Title: Quantum Walks and algorithmic speedup

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Abstract:

Quantum Walks and Algorithmic Speedup

Richard Cleve (Waterloo)

Joint work with: Andrew Childs (MIT)
Enrico Deotto (MIT)
Eddie Farhi (MIT)
Sam Gutmann (Northeastern)
Dan Spielman (MIT)

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Motivation

Find new quantum algorithms

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Shor's algorithms:
based on the ***Quantum Fourier Transform***

[Shor '94]

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Grover's algorithm:
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[Shor '94][Grover '96]

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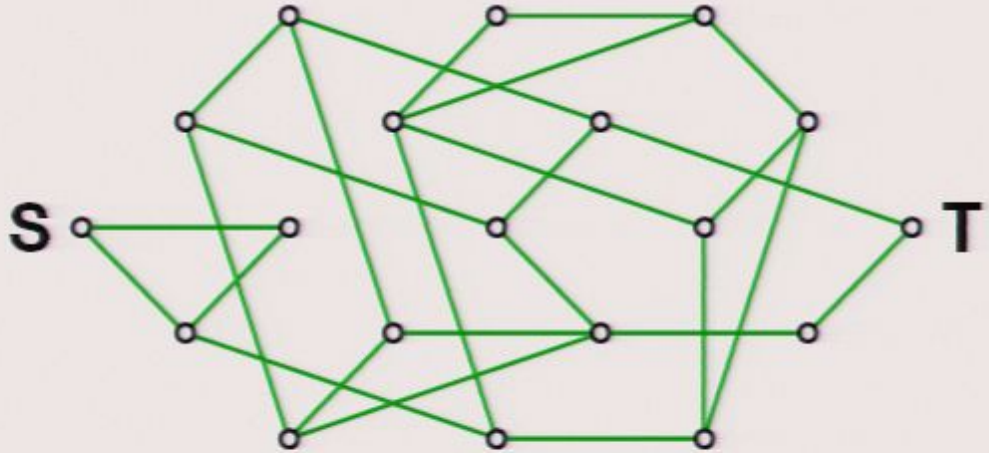
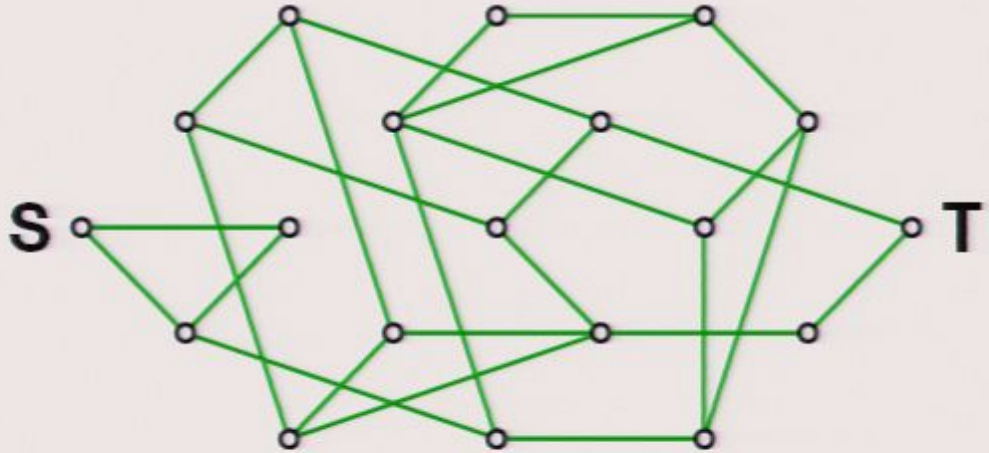
Are there other useful algorithmic techniques?

Classical random walks



Algorithmic tools for searching and sampling

Classical random walks

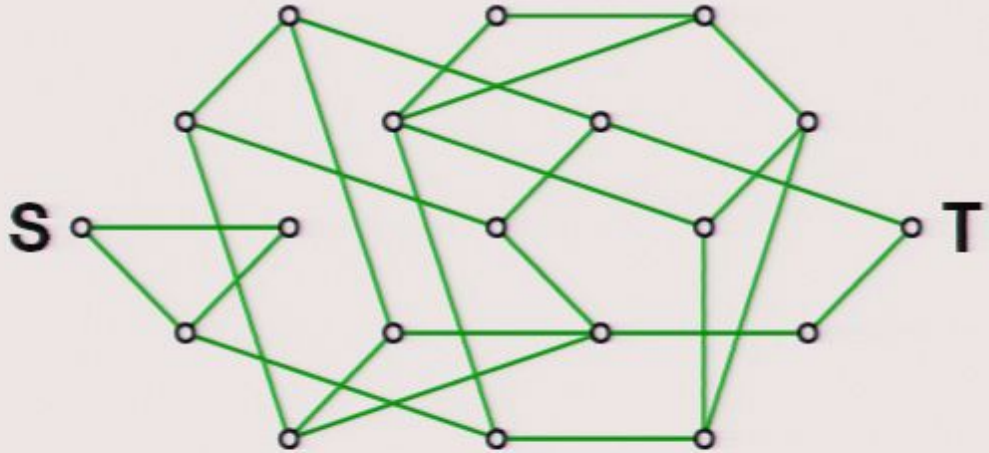


Algorithmic tools for searching and sampling

Example 1: ST-connectivity (undirected graphs)

[Aleliunas, Karp, Lipton, Lovász & Rackoff '79]

Classical random walks



Algorithmic tools for searching and sampling

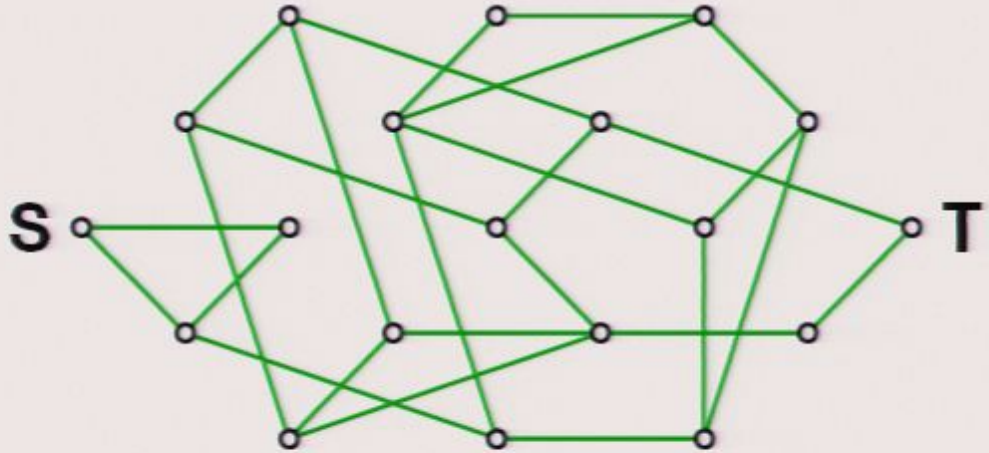
Example 1: ST-connectivity (undirected graphs)

A very space efficient algorithm:

1. perform a random walk starting from **S**
2. if **T** is ever reached then **YES**, else **NO**

[Aleliunas, Karp, Lipton, Lovász & Rackoff '79]

Classical random walks



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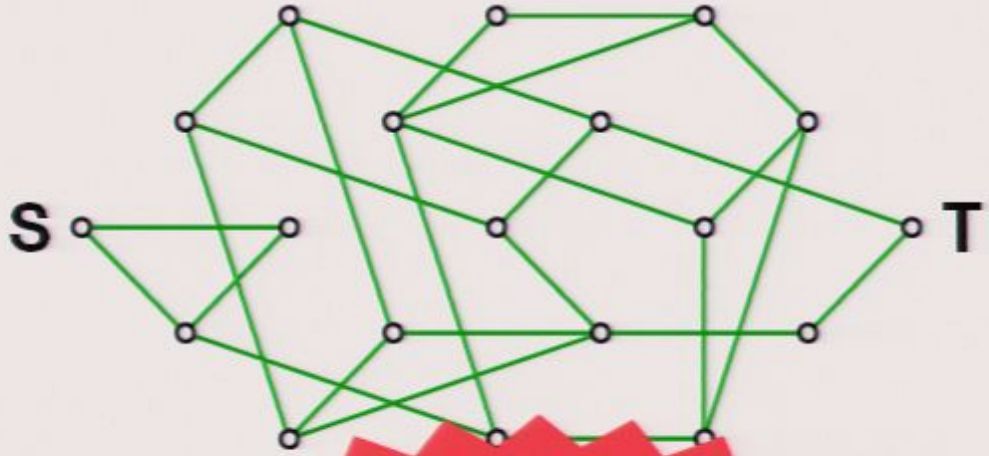
Space $O(\log n)$ and time $O(n^3)$ for ***any*** n-vertex graph,

[Aleliunas, Karp, Lipton, Lovász & Rackoff '79] 3

Classical random walks



Algorithmic tools fo

Example 1: ST-

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1. perform a rando
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November 8, 2004:
deterministic algorithm
using similar space/time

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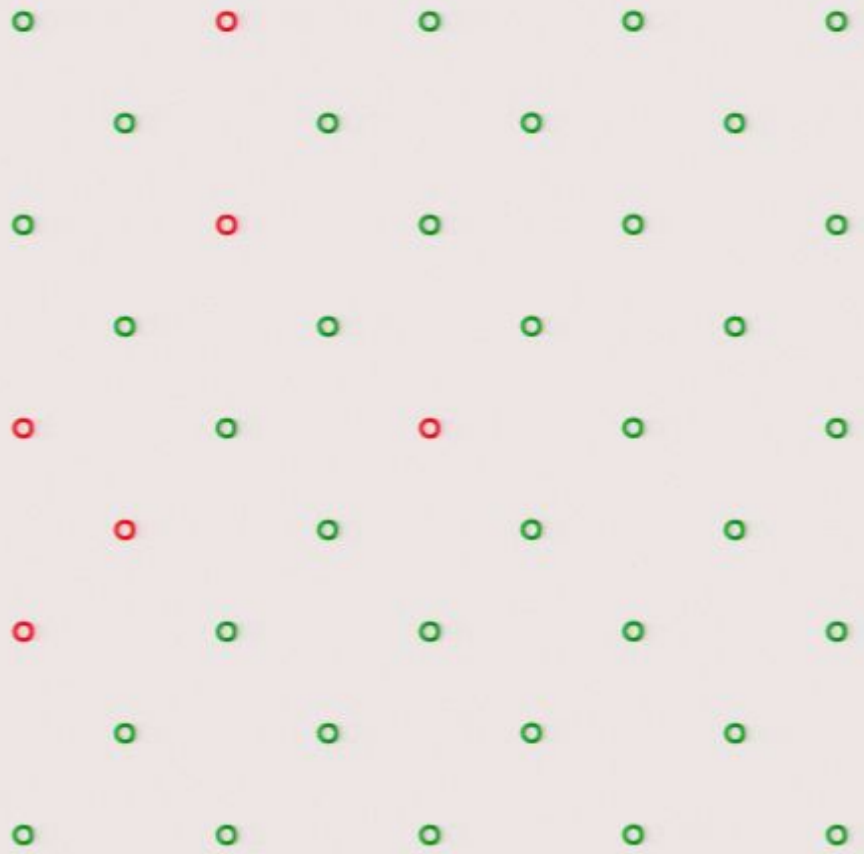
[Aleliunas, Karp, Lipton, Lovász & Rackoff '79] [Reingold '04]

Random walks as algorithmic tools

Example 2: uniform samping on a subset

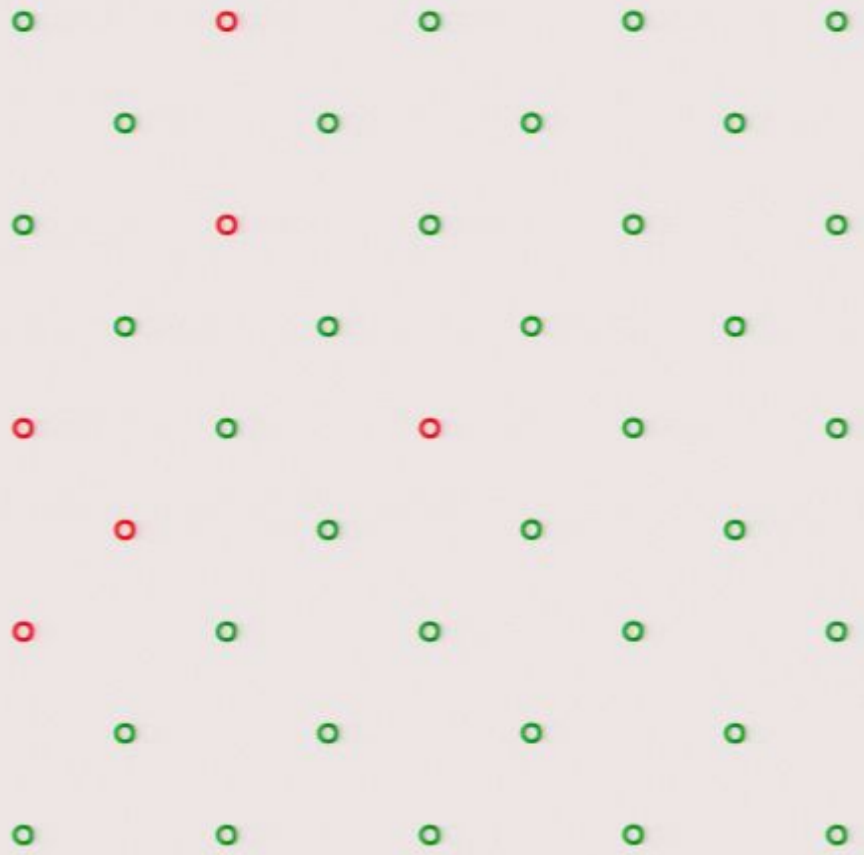


[Broder '86][Jerrum & Sinclair '89][L-S & Vigoda '01] 4

Random walks as algorithmic tools

Example 2: uniform samping on a subset

Execute random walk on a related graph that quickly approaches uniform distribution



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Random walks as algorithmic tools

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Random walks as algorithmic tools

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Basis of some very good approximation algorithms

[Broder '86][Jerrum & Sinclair '89][L-S & Vigoda '01] 4

Random walks as algorithmic tools

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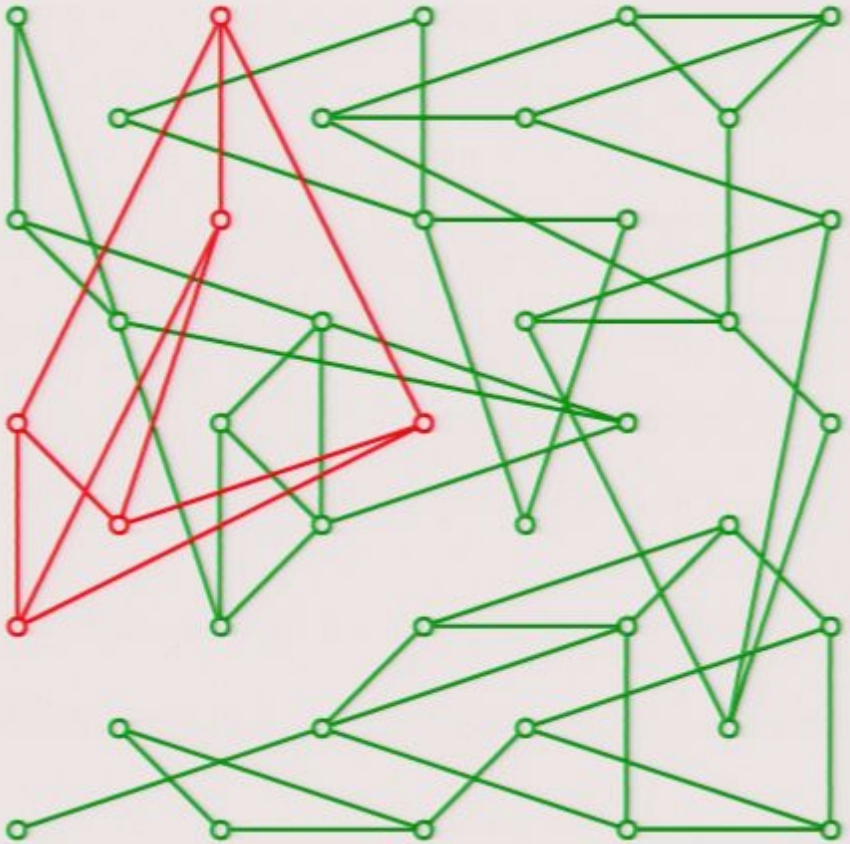
Basis of some very good approximation algorithms

E.g., approximating the ***permanent*** of a matrix



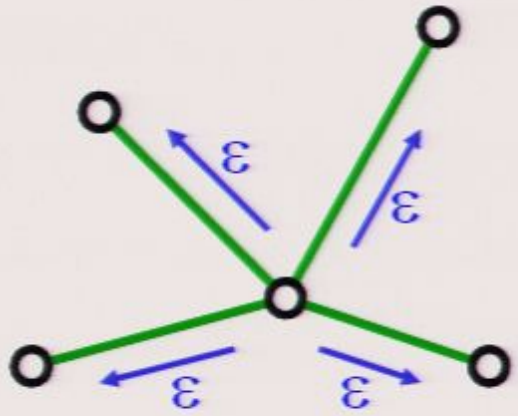
[Broder '86][Jerrum & Sinclair '89][J., S. & Vigoda '01] 4

Quantum walks on graphs

Quantum walks on graphs

Continuous-time *random* walk:

in each small ε time interval, move to each neighbor with probability ε (lim $\varepsilon \to 0$, like a Poisson process)



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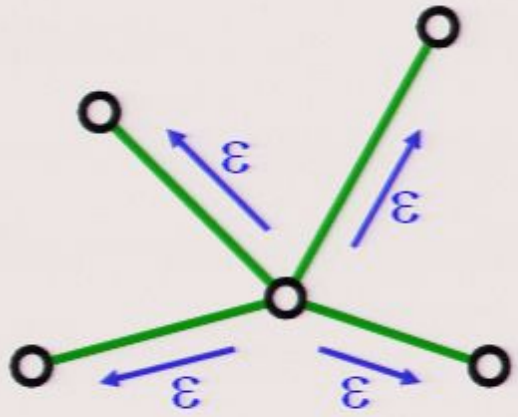
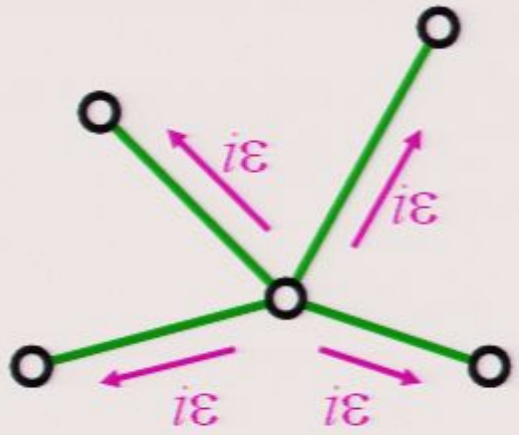


Quantum analog:

in ε time interval, move to neighbors with amplitude $i\varepsilon$ (lim $\varepsilon \to 0$)



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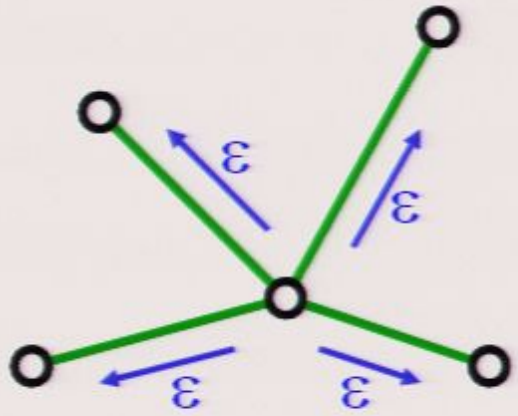

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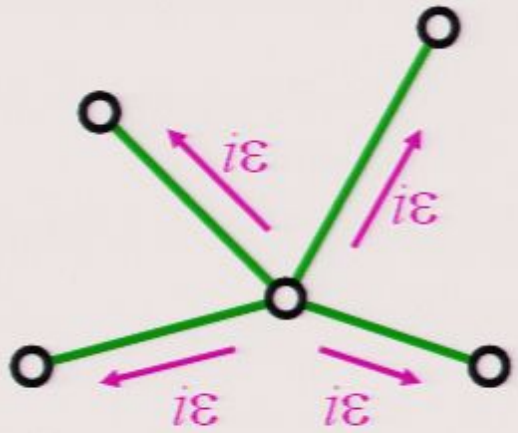
Evolving for time t: e^{-iAt} |initial state⟩

where $A_{xy} = \begin{cases} 1 & \text{if } (x,y) \text{ edge} \\ 0 & \text{otherwise} \end{cases}$

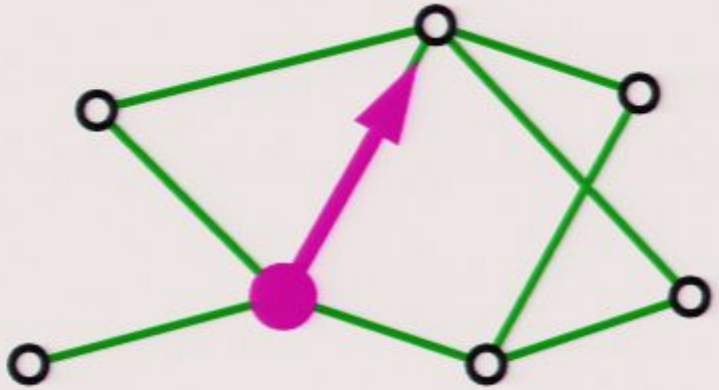


Quantum walks on graphs

***Discrete-time* version:**

Acts on extended Hilbert space

$|\text{vertex}\rangle \otimes |\text{orientation}\rangle$



[Aharonov, Ambainis, Kempe & Vazirani '01][A. Bach, Nayak, Vishwanath & Wotters '01] 6

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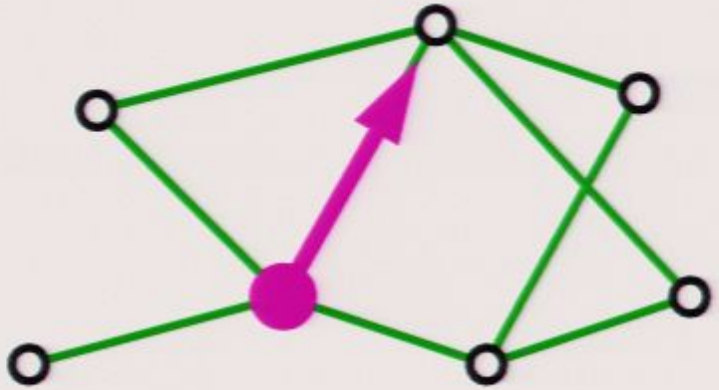

Quantum walk on unbounded line

Quantum walks on graphs

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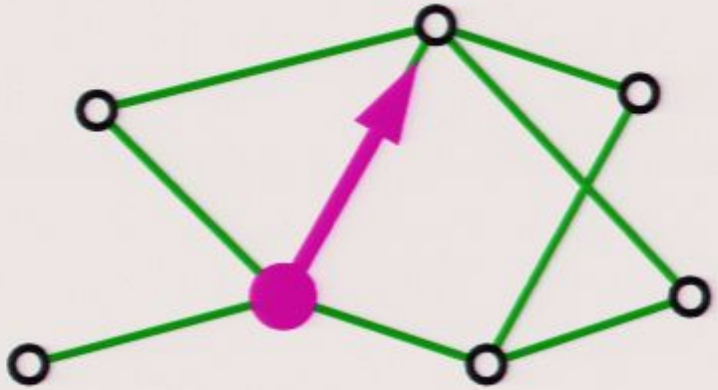
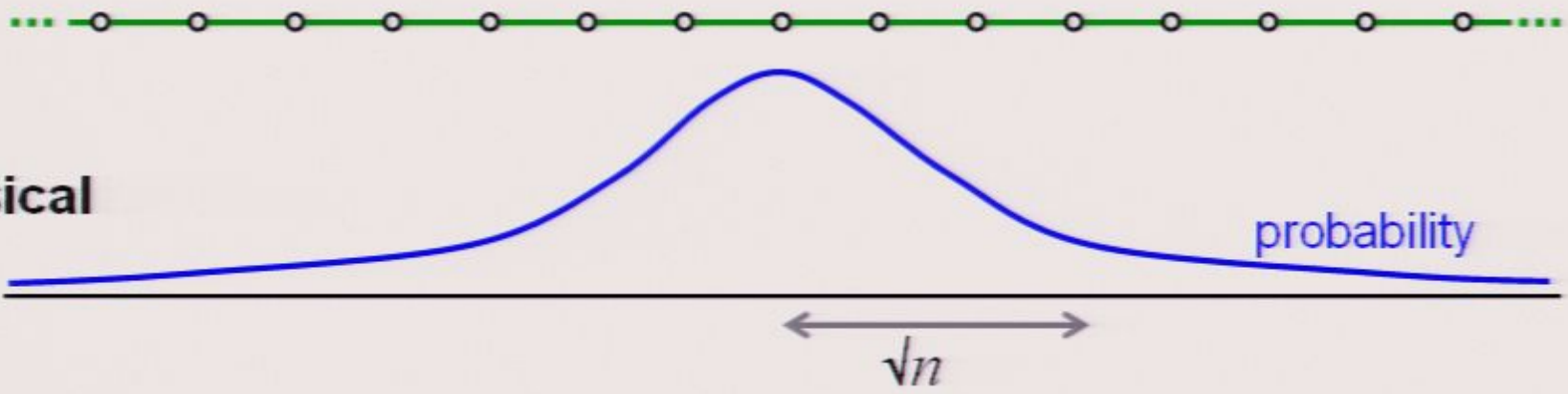


Quantum walk on unbounded line



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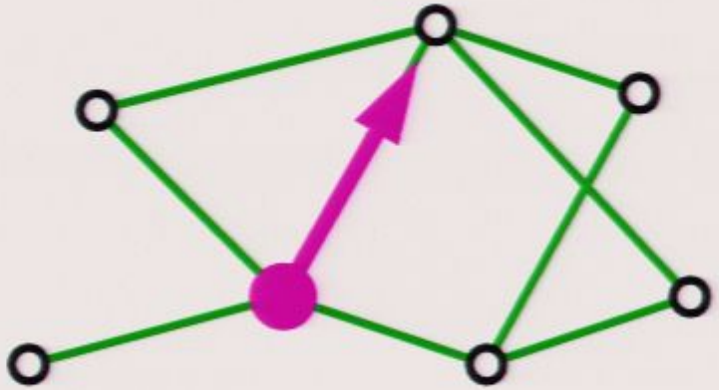

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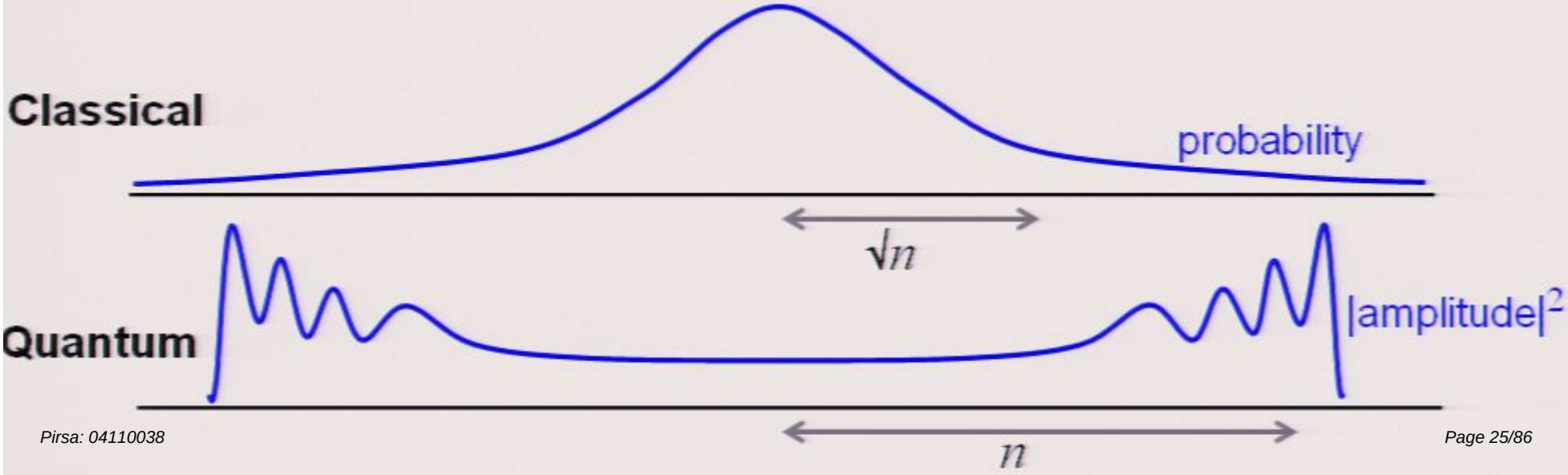

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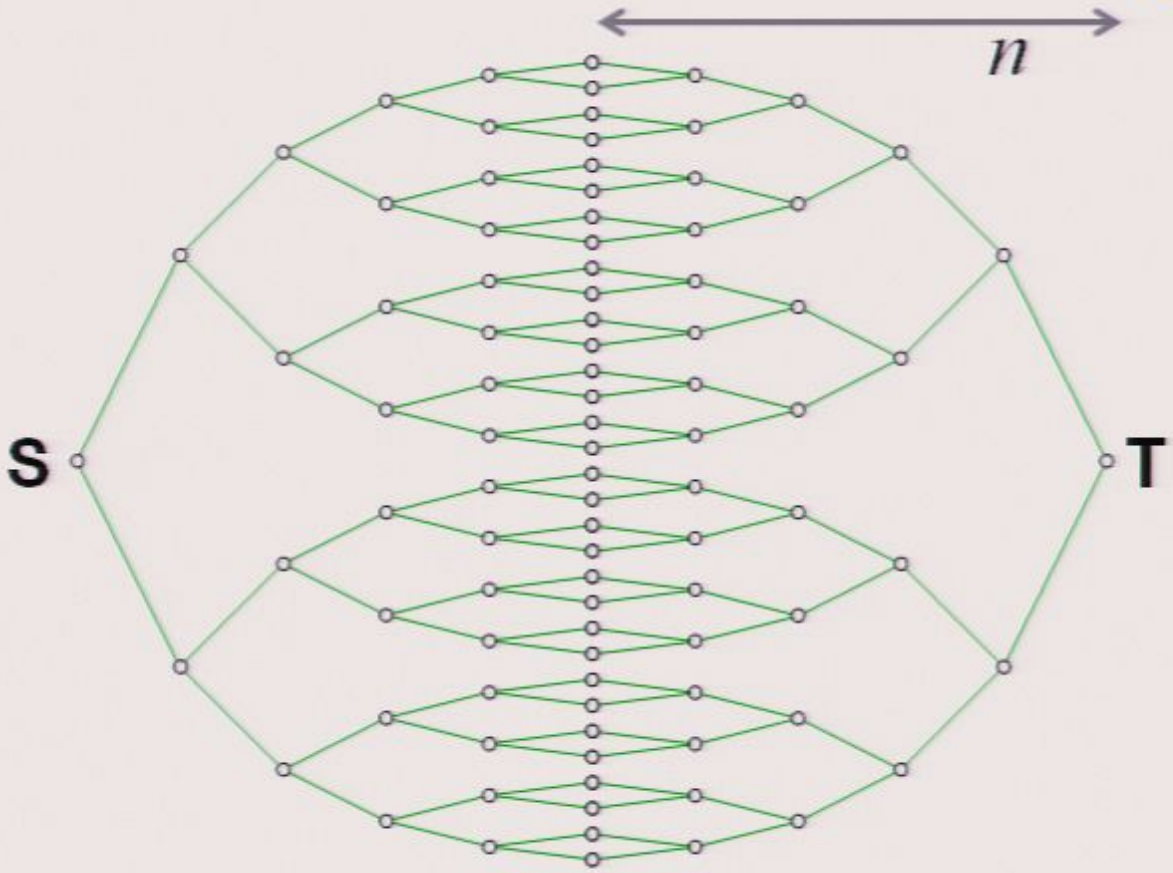
Quantum vs classical hitting times



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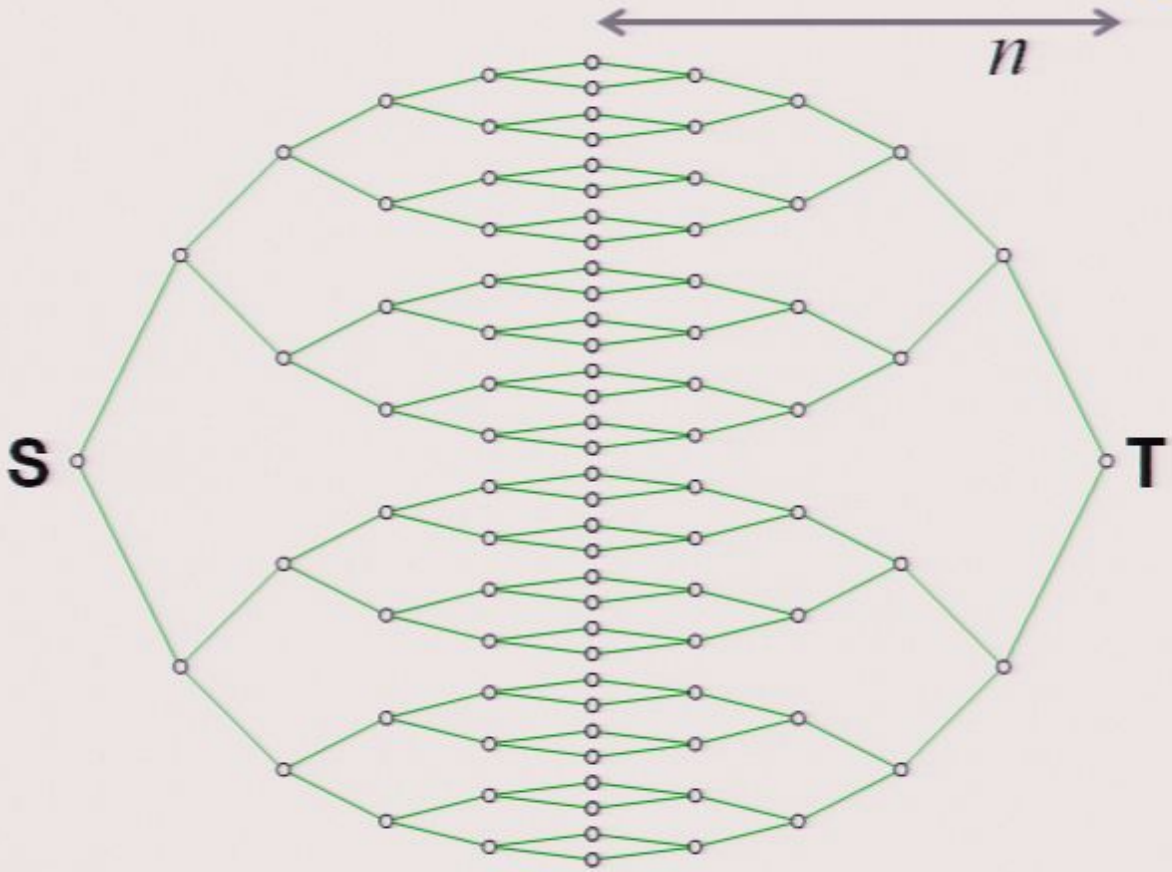

A ***random*** walk takes ***exponential*** time to get from **S** to **T**

[Childs, Farhi & Gutmann '01]

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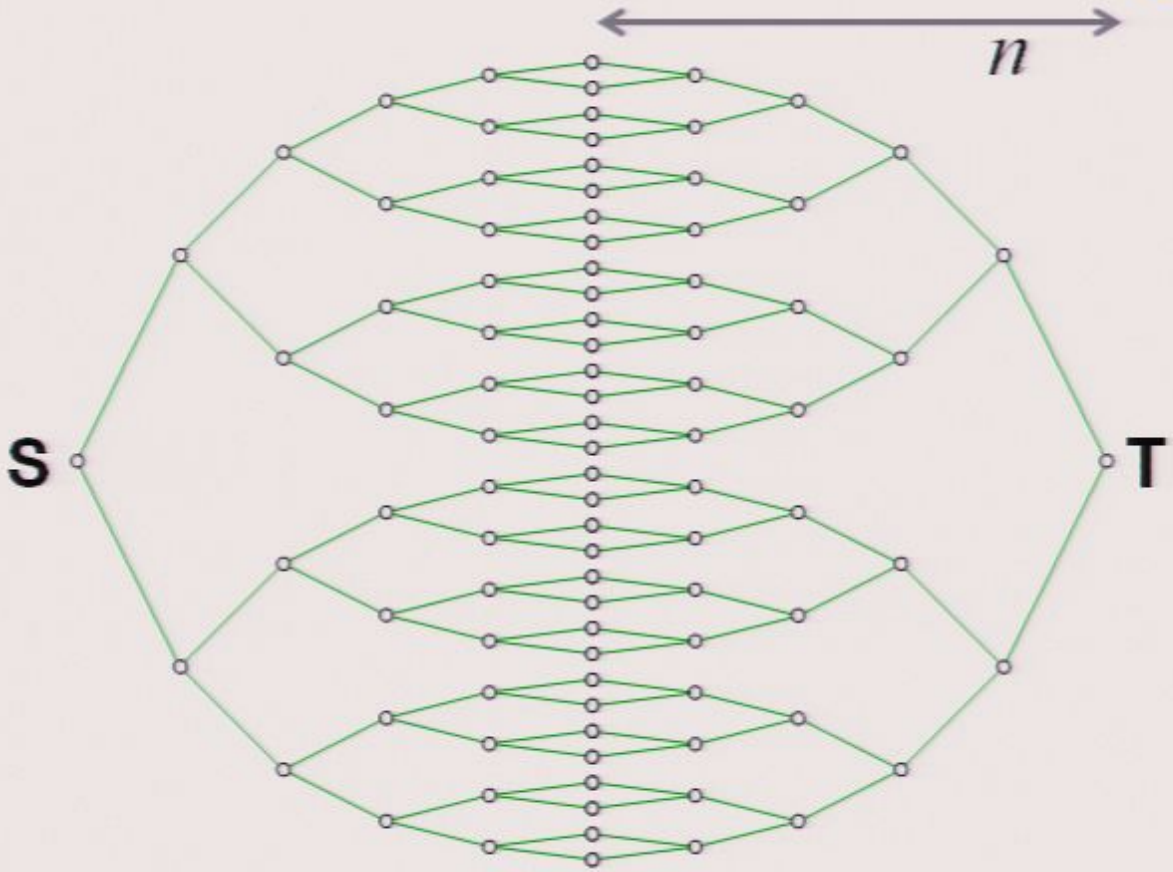

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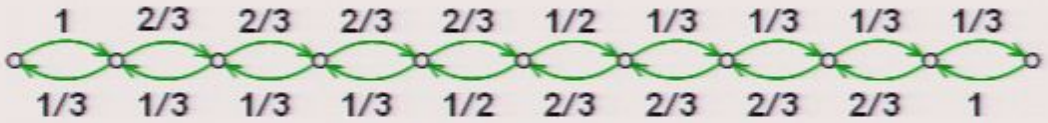

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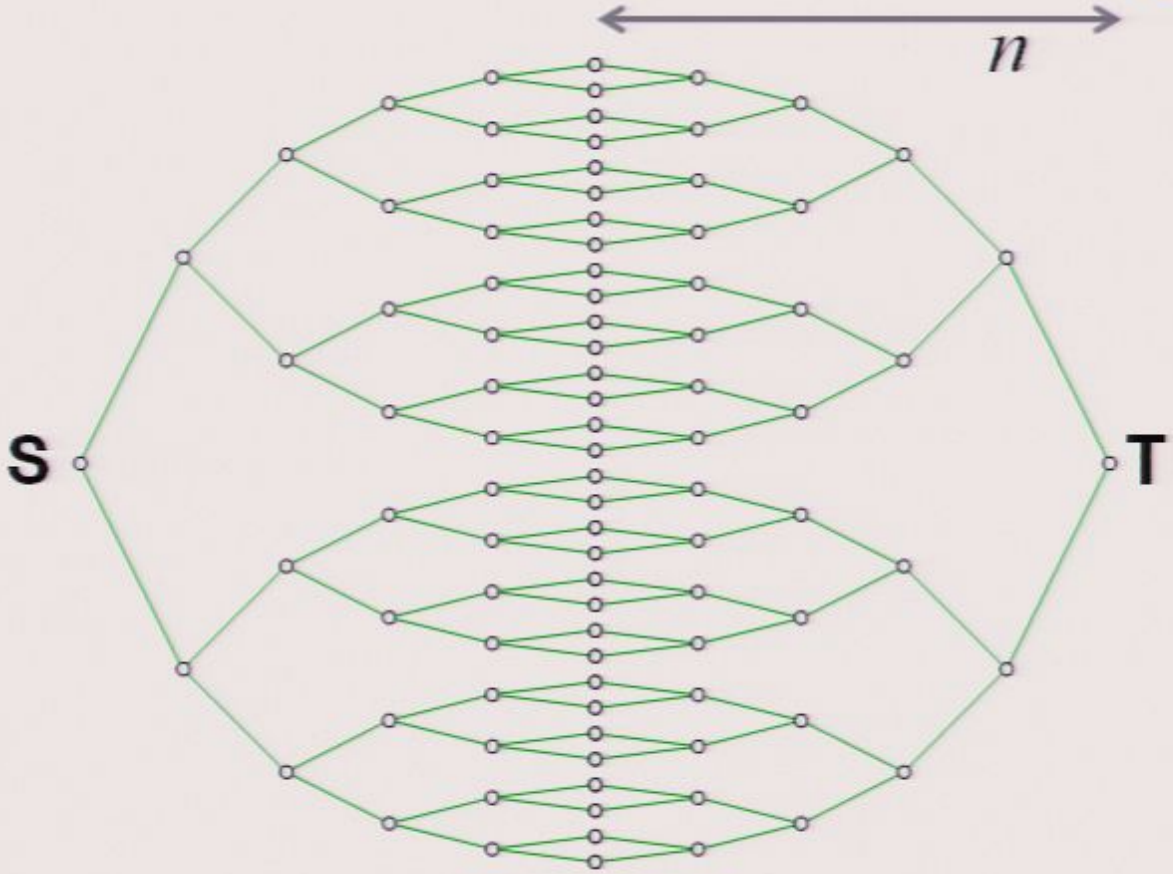

A ***quantum*** walk takes ***polynomial*** time to get from **s** to **t**

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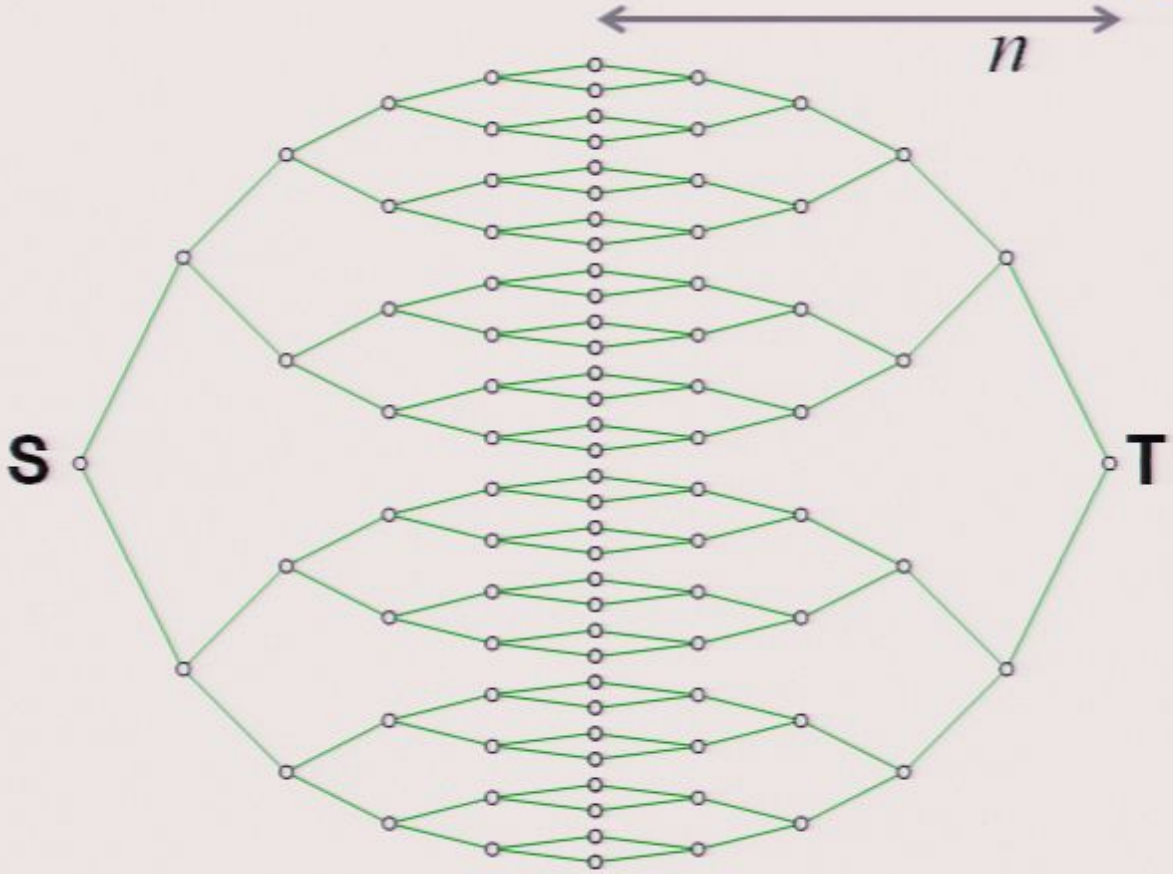

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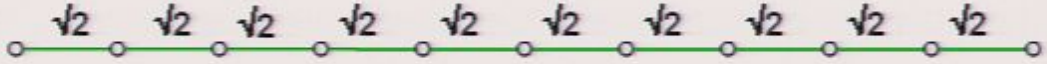
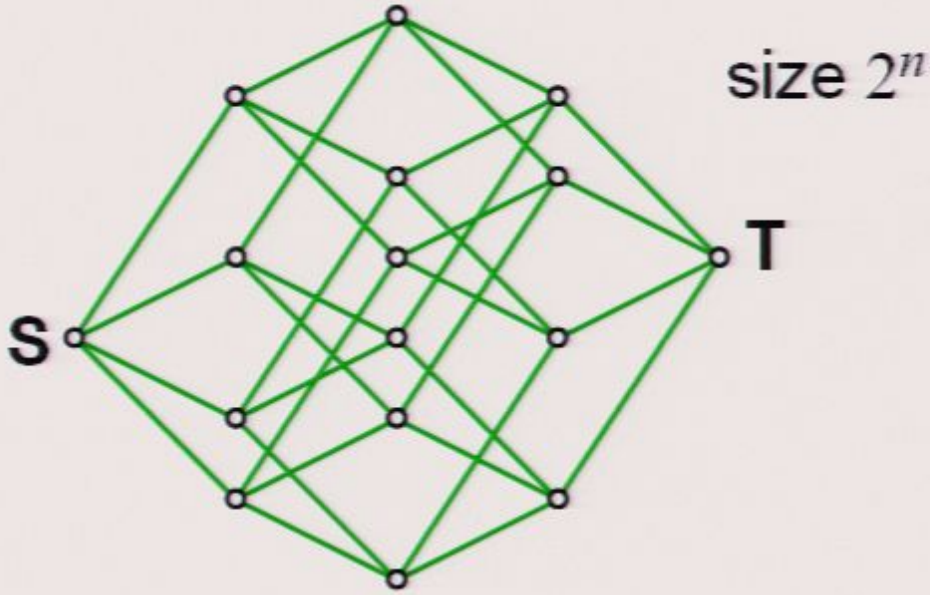


[Childs, Farhi & Gutmann '01]

n-dimensional hypercube



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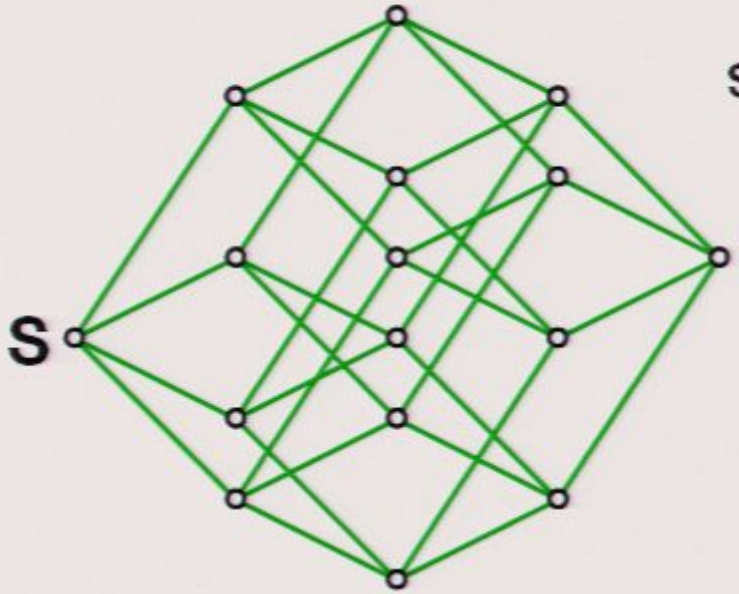

- Random walk takes ***exponential*** time to get from **S** to **T**

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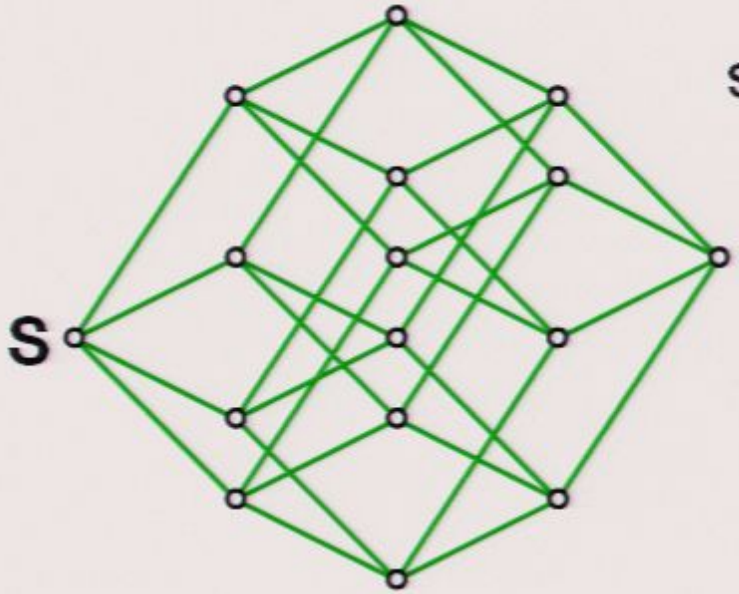

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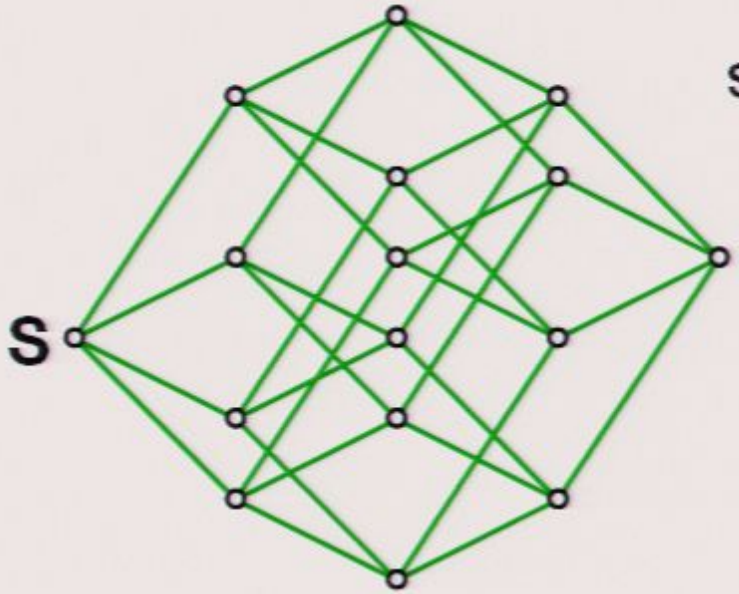
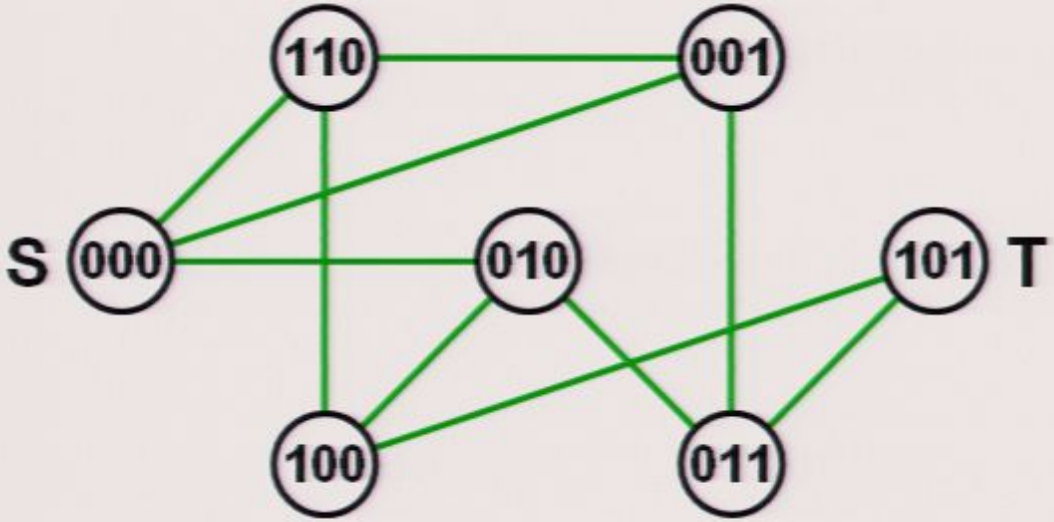


- Random walk takes ***exponential*** time to get from **S** to **T**
- Quantum walk takes ***polynomial*** time to get from **S** to **T**

ST-traversal problem: starting at node **S**, "find" node **T**

Representing graphs



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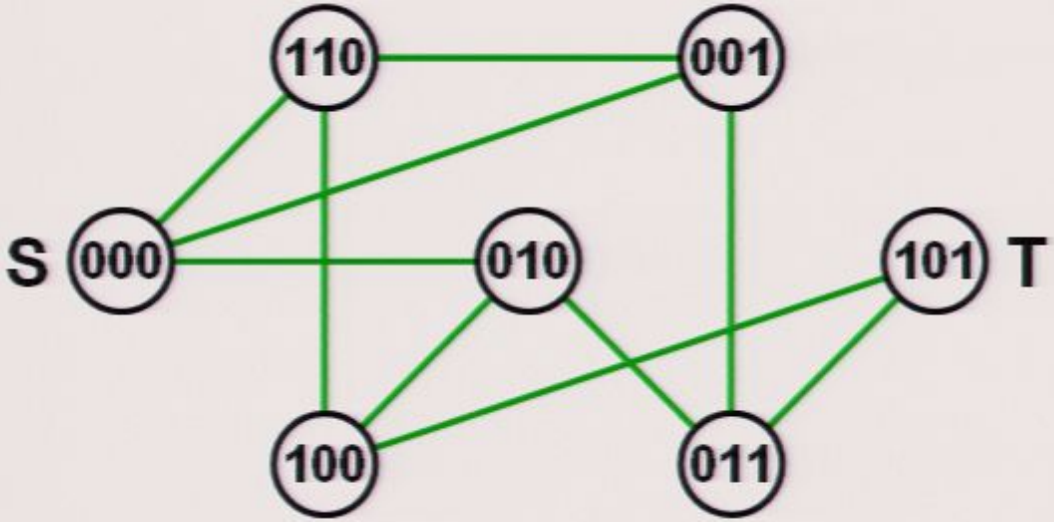
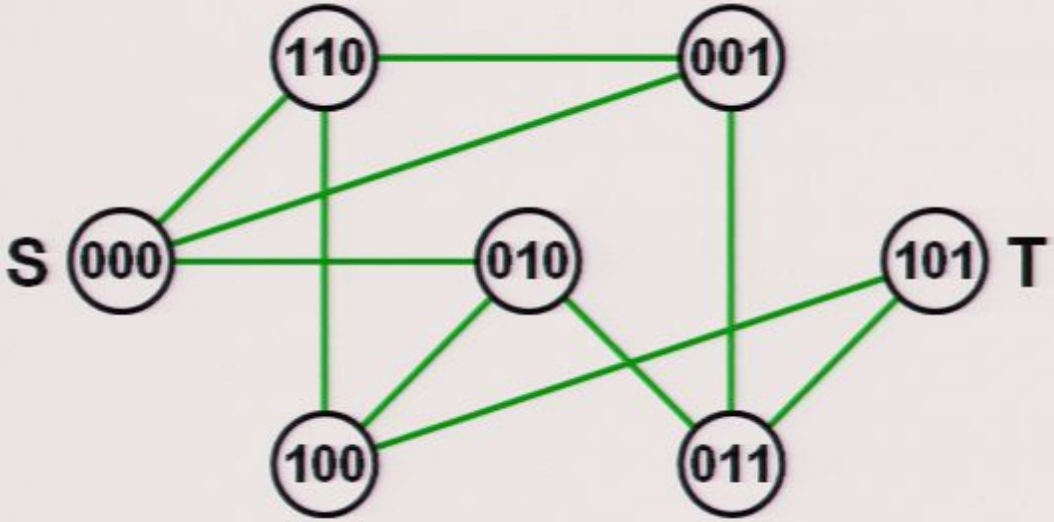


node	neighbors		
000	110	001	100
001	000	011	110
010	000	011	100
011	010	101	010
100	110	010	101
101	011	100	111
110	100	000	001
111	111	111	111

Representing graphs



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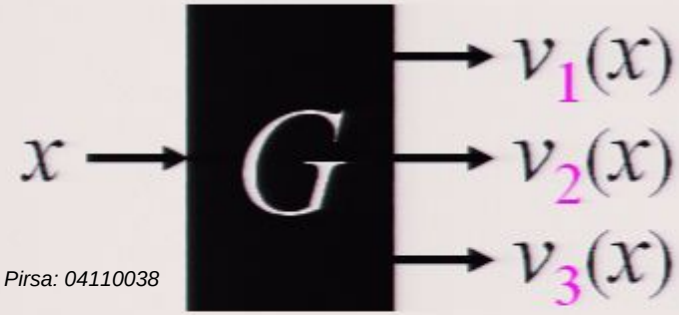

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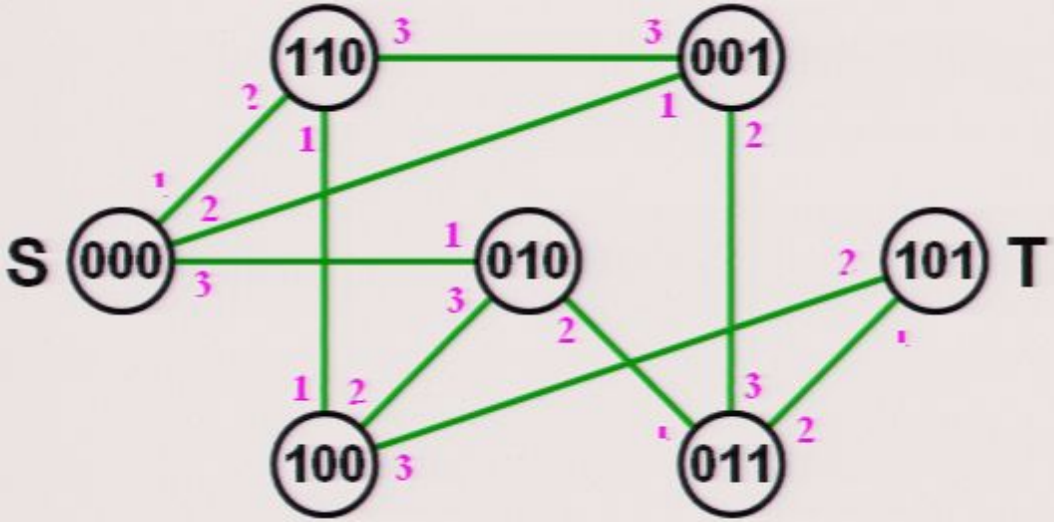

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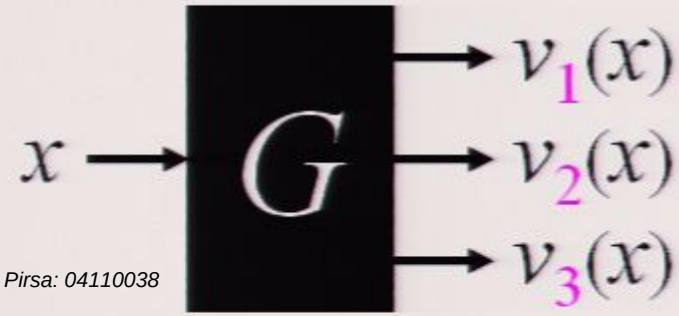

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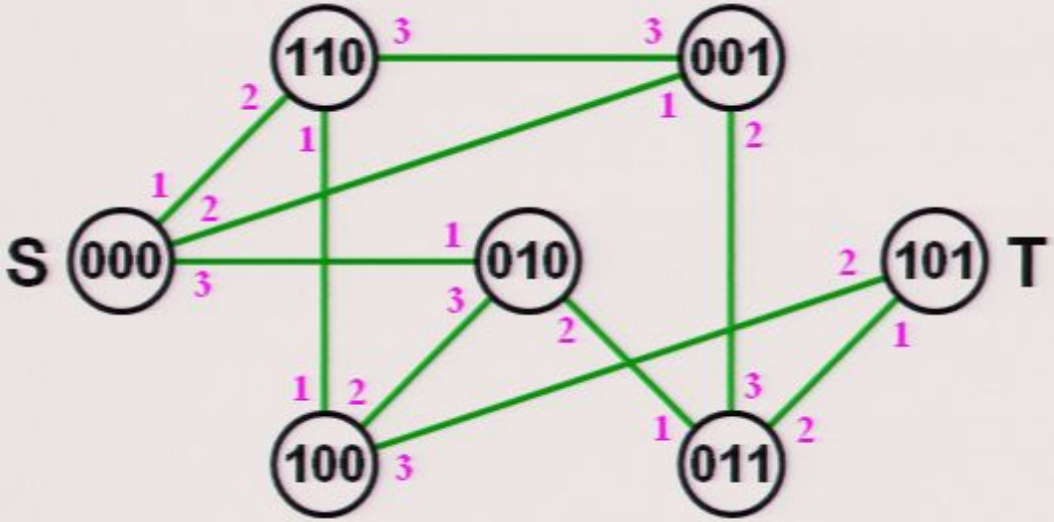

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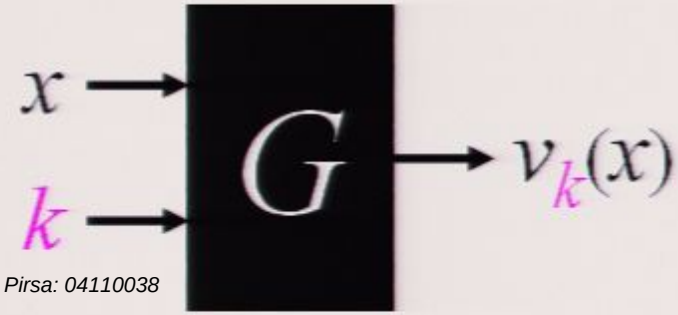

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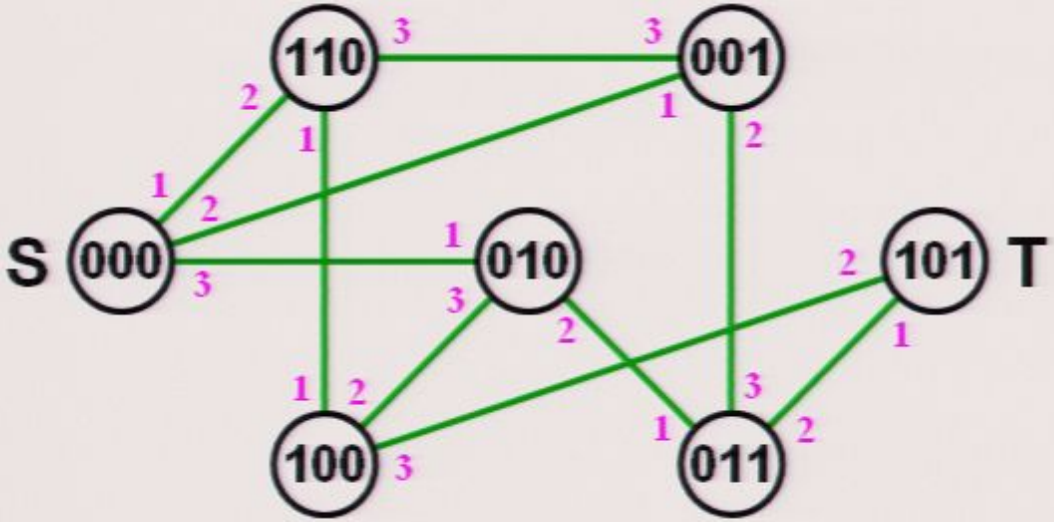

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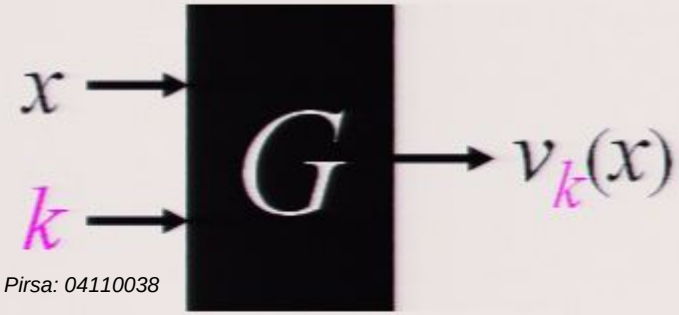
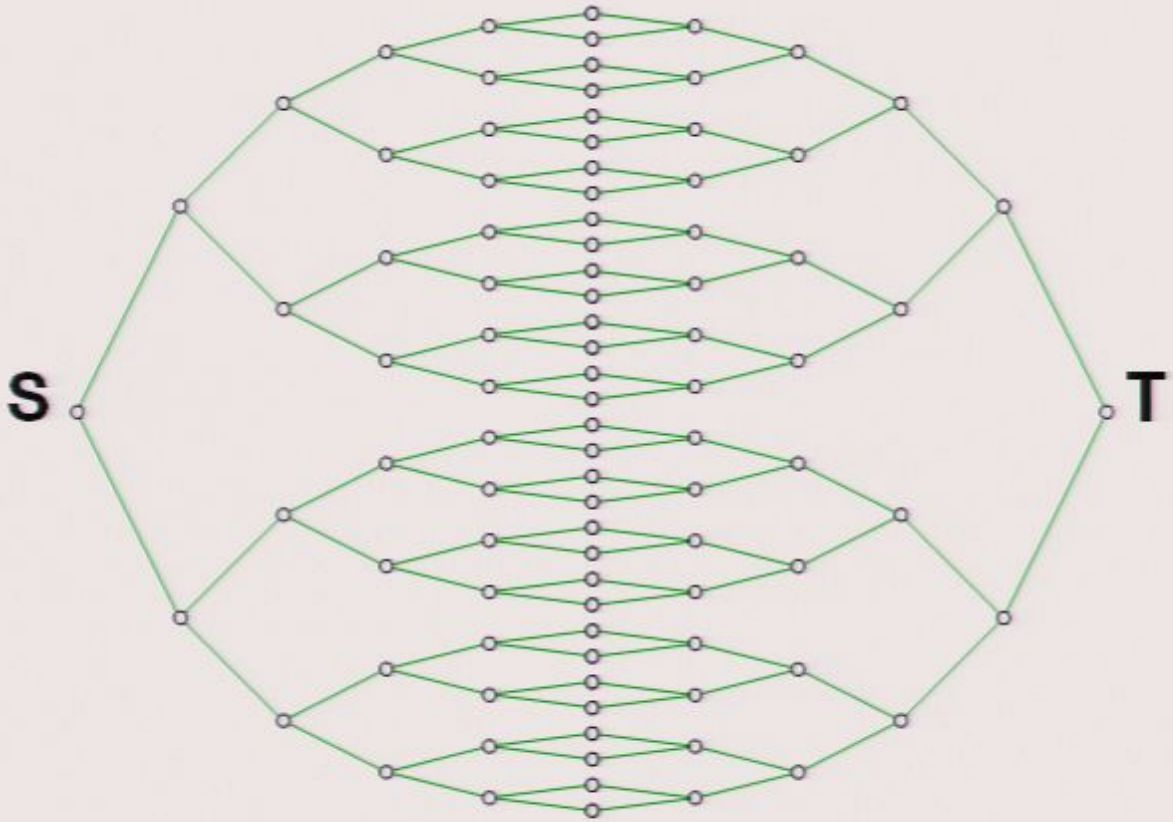


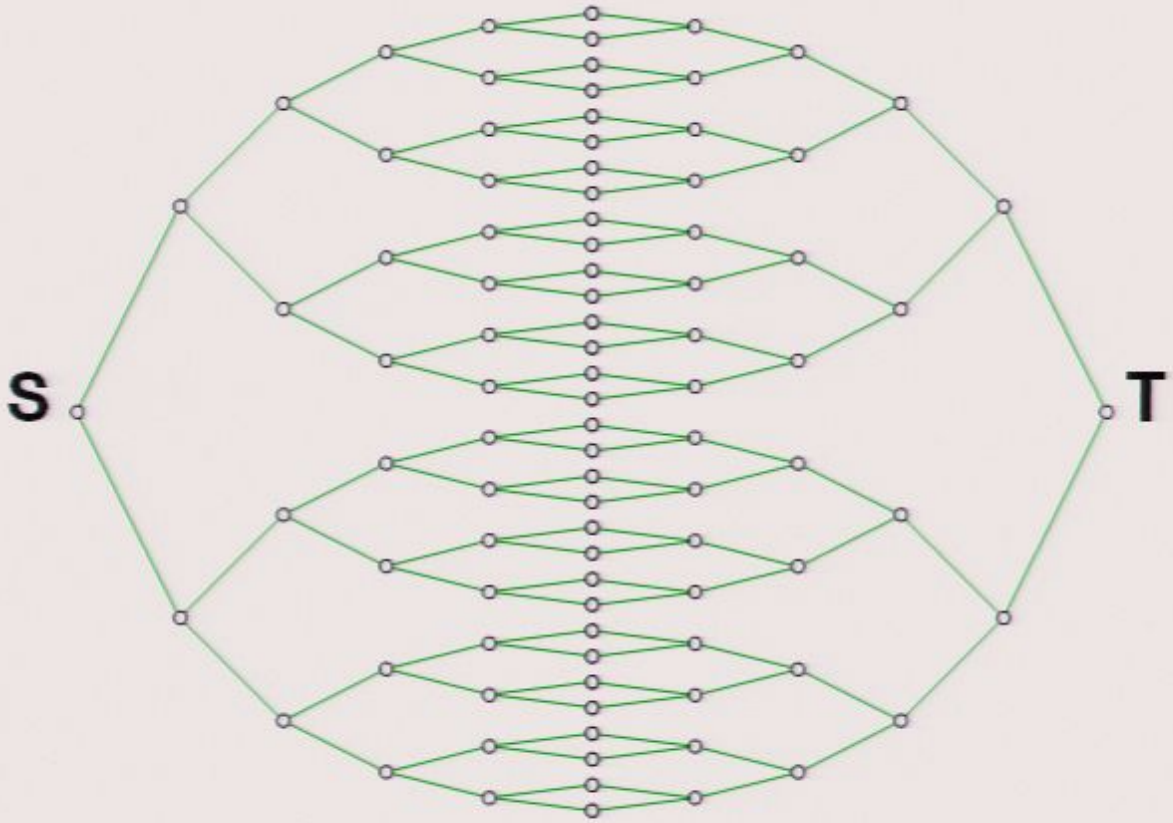
ST-traversal problem: given a black box for G and the name of node **S**, find the ***name*** of node **T**

"virtual labyrinth"

ST-traversal on GFG-graph

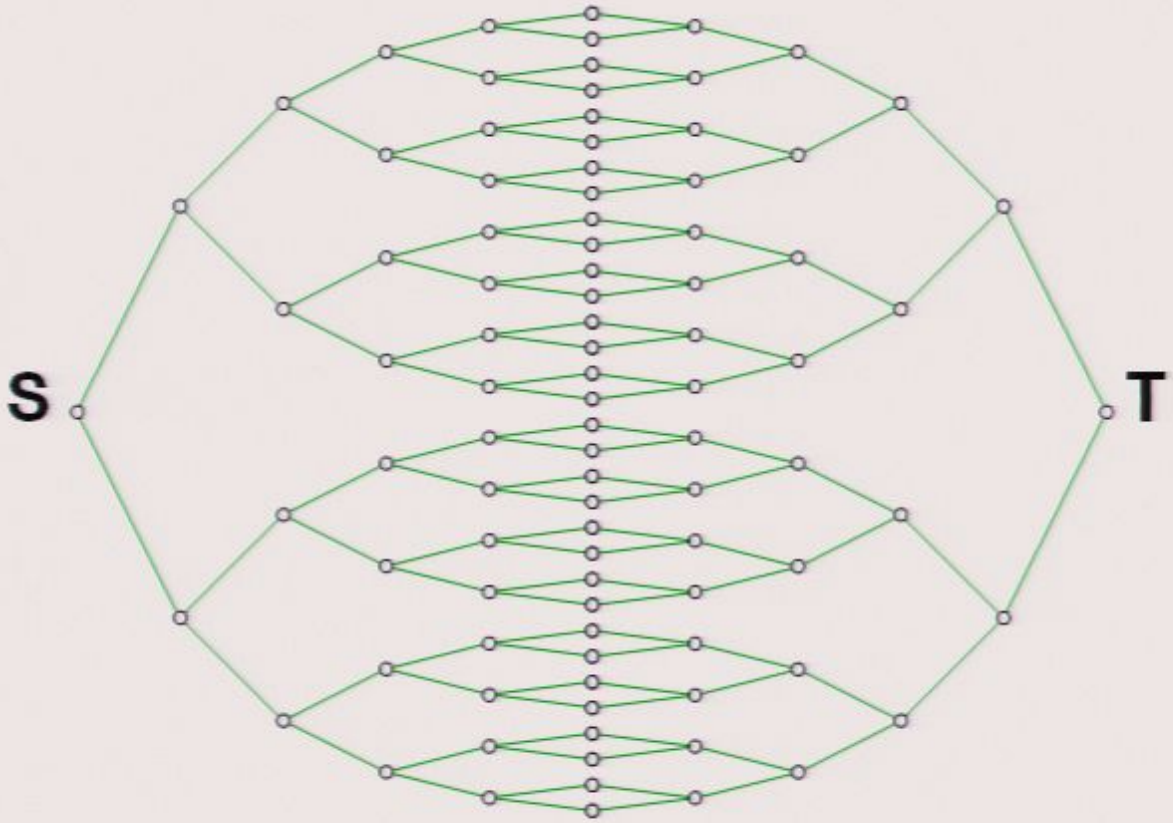


ST-traversal on GFG-graph



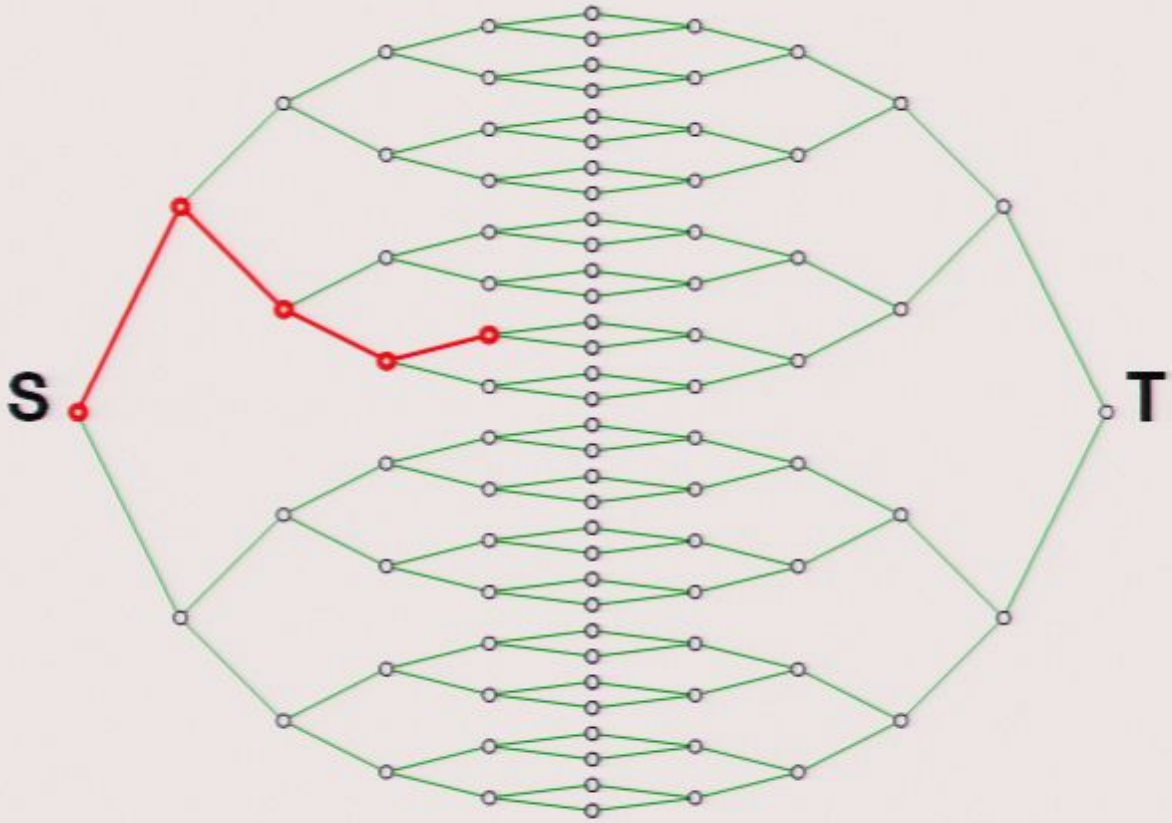
- Random walk costs $2^{O(n)}$

ST-traversal on GFG-graph



- Random walk costs $2^{O(n)}$
- Quantum walk with success probability $1/n$ costs $O(n^4)$

ST-traversal on GFG-graph



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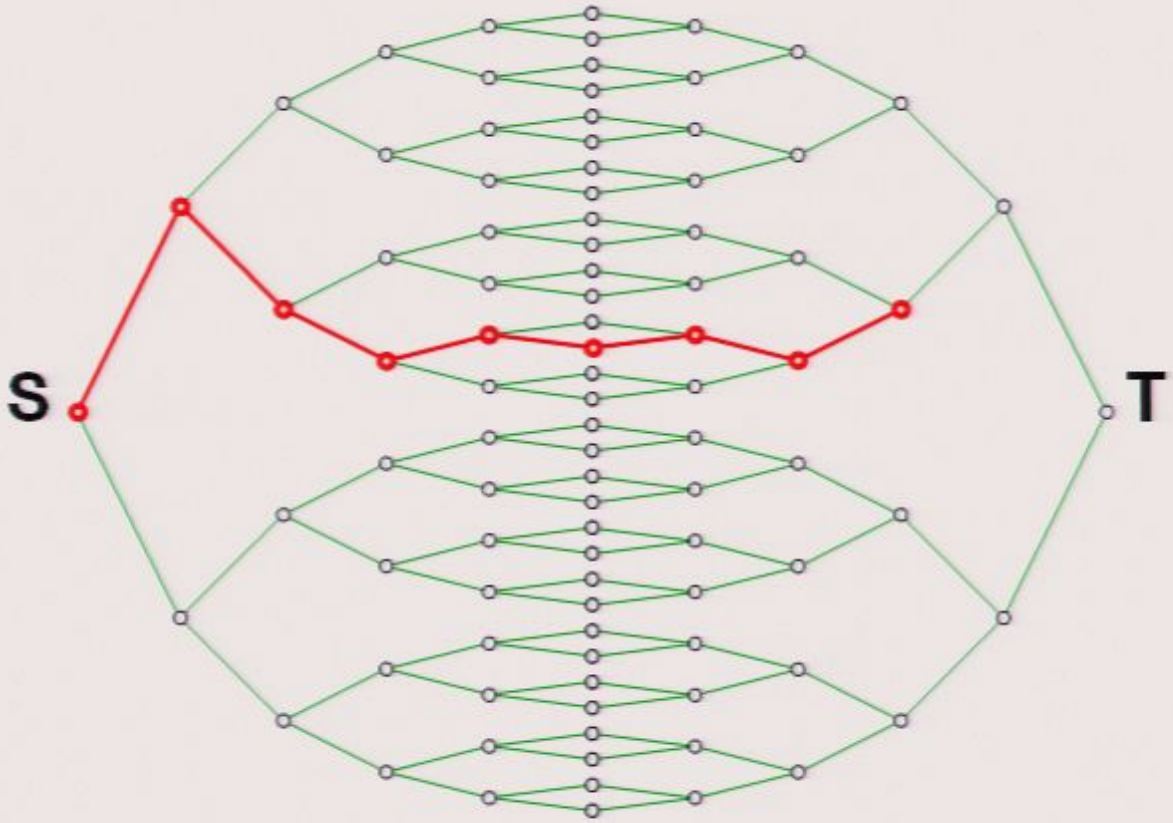

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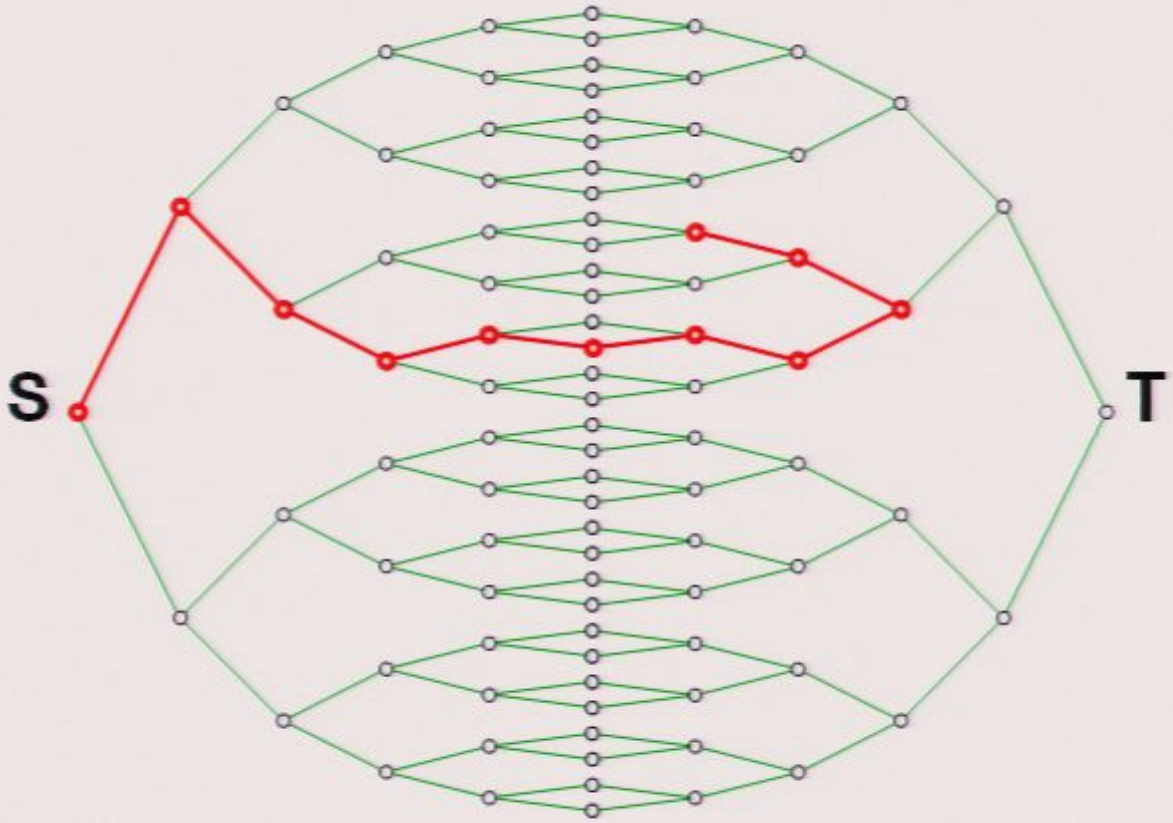

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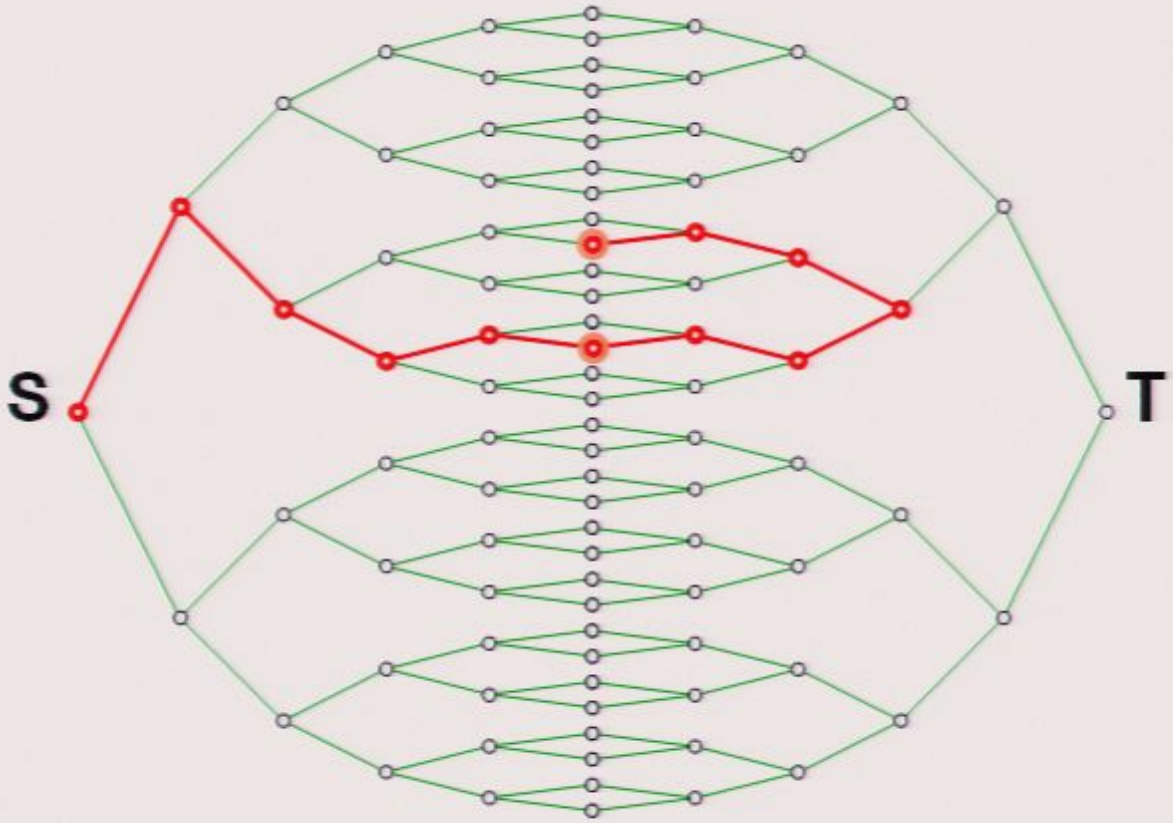

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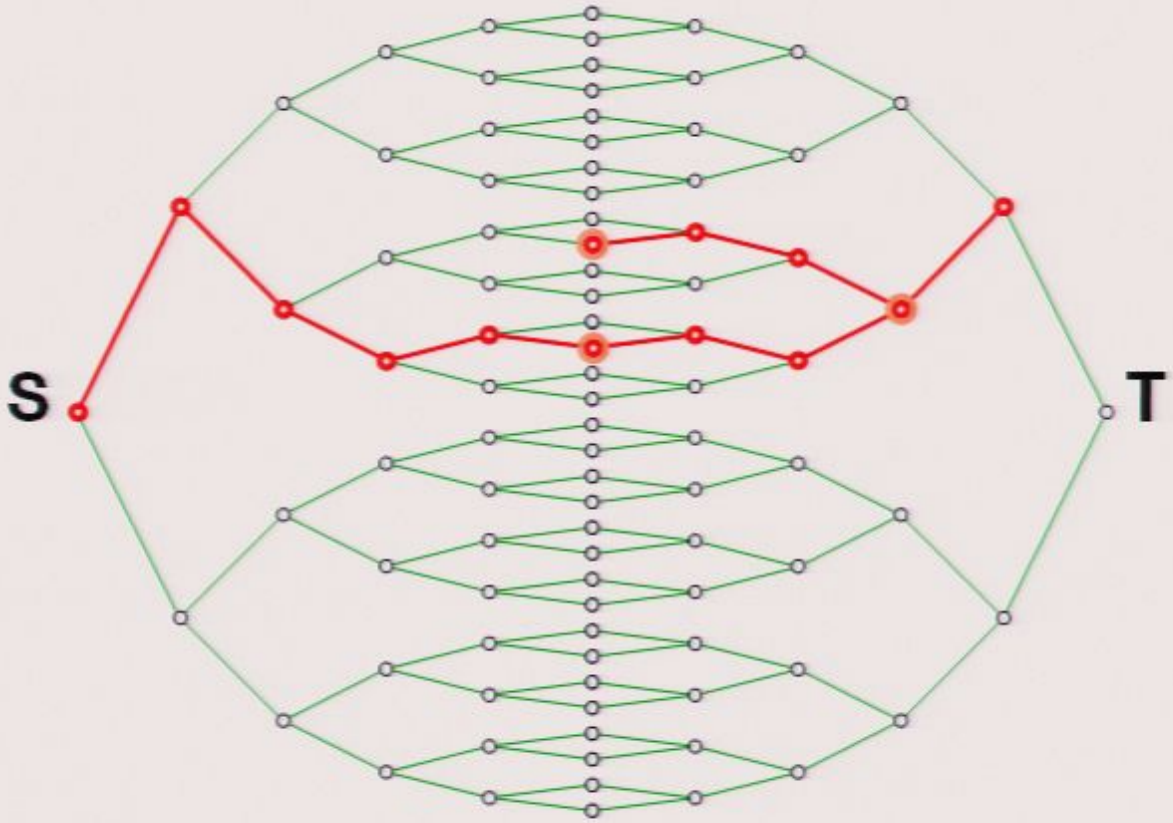

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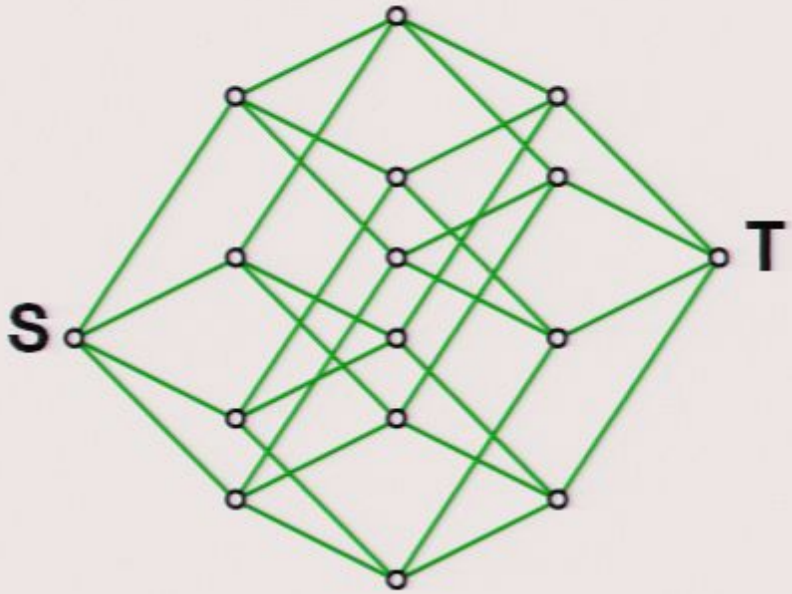
ST-traversal on hypercube



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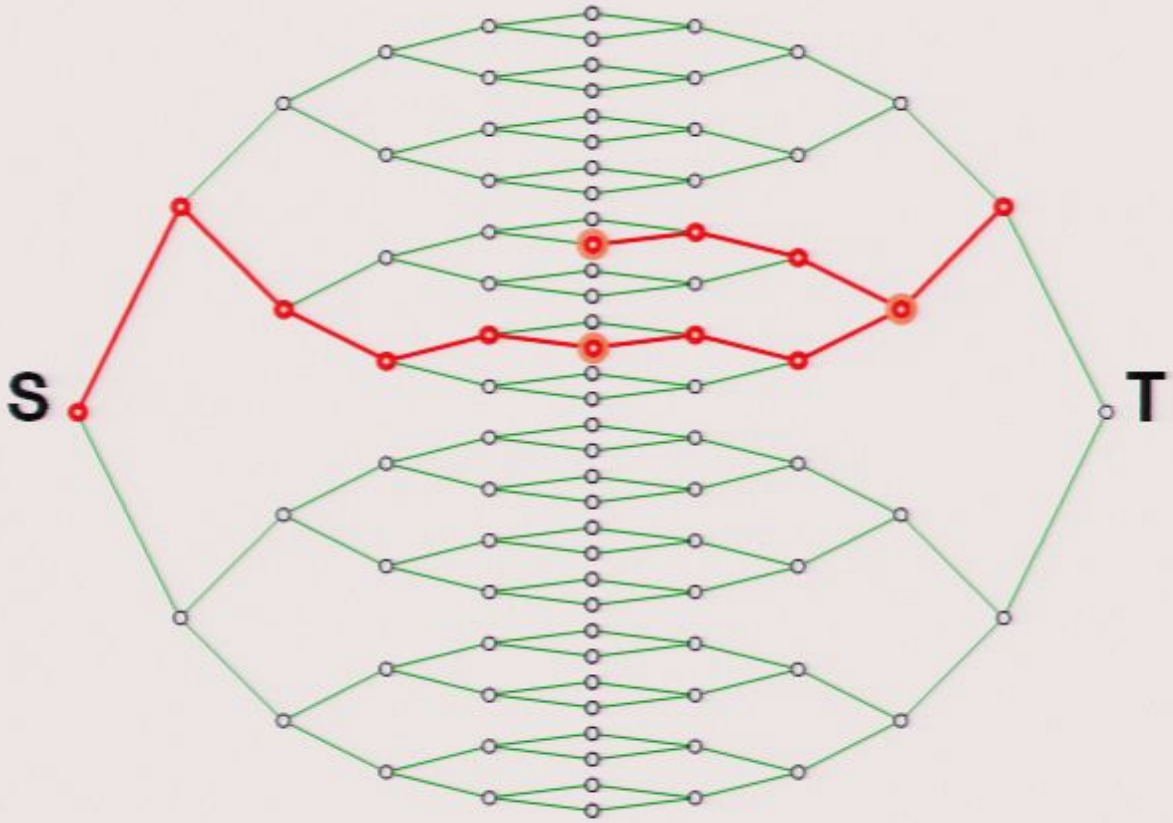

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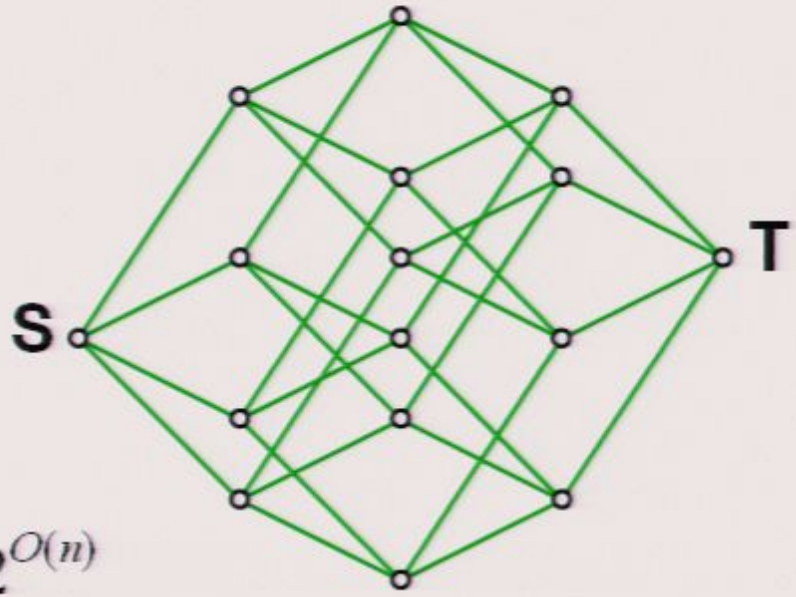

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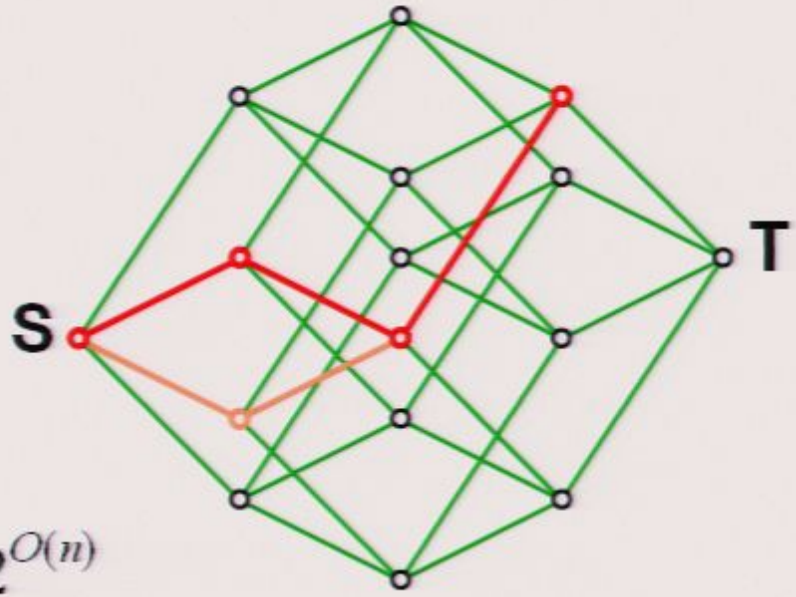

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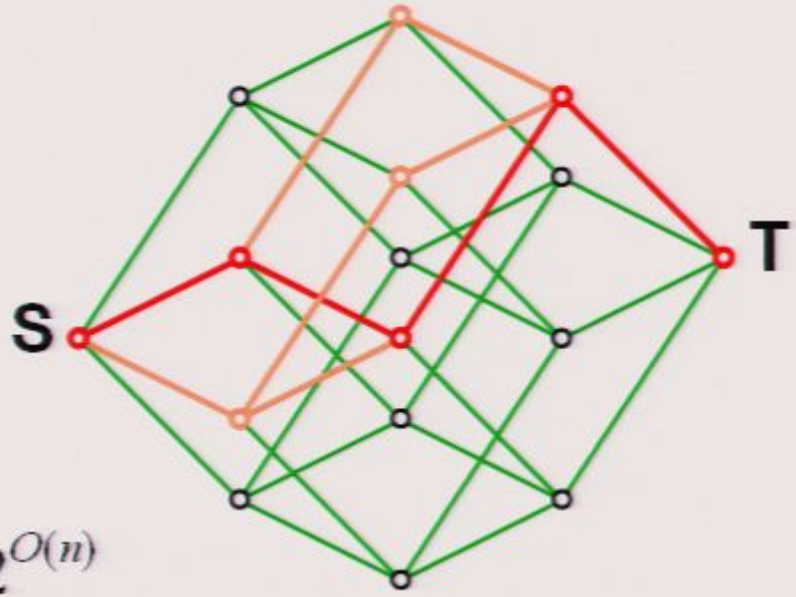

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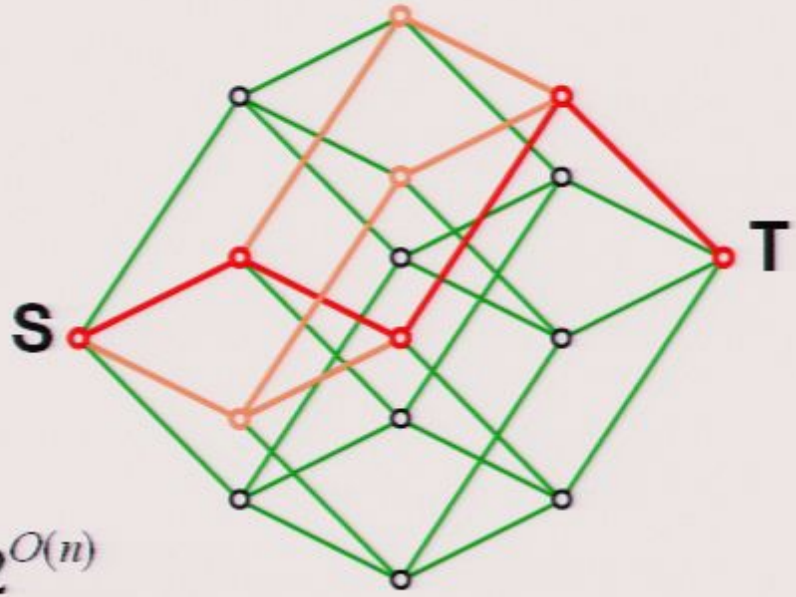

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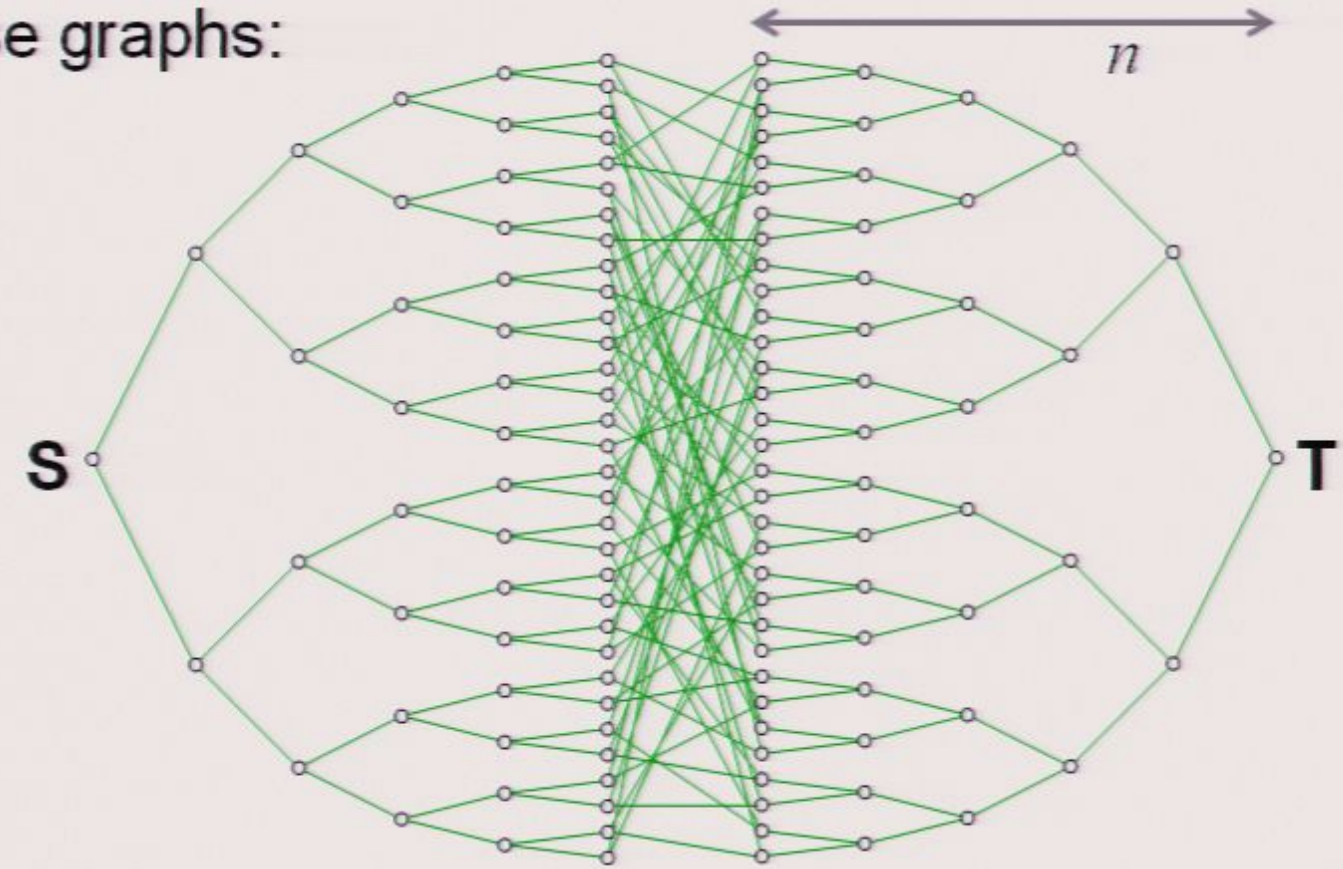
Question: is there a graph for which a quantum walk finds **T** exponentially faster than ***any*** classical algorithm?

Answer: yes

Answer: yes

For these graphs:



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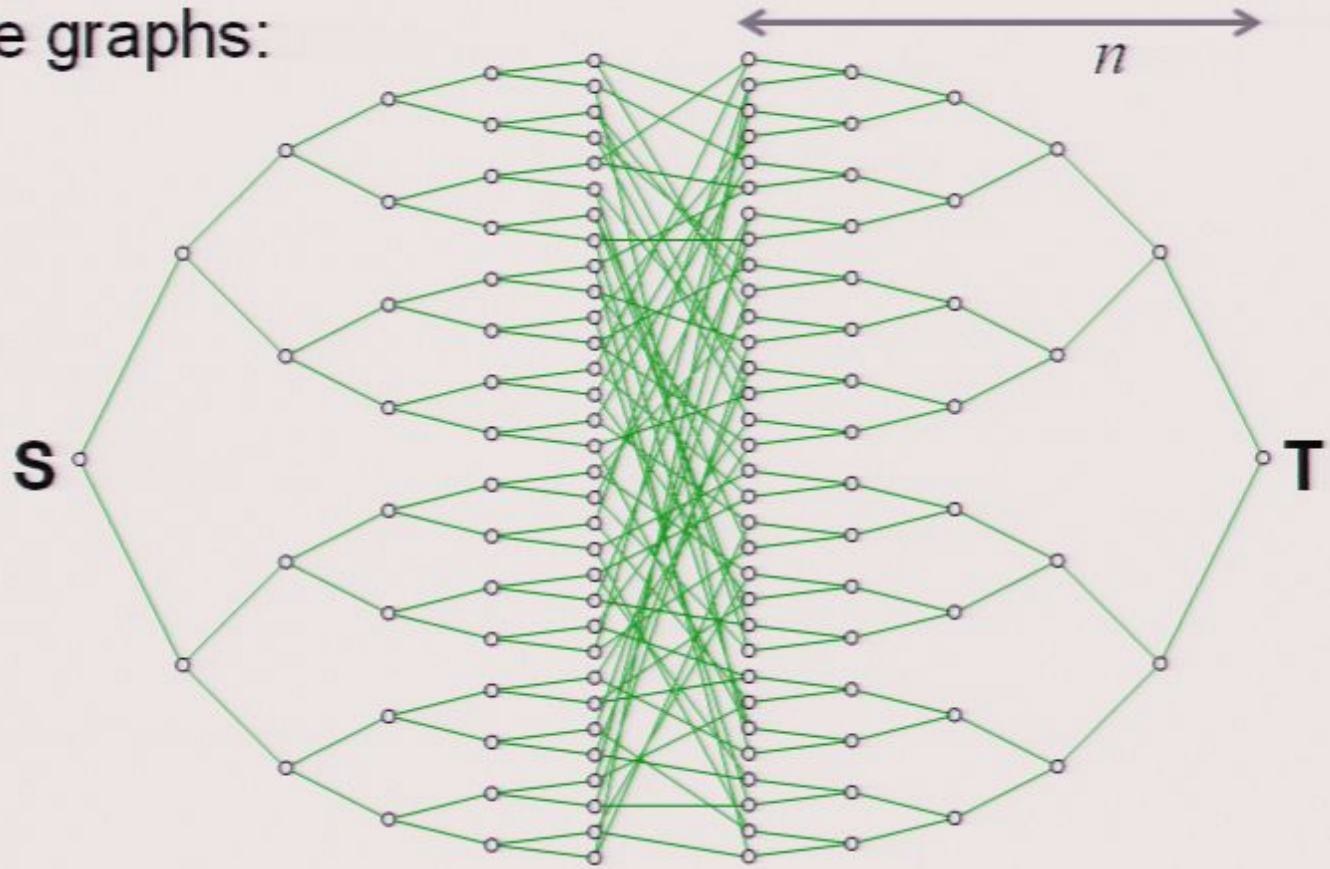

- ***Any*** classical **ST**-traversal algorithm takes exponential time

Answer: yes

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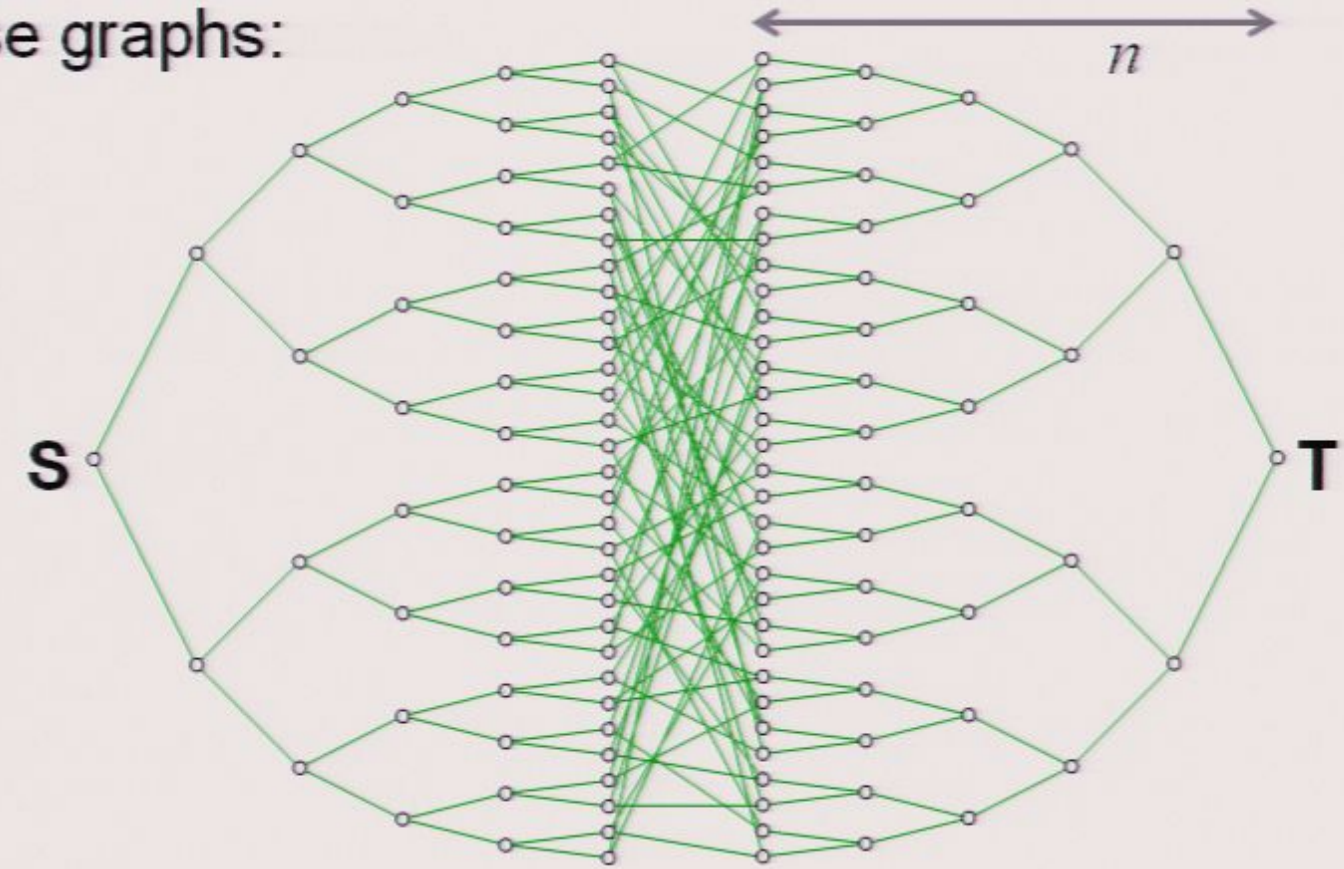
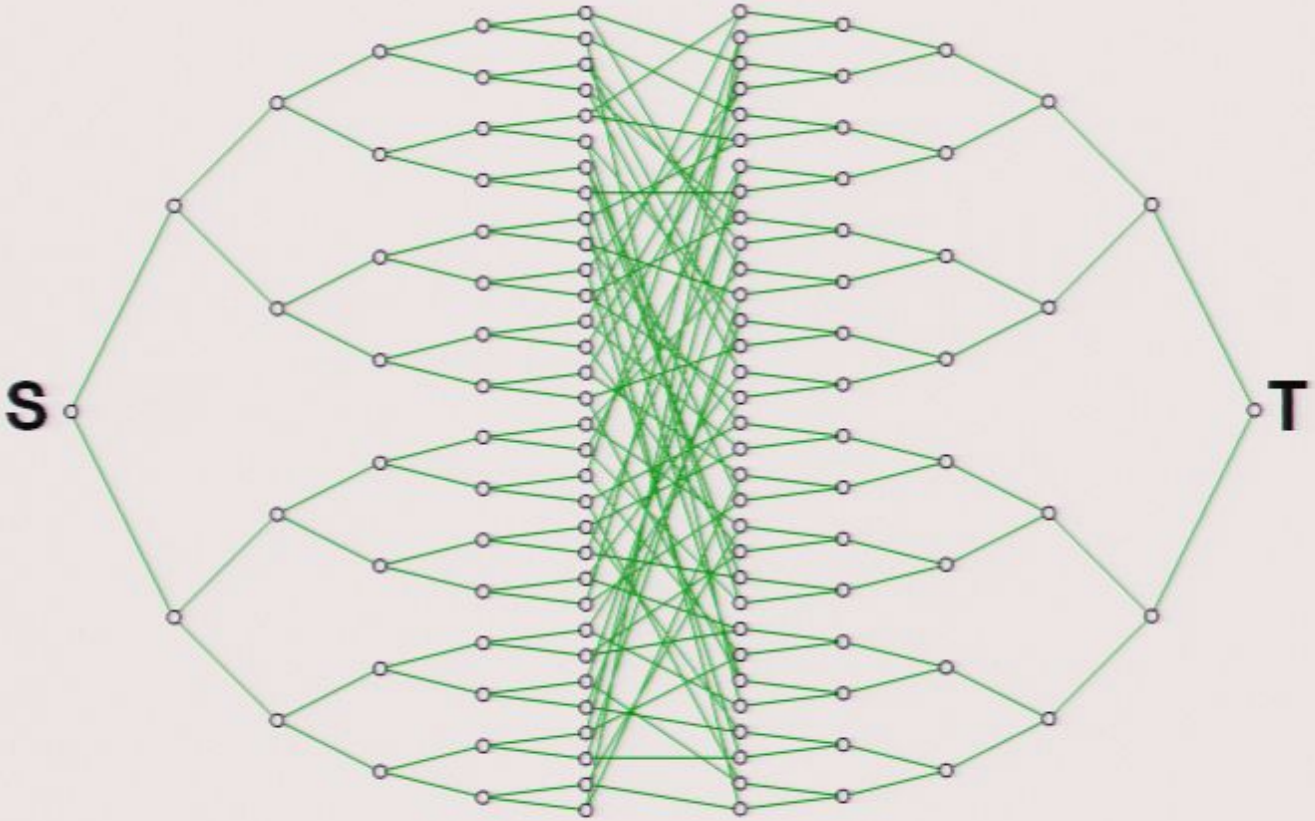
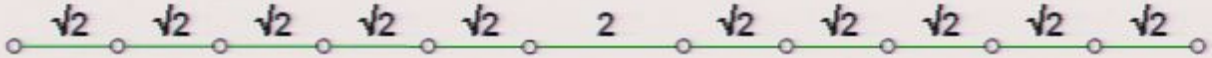


- ***Any*** classical **ST**-traversal algorithm takes exponential time
- Algorithm based on ***quantum walks*** takes polynomial time

Analysis of continuous-time QW



Column subspace:



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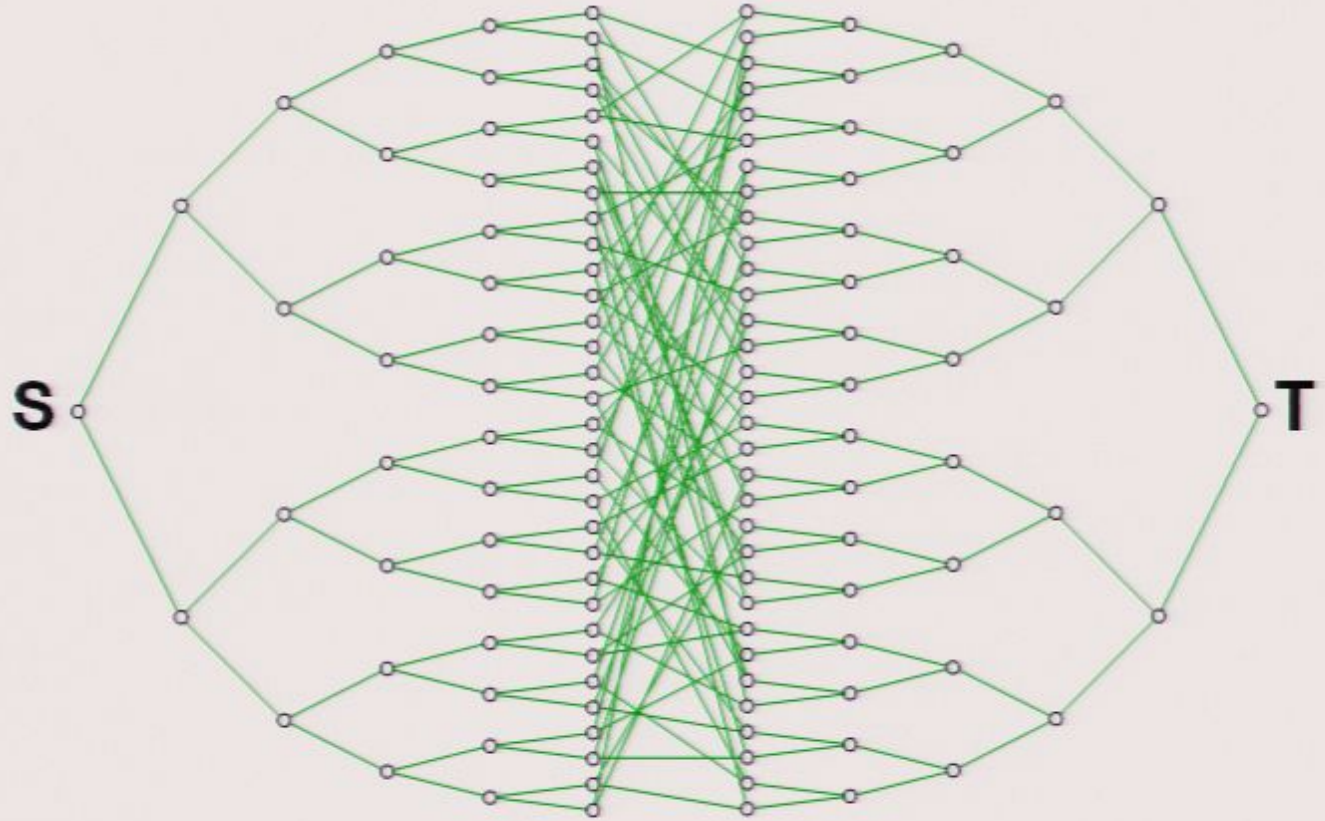

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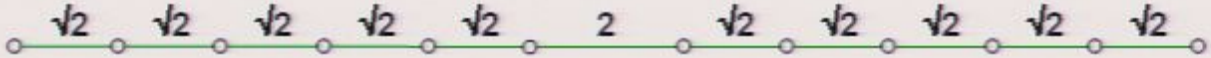

Choose $t \in [0, O(n^4)]$ uniformly, run QW for time t, and measure

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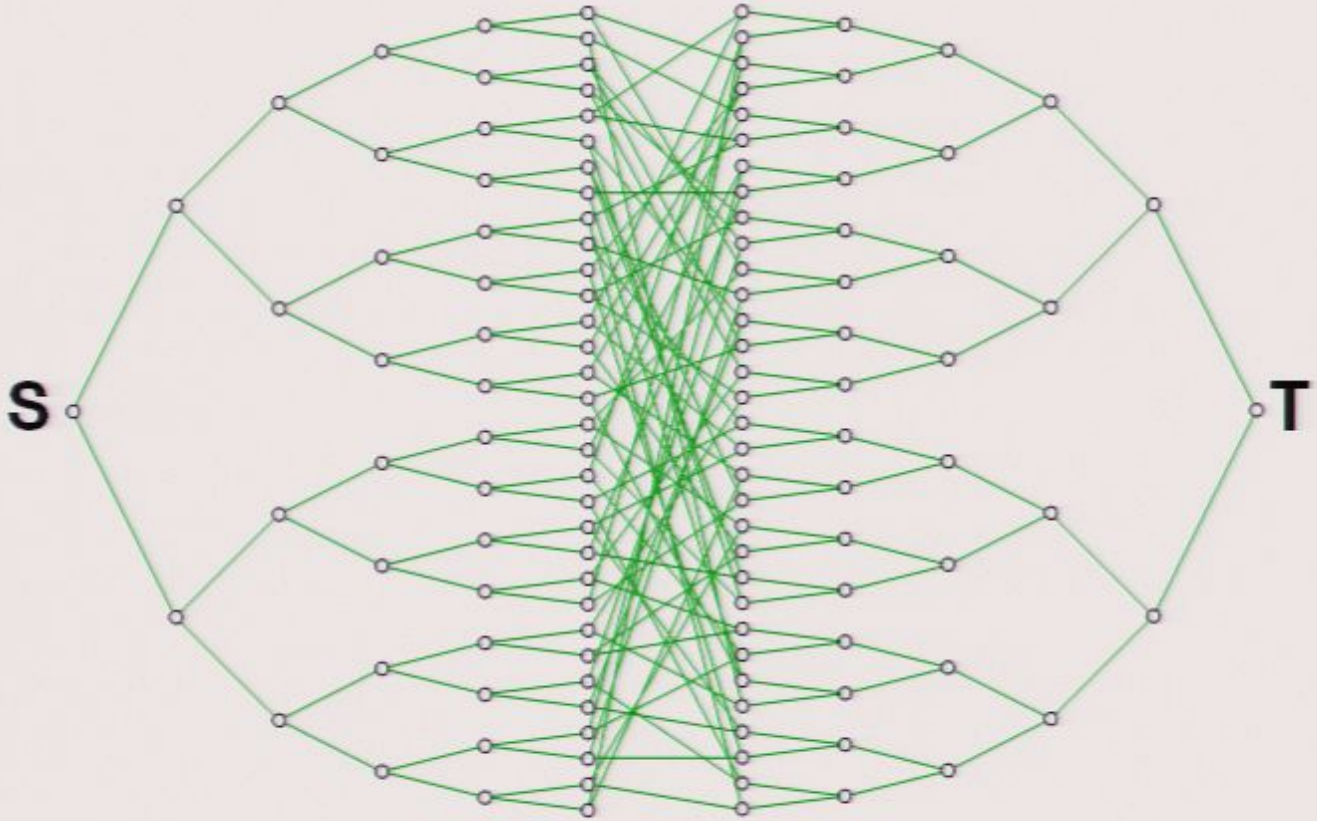

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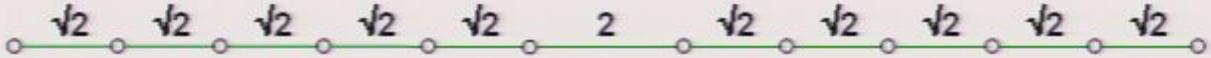

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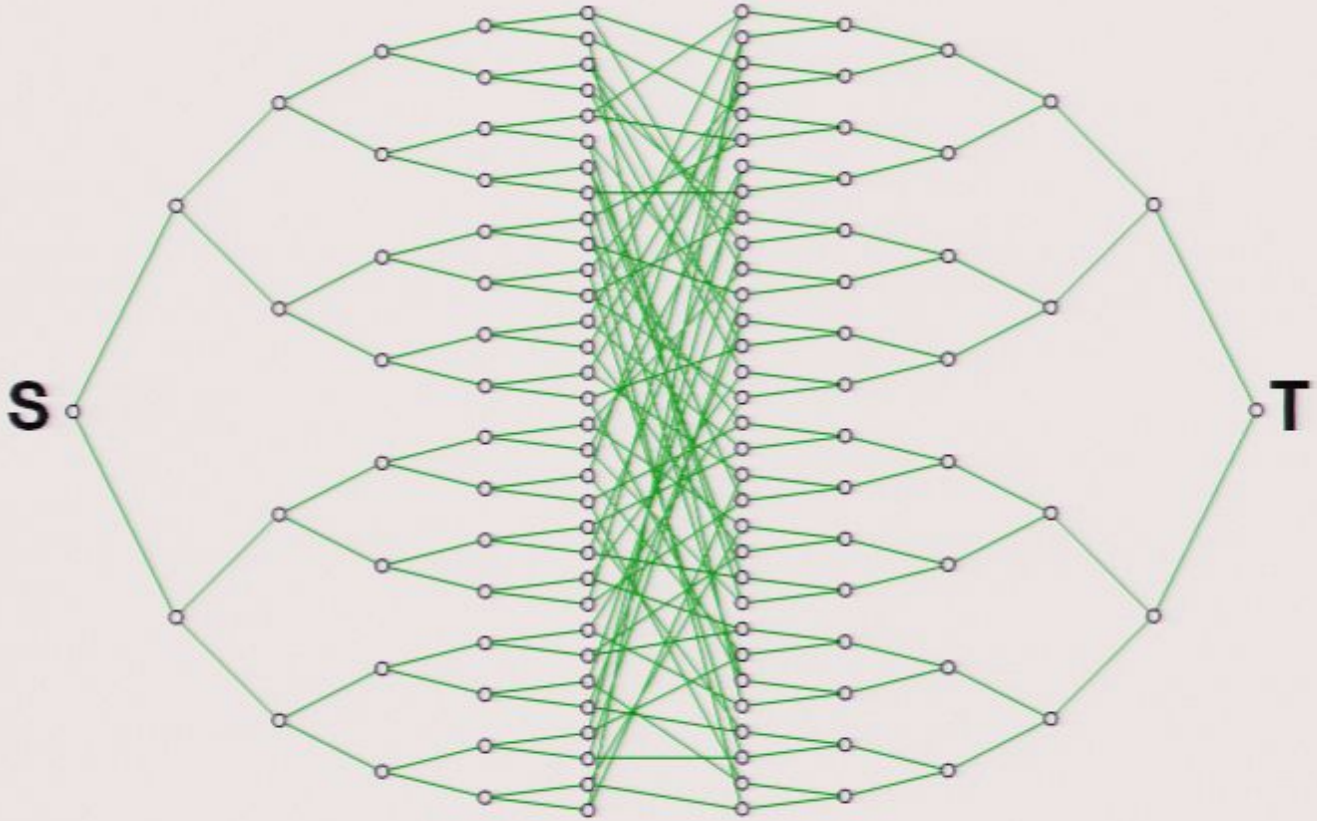
Theorem: the outcome is T with probability $\Omega(1/n)$

How to simulate the quantum walk

Analysis of continuous-time QW



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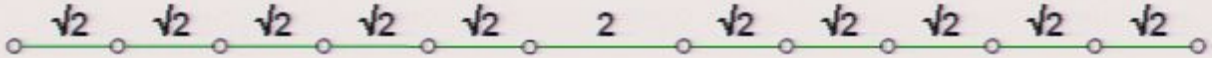

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How to simulate the quantum walk

Goal: simulate $e^{-iHt}|\mathbf{s}\rangle$, where $\langle x|H|y\rangle = \begin{cases} 1 & \text{if } (x,y) \text{ edge} \\ 0 & \text{otherwise} \end{cases}$

given **s** and a black box for G

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Tensor product simulations of $H_1, H_2, \ldots, H_n$ and a method for efficiently computing product of their eigenvalues enables simulation of $H_1 \otimes H_2 \otimes \ldots \otimes H_n$

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Unitary conjugation simulation of H and U, $U^\dagger$ efficiently computable enables simulation of $U^\dagger H U$ (that is, $e^{-iU^\dagger HUt}$)

Linear combination simulation of $H_1, H_2, \ldots, H_n$ enables simulation of $H_1+H_2+\ldots+H_n$ (that is, $e^{-i(H_1+H_2+\ldots+H_n)t}$)

Simulation of quantum walk for *symmetric* black-boxes

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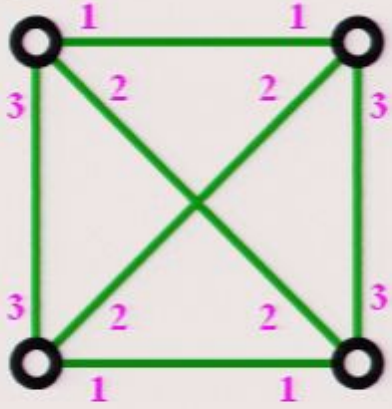


Symmetric: $v_k(x) = y$ iff $v_k(y) = x$

Define **SWAP** $|x\rangle|y\rangle = |y\rangle|x\rangle$

Simulate evolution by **SWAP** (tensor product)

Simulate evolution by $\boldsymbol{V}_k^{\dagger}\textbf{SWAP}\boldsymbol{V}_k$ (unitary conjugation)

$$\boldsymbol{V}_k^{\dagger}\textbf{SWAP}\boldsymbol{V}_k\,|x\rangle|0\rangle = \boldsymbol{V}_k^{\dagger}\textbf{SWAP}\,|x\rangle|v_k(x)\rangle \quad (\boldsymbol{V}_k^{\dagger} \text{ reversible black-box})$$

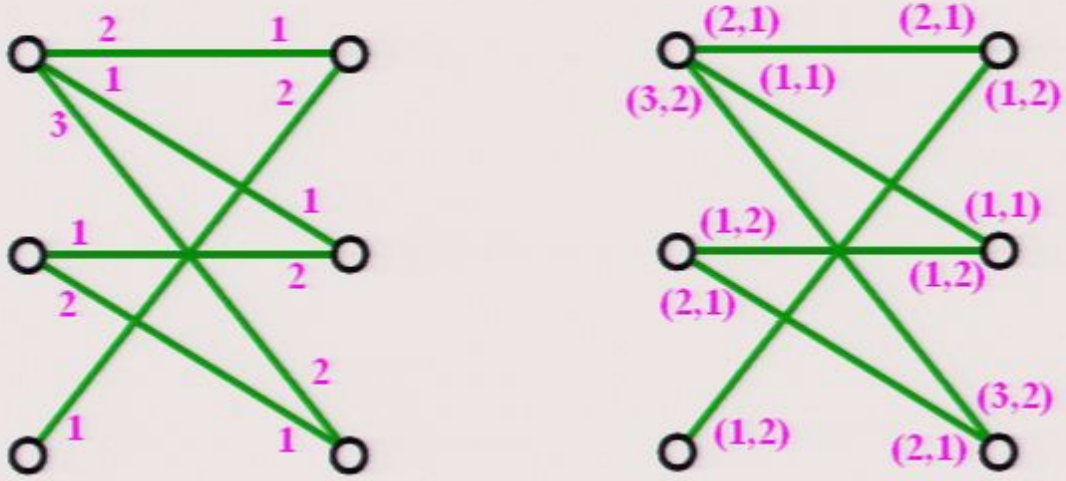
$$= \boldsymbol{V}_k^{\dagger}|v_k(x)\rangle|x\rangle$$

$$= |v_k(x)\rangle|x \oplus v_k(v_k(x))\rangle \;=\; |v_k(x)\rangle|0\rangle$$

Simulate evolution by $A \otimes I = \sum_k \boldsymbol{V}_k^{\dagger}\textbf{SWAP}\boldsymbol{V}_k$ (linear combination)

Simulation for *bipartite* graphs

Idea: reduce to the symmetric case



Conclusion: there is a quantum walk algorithm that solves **ST**-traversal with polynomially many queries and other operations

Simulation for *arbitrary* graphs

[Aharonov & Ta-Shma '03]: $O(n^9 d^3 t^{1.5}/\sqrt{\varepsilon})$ queries, where:

- $n = \log|G|$
- d is the maximum degree
- t is the time duration
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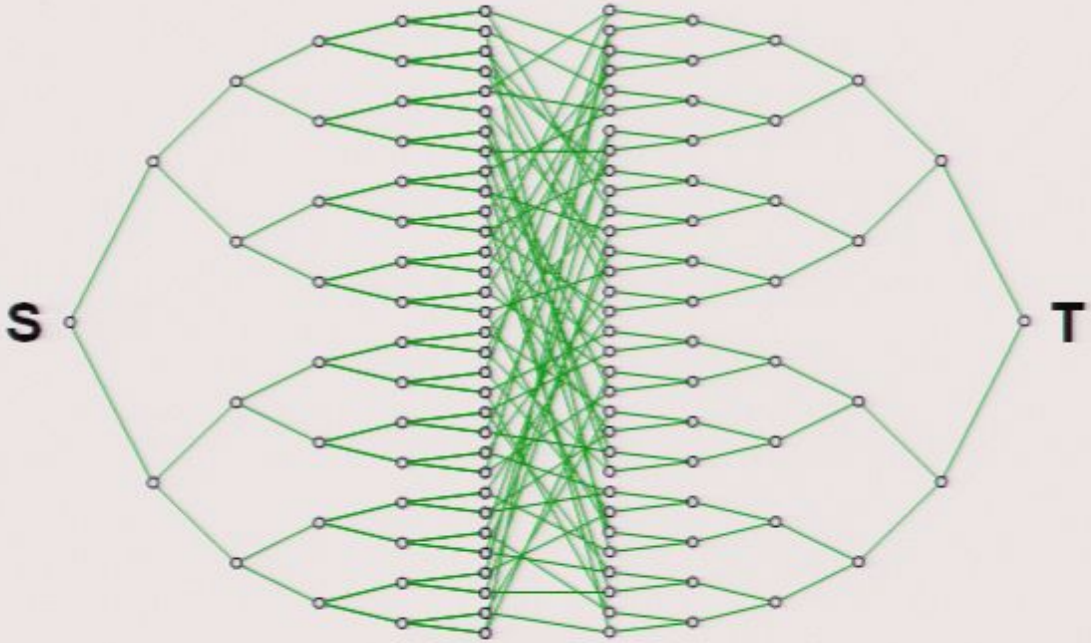
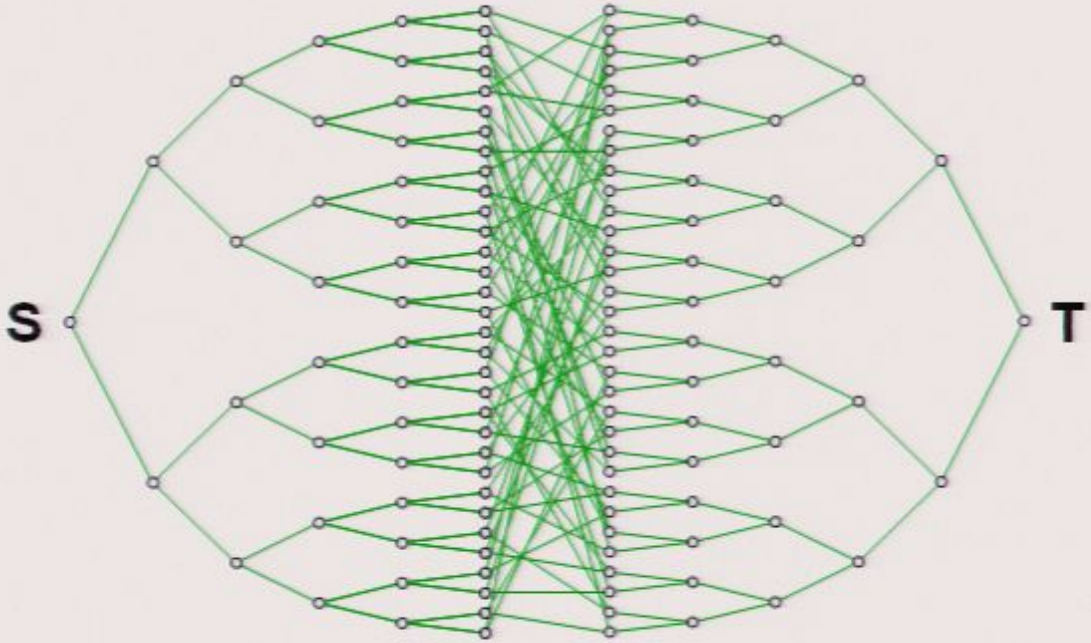
Consequence: for algorithms that employ continuous-time quantum walks, the overhead for the QW simulation can be made negligible in many cases

Classical hardness of ST-traversal



Classical hardness of ST-traversal

Define ***success*** as either finding T or finding a cycle



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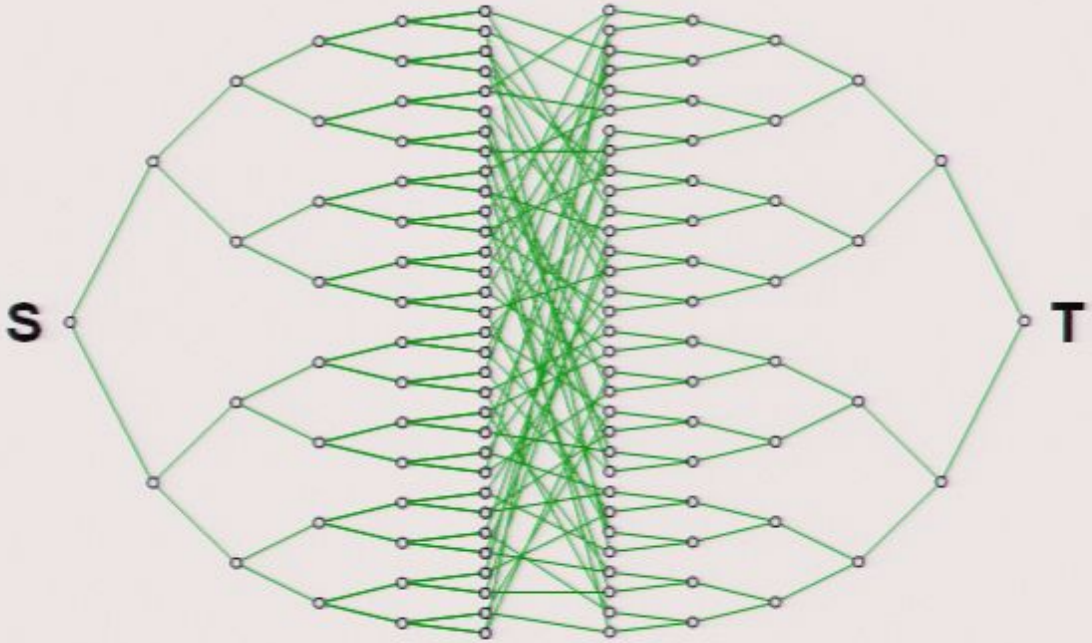

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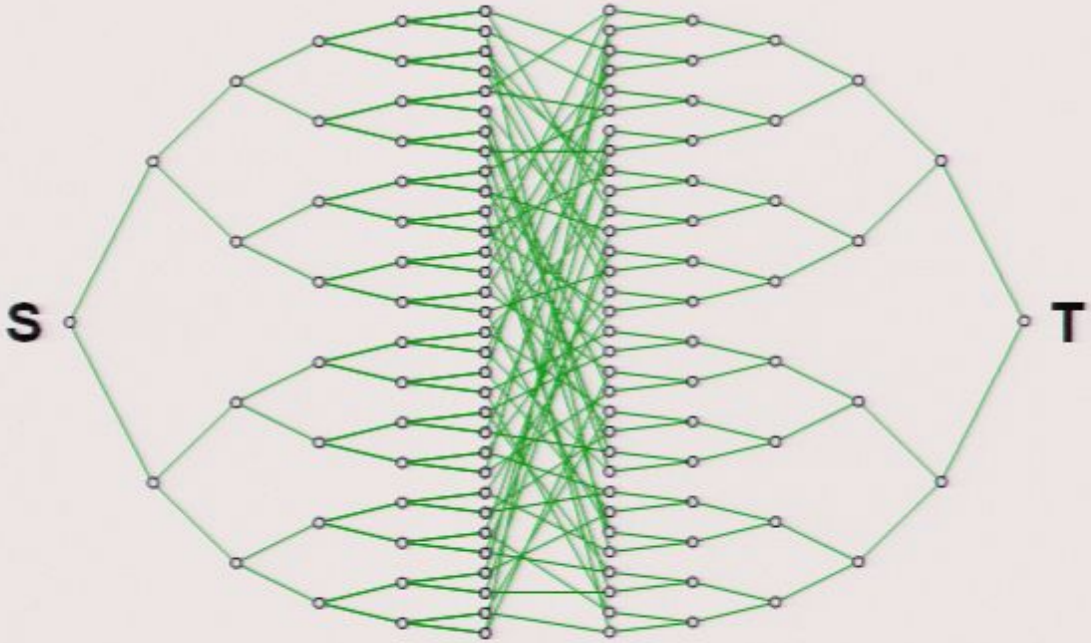
Define ***success*** as either finding T or finding a cycle

Then, prior to success, algorithm is growing a tree on G



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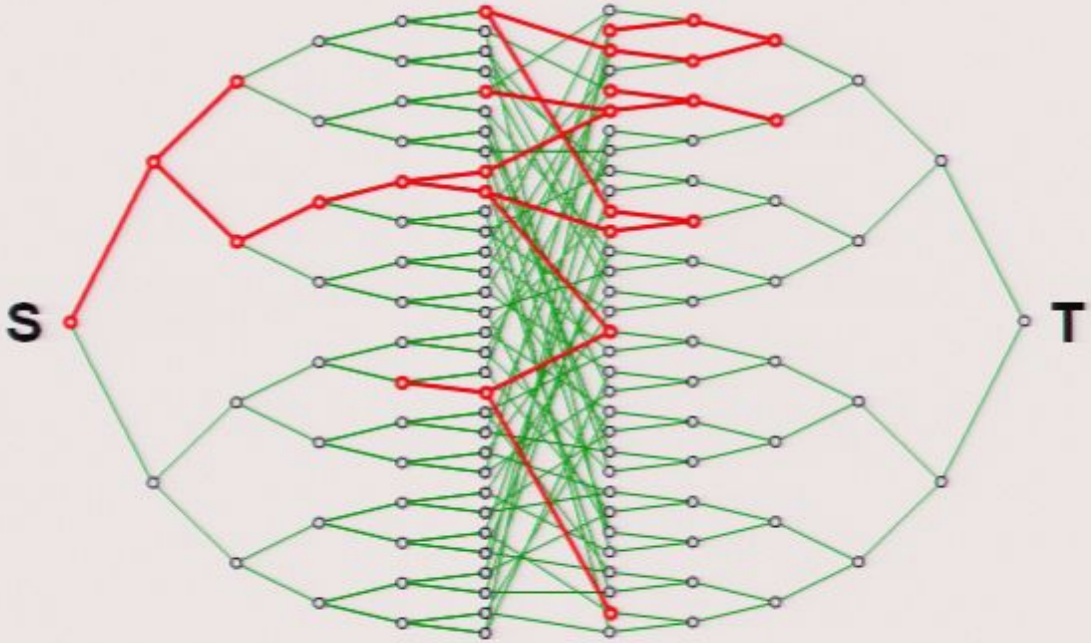

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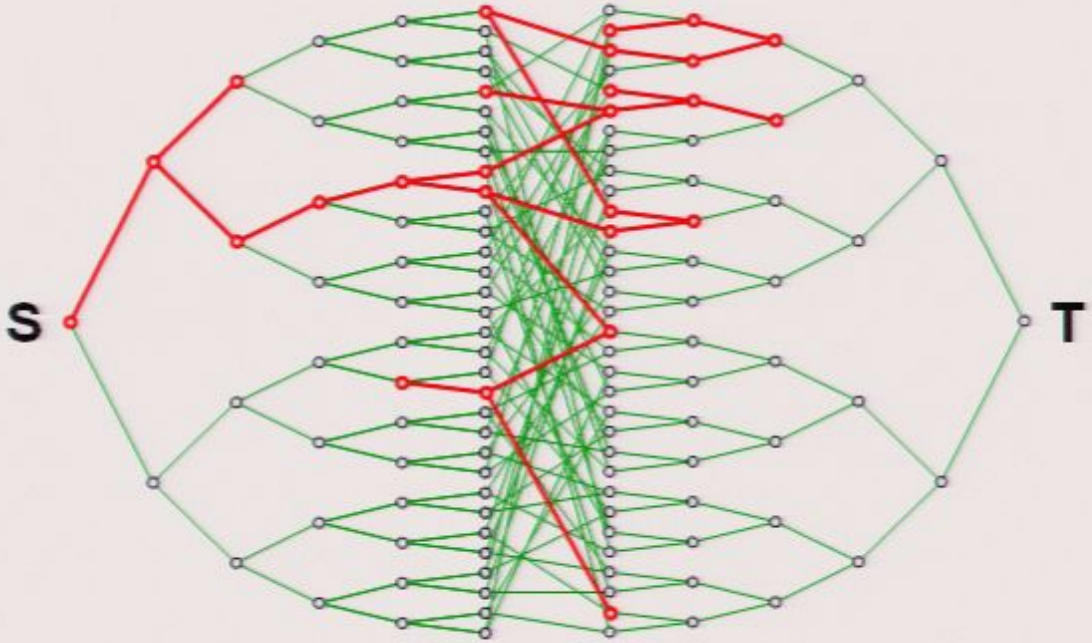

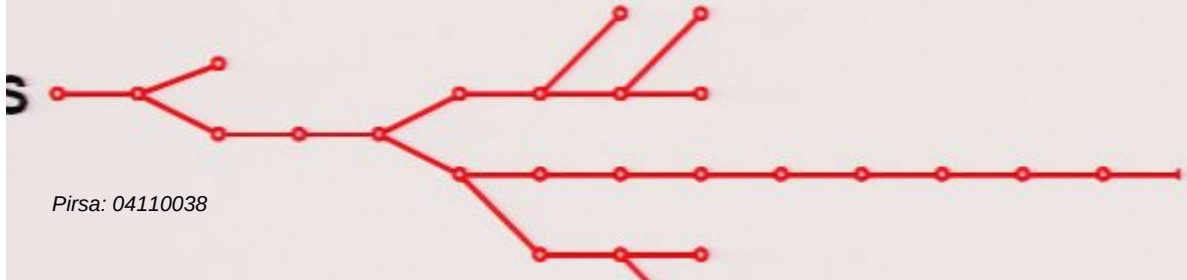

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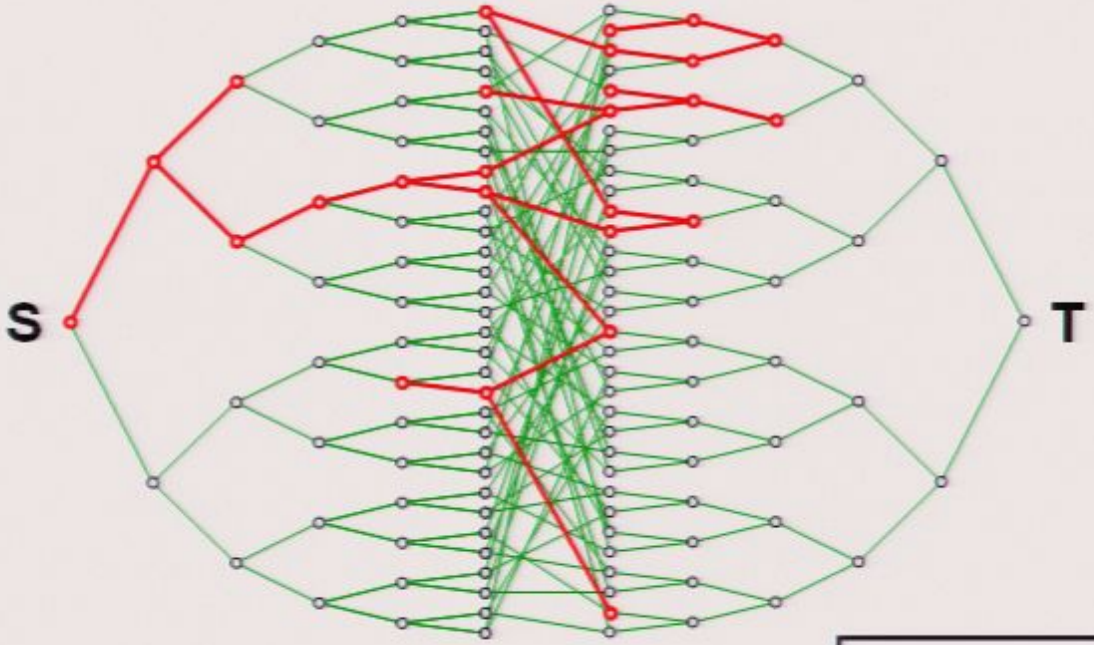
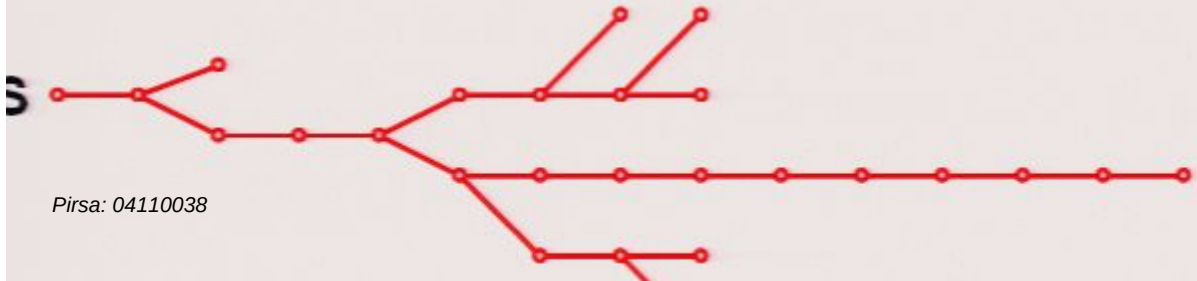
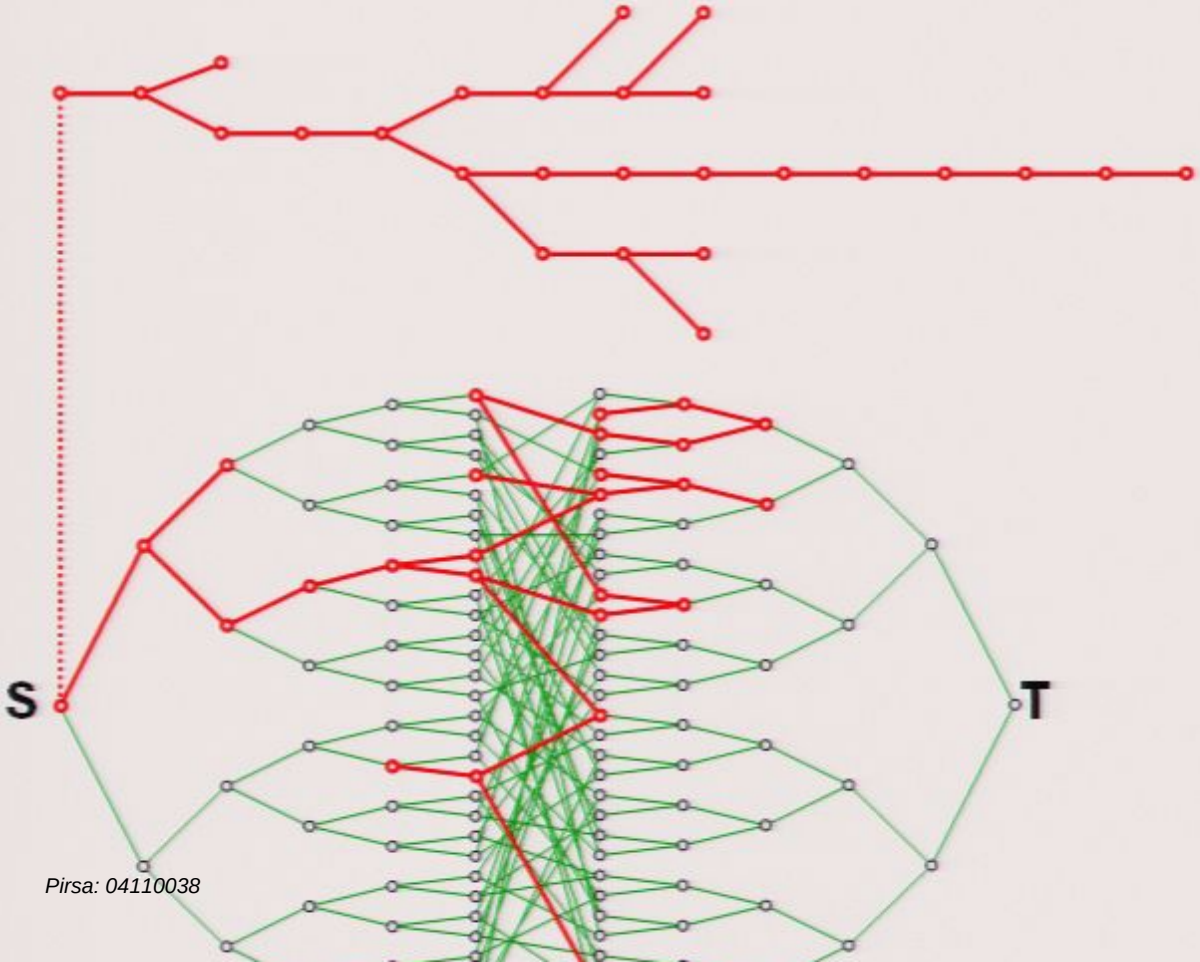




Can assume tree is pre-computed, since black box is random

Classical hardness of ST-traversal

Lemma: randomly embedding ***any*** tree of size m into G "succeeds" with probability at most $m^2/2^{n/2}$



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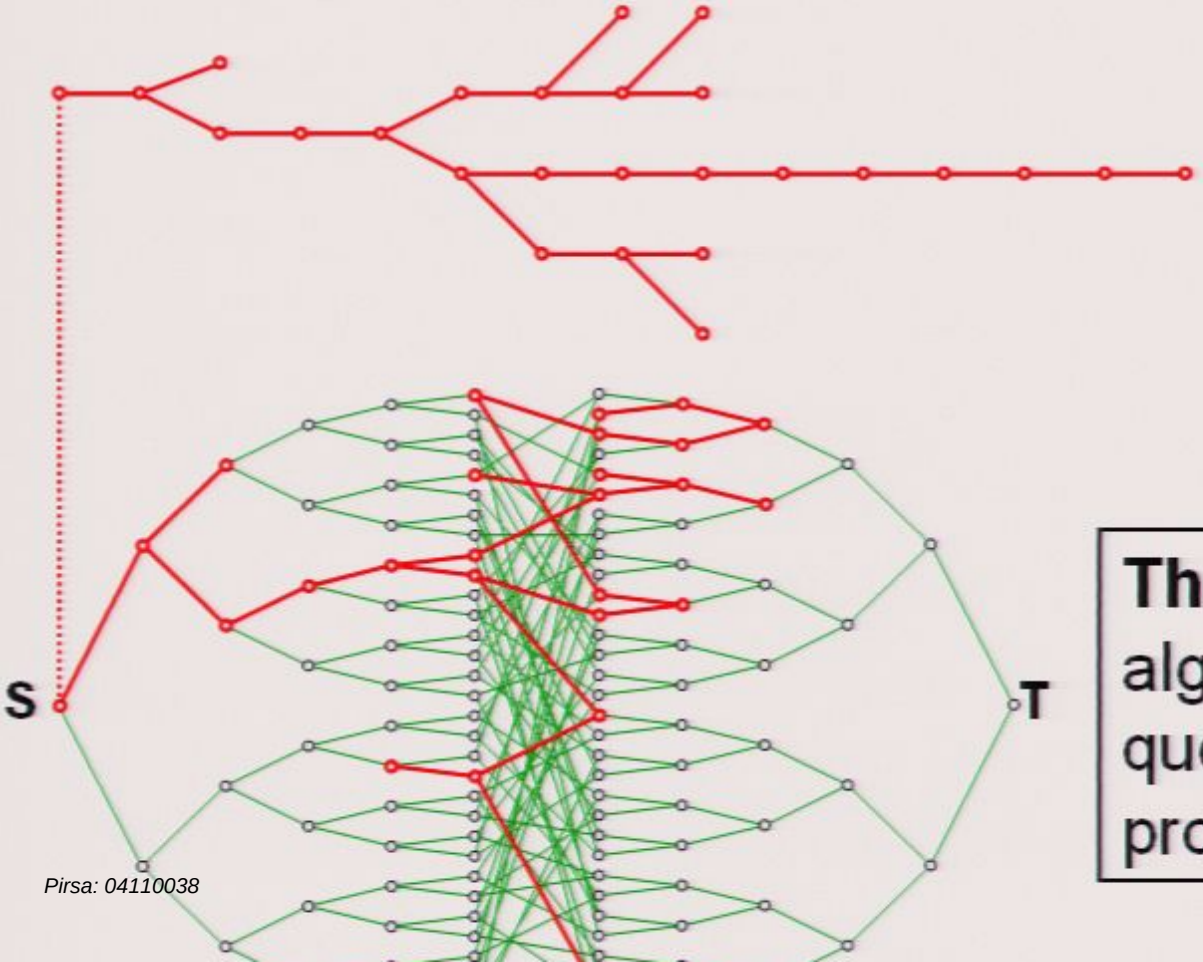


Theorem: *any* classical algorithm making $\leq 2^{n/6}$ queries succeeds with probability $\leq 4 \cdot 2^{-n/6}$

Concluding remarks

- Exponential quantum vs. classical speedup using a quantum walk algorithm
- Results easily adaptable to ***ST-connectivity*** problem
- Quantum algorithm solves **ST**-traversal without finding a path from **S** to **T**!

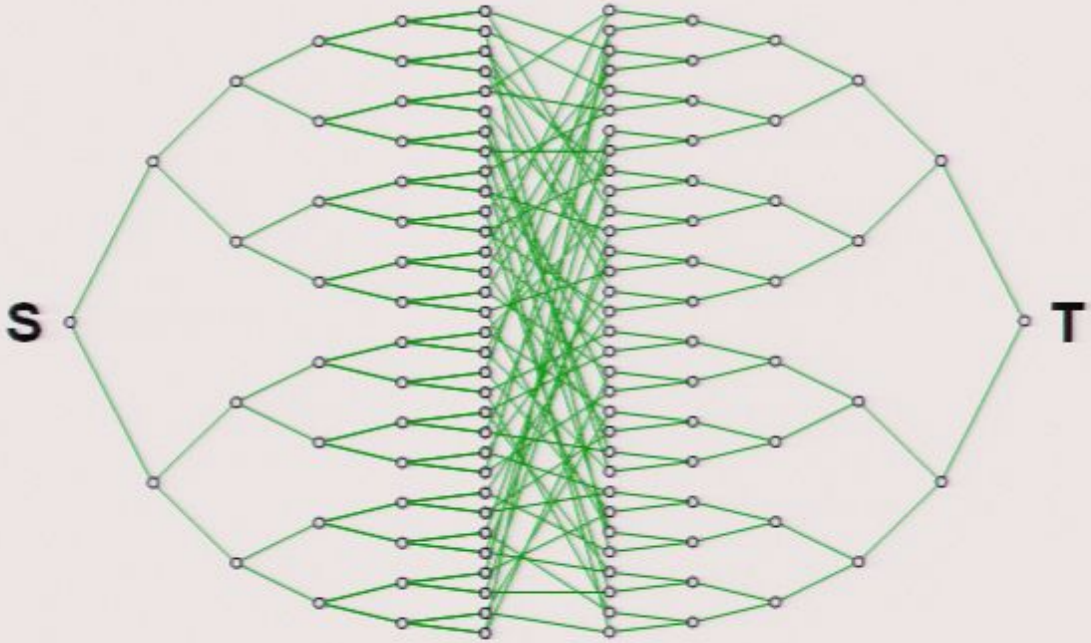
Open questions:

- Are there ***useful*** problems that can solved efficiently by quantum walks? Progress by [Ambainis '04]
- Is there a graph where **ST**-connectivity can be ***decided*** efficiently, but an **ST**-path cannot be found efficiently?

Classical hardness of ST-traversal

Define ***success*** as either finding T or finding a cycle

Then, prior to success, algorithm is growing a tree on G



Classical hardness of ST-traversal