}{4\pi c_0}\log\left[1 + 2\cos\left(\frac{\pi}{2}[Dq-2]\right)\right]$$

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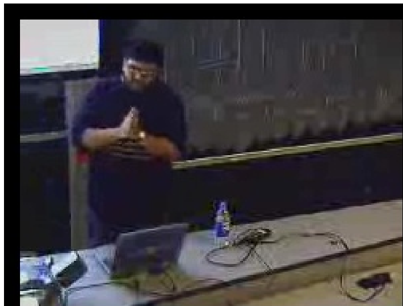


Applications

$\boxed{(D+2)\text{-dimensional Schwarzschild}}$

$$f(r) = g(r) = 1 - \left(\frac{r_h}{r}\right)^{D-1} \; ; \; T_H = \frac{(D-1)}{4\pi r_h}$$

$\boxed{(D+2)\text{-dimensional Extremal RN}}$

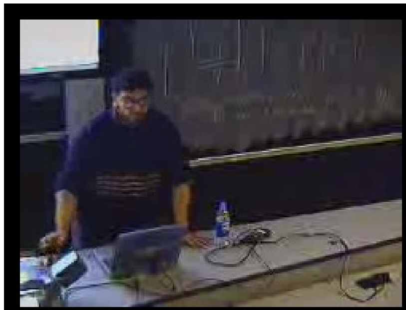
$$f(r) = g(r) = \frac{(r^{D-1} - r_h^{D-1})^2}{r^{2(D-1)}} \; ; \; T_H = 0$$

$\boxed{\text{2-Dim Stringy BH}}$

$$f(r) = g(r) = 1 - \frac{M}{\lambda}\, e^{2\lambda r}$$

$$r_H = \frac{1}{2\lambda} \ln\left(\frac{\lambda}{M}\right) \; ; \; T_H = \frac{\lambda}{2\pi}$$

E. Witten, Phys. Rev. **D44** (1991) 314;
G. Mandal et al, Mod. Phys. Lett. **A6** (1991) 1685;
CGHS hep-th/9111056



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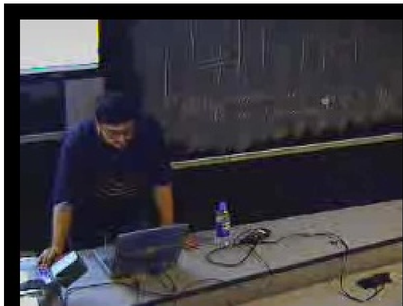


$\boxed{\text{4-Dim Stringy BH}}$

$$
\begin{aligned}
f(r) &= g(r) = \left(1-\frac{r_h}{r}\right)\left(1-\frac{r_0}{r}\right)^{(1-\alpha^2)/(1+\alpha^2)}; \\
\rho(r) &= r\left(1-\frac{r_0}{r}\right)^{\alpha^2/(1+\alpha^2)} \qquad [\alpha=0 \to \ \mathrm{RN}] \\
M &= \frac{r_h}{2}+\left[\frac{1-\alpha^2}{1+\alpha^2}\right]\frac{r_0}{2}; \quad Q=\left[\frac{r_h\, r_0}{1+\alpha^2}\right]^{1/2} \\
T_H &= \frac{1}{4\pi r_h}\left(1-\frac{r_0}{r_h}\right)^{(1-\alpha^2)/(1+\alpha^2)}
\end{aligned}
$$

Garfinkle, Horowitz, Strominger, Phys. Rev. **D43** (1991) 3140 etc

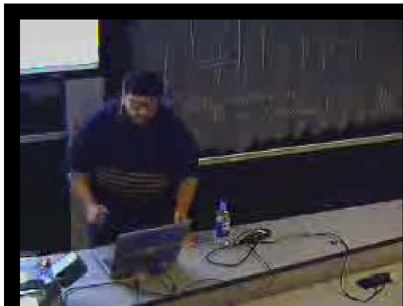
$\boxed{\text{5-Dim Stringy BH}}$

$$
\begin{aligned}
f(r) &= F^{-2/3}\left(1-\frac{r_h^2}{r^2}\right); \quad \rho(r)=F^{1/6}r; \\
g(r) &= F^{-1/3}\left(1-\frac{r_h^2}{r^2}\right).
\end{aligned}
$$

and

$$
F=\left[1+\frac{r_h^2\sinh^2\alpha}{r^2}\right]\left[1+\frac{r_h^2\sinh^2\gamma}{r^2}\right]\left[1+\frac{r_h^2\sinh^2\sigma}{r^2}\right].
$$

$$
\begin{aligned}
Q_1 &= \frac{Vr_h^2}{2g}\sinh 2\alpha\ ;\ Q_5=\frac{r_h^2}{2g}\sinh 2\gamma \quad (\#\text{ of } 1\&5-\text{branes}) \\
n &= \frac{R^2Vr_h^2}{2g^2}\sinh 2\sigma \qquad (\text{Momentum on 1-brane})
\end{aligned}
$$



$$T_H = \frac{1}{4\pi r_h}\left(1-\frac{r_0}{r_h}\right)^{(1-\alpha^2)/(1+\alpha^2)}$$

Garfinkle, Horowitz, Strominger, Phys. Rev. **D43** (1991) 3140 etc

5-Dim Stringy BH

$$f(r) = F^{-2/3}\left(1-\frac{r_h^2}{r^2}\right); \quad \rho(r) = F^{1/6}r;$$
$$g(r) = F^{-1/3}\left(1-\frac{r_h^2}{r^2}\right),$$

and

$$F = \left[1+\frac{r_h^2\sinh^2\alpha}{r^2}\right]\left[1+\frac{r_h^2\sinh^2\gamma}{r^2}\right]\left[1+\frac{r_h^2\sinh^2\sigma}{r^2}\right].$$

$$Q_1 = \frac{Vr_h^2}{2g}\sinh 2\alpha\ ;\ Q_5 = \frac{r_h^2}{2g}\sinh 2\gamma \quad (\#\text{ of }1\&5-\text{branes})$$
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$$\sigma = 0 \Rightarrow \ r=0 \ \text{singular} \ \rightarrow \ T_H = (2\pi r_h\cosh\alpha\cosh\gamma)^{-1}$$

Horowitz, Maldacena, Strominger, hep-th/9603109

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Space-Time	Near horizon properties		Near singularity properties			$\Re[\omega_{\mathrm{QNM}}]$
	κ	c_0	p	q	$(Dq-2)/2$	
Non-degenerate Horizons	$\frac{\sqrt{f'(r_h)g'(r_h)}}{2}$	$\frac{1}{2\kappa}=\frac{1}{4\pi T_H}$	$p>q-2$	$q>0$	Any real value	$\frac{1}{4\pi c_0}\log\left[1+2\cos\left(\frac{\pi(Dq-2)}{2}\right)\right]$
(D + 2)-dimens. Schwarzchild	$\frac{(D-1)}{2r_h}$	$\frac{r_h}{D-1}$	$\frac{1-D}{D}$	$\frac{2}{D}$	0	$T_H \log 3$
4-dimensional stringy BH	$\frac{1}{2r_h}\left[1-\frac{r_0}{r_h}\right]^{\frac{1-\alpha^2}{1+\alpha^2}}$	$\frac{1}{2\kappa}$	$\frac{1-\alpha^2}{2\alpha^2}$	1	0	$T_H \log 3$
5-dimensional stringy BH	$\frac{r_h^{-1}}{\cosh\alpha\cosh\gamma}$	$\frac{1}{2\kappa}$	$-\frac{2}{3}$	$\frac{2}{3}$	0	$T_H \log 3$
2-dimensional stringy BH	1	$\frac{1}{2}$	1	0	-1	0
(D + 2)-dimens. Extremal RN	0	$\frac{D}{(D-1)^2}r_h$	$-2\left(\frac{D-1}{2D-1}\right)$	$\frac{2}{2D-1}$	$\left(\frac{1-D}{2D-1}\right)$	$\frac{1}{8\pi r_h}\log 2$

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$$\boxed{\Re\left(\omega_{QNM}\right) = \pm\frac{1}{4\pi c_0}\log\left[1+2\cos\left(\frac{\pi}{2}[Dq-2]\right)\right]}$$

$\uparrow$ Horizon $\qquad\qquad$ $\uparrow$ Singularity (no p)

Note:

$\Re\left(\omega_{QNM}\right) =$ Integer, iff

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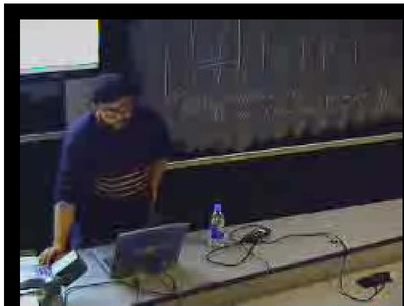

$$\boxed{\text{Solution (with } R(x) \sim e^{\pm i\omega x} \quad \text{as} \quad x \to \mp\infty)}$$

$$R(z) = A_+ c_+ \sqrt{\omega z} J_{+\nu}(\omega z) + A_- c_- \sqrt{\omega z} J_{-\nu}(\omega z) \quad (\nu = (D\,q - 2)/4)$$

$$c_\pm \sqrt{\omega z} J_{\pm\nu}(\omega z) \sim 2\cos(\omega z - \alpha_\pm) \quad \text{as} \quad \omega z \to \infty\,; \quad (\alpha_\pm \equiv \tfrac{\pi}{4}[1 \pm 2\nu]\ )$$

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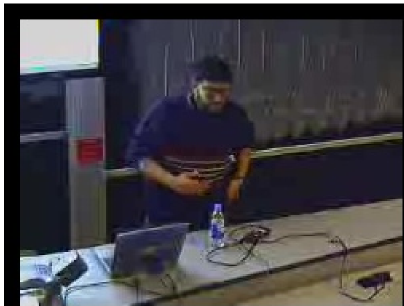

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Implications for LQG

$$\Delta A = 4 \underbrace{\frac{\Re(\omega_{QNM})}{T_H}}_{\text{First Law}} = 4 \log\left[1 + 2\cos\left(\frac{\pi(Dq-2)}{2}\right)\right]$$

$$\Delta A = 8\pi\gamma\sqrt{j_m(j_m+1)} \quad \text{(LQG)}$$

$$\gamma = \frac{\log\left[1 + 2\cos\left(\pi(q-1)\right)\right]}{2\pi\sqrt{j_m(j_m+1)}} \quad \text{(Immirzi)}$$

$$N = A/\Delta A$$

$$\begin{aligned} S &= N \log(1 + 2j_m) \\ &= \frac{A}{4} \frac{\ln(1+2j_m)}{\log\left[1 + 2\cos\left(\pi(q-1)\right)\right]} \end{aligned}$$

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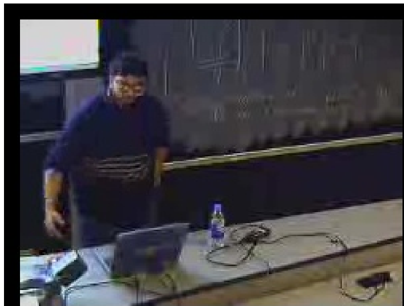

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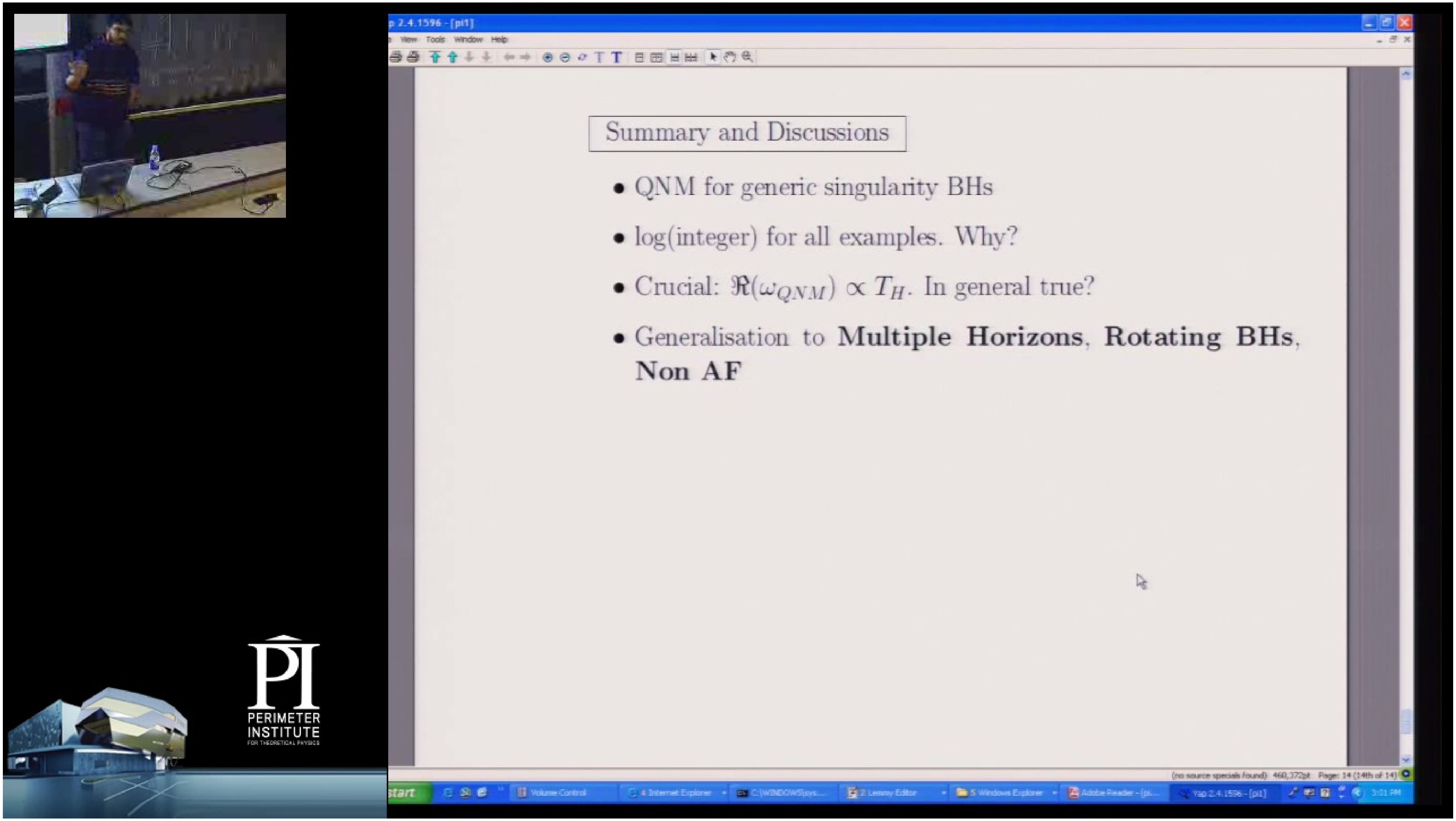
Summary and Discussions
• QNM for generic singularity BHs
• log(integer) for all examples. Why?
• Crucial: $\Re(\omega_{QNM}) \propto T_H$. In general true?
• Generalisation to **Multiple Horizons**, **Rotating BHs**, **Non AF**
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