Title: The Picard-Fuchs equation of the A_n family of Calabi-Yau varieties

Date: Nov 23, 2004 09:10 AM

URL: http://pirsa.org/04110034

Abstract:

①

The Picard-Fuchs equation of the A_n-family of Calabi-Yau Varieties

(work in progress)

Definition

The A_n-family of Calabi-Yaus (also called "Darboux varieties" (Borcea))

Def 1: $\mathcal{X}^{a_1, a_2, \ldots, a_{n+2}}$ is the resolution of:

$$\boxed{\left(X_1 + X_2 + \cdots + X_{n+1}\right)\left(\frac{a_1}{X_1} + \cdots + \frac{a_{n+1}}{X_{n+1}}\right) = a_{n+2}}$$

where $X_1 \ldots X_{n+1}$ co-ords of $\mathbb{P}^n$

parameters $a_1 \ldots a_{n+2}$

The Picard-Fuchs equation of the A_n-family of Calabi-Yau Varieties

①

(work in progress)

Definition

The A_n-family of Calabi-Yaus (also called "Darboux varieties" (Borcea))

Def 1: $\mathcal{X}^{a_1, a_2, \ldots, a_{n+2}}$ is the resolution of:

$$\boxed{(X_1 + X_2 + \cdots + X_{n+1})\left(\frac{a_1}{X_1} + \cdots + \frac{a_{n+1}}{X_{n+1}}\right) = a_{n+2}}$$

where $X_1 \ldots X_{n+1}$ co-ords of $\mathbb{P}^n$
parameters $a_1 \ldots a_{n+2}$

②

Def 2: $\mathcal{X}^{a_1 \ldots a_{n+2}}$ is a subfamily of the family of toric Calabi-Yau hypersurfaces constructed via Batyrev's construction from the Reflexive polytope with vertices $e_i - e_j$, simple roots of A_n root lattice.

Fan of the toric variety is Weyl chambers.

general member of family:

$$\sum a_{ij} X_i X_j^{-1} = t$$

The Picard - Fuchs equation of the A_n - family of Calabi - Yau Varieties

(1)

(work in progress)

Definition

The A_n - family of Calabi - Yaus (also called "Darboux varieties" (Borcea))

Def1: $\mathcal{X}^{a_1, a_2, \dots, a_{n+2}}$ is the resolution of:

$$\boxed{(X_1 + X_2 + \cdots + X_{n+1})\left(\frac{a_1}{X_1} + \cdots + \frac{a_{n+1}}{X_{n+1}}\right) = a_{n+2}}$$

where $X_1 \dots X_{n+1}$ co-ords of $\mathbb{P}^n$

parameters $a_1 \dots a_{n+2}$

(2)

Def2: $\mathcal{X}^{a_1 \dots a_{n+2}}$ is a subfamily of the family of toric Calabi-Yau hypersurfaces constructed via Batyrev's construction from the Reflexive polytope with vertices $e_i - e_j$, simple roots of A_n root lattice.

Fan of the toric variety is Weyl chambers.

general member of family:

$$\sum a_{ij} X_i X_j^{-1} = t$$

The Picard-Fuchs equation of the A_n-family of Calabi-Yau Varieties

①

(work in progress)

Definition

The A_n-family of Calabi-Yaus (also called "Darboux varieties" (Borcea))

Def 1: $\mathcal{X}^{a_1, a_2, \ldots, a_{n+2}}$ is the resolution of:

$$\boxed{(X_1 + X_2 + \cdots + X_{n+1})\left(\frac{a_1}{X_1} + \cdots + \frac{a_{n+1}}{X_{n+1}}\right) = a_{n+2}}$$

where $X_1 \ldots X_{n+1}$ co-ords of $\mathbb{P}^n$

parameters $a_1 \ldots a_{n+2}$

②

Def 2: $\mathcal{X}^{a_1 \ldots a_{n+2}}$ is a subfamily of the family of toric Calabi-Yau hypersurfaces constructed via Batyrev's construction from the Reflexive polytope with vertices $e_i - e_j$, simple roots of A_n root lattice. Fan of the toric variety is Weyl chambers.

general member of family:

$$\sum a_{ij} X_i X_j^{-1} = t$$

The Picard - Fuchs equation
of the A_n - family of Calabi-Yau Varieties ①

(work in progress)

Definition

The A_n - family of Calabi-Yaus
(also called "Darboux varieties" (Borcea))

Def 1: $\mathcal{X}^{a_1, a_2, \dots, a_{n+2}}$ is the resolution of:

$$\boxed{(X_1 + X_2 + \cdots + X_{n+1})\left(\frac{a_1}{X_1} + \cdots + \frac{a_{n+1}}{X_{n+1}}\right) = a_{n+2}}$$

where $X_1 \dots X_{n+1}$ co-ords of $\mathbb{P}^n$
parameters $a_1 \dots a_{n+2}$

②

Def 2: $\mathcal{X}^{a_1 \dots a_{n+2}}$ is a subfamily of
the family of toric Calabi-Yau hypersurfaces
constructed via Batyrev's construction
from the Reflexive polytope
with vertices $e_i - e_j$, simple roots of A_n root lattice.
Fan of the toric variety is Weyl chambers.

general member of family:

$$\sum a_{ij} X_i X_j^{-1} = t$$

The Picard-Fuchs equation

of the A_n-family of Calabi-Yau Varieties

(work in progress)

Definition

The A_n-family of Calabi-Yaus (also called "Darboux varieties" (Borcea))

Def 1: $\mathcal{X}^{a_1, a_2, \ldots, a_{n+2}}$ is the resolution of:

$$\boxed{(X_1 + X_2 + \cdots + X_{n+1})\left(\frac{a_1}{X_1} + \cdots + \frac{a_{n+1}}{X_{n+1}}\right) = a_{n+2}}$$

where $X_1 \ldots X_{n+1}$ co-ords of $\mathbb{P}^n$

parameters $a_1 \ldots a_{n+2}$

Def 2: $\mathcal{X}^{a_1 \ldots a_{n+2}}$ is a subfamily of the family of toric Calabi-Yau hypersurfaces constructed via Batyrev's construction from the Reflexive polytope with vertices $e_i - e_j$, simple roots of A_n root lattice.

Fan of the toric variety is Weyl chambers.

general member of family:

$$\sum a_{ij} X_i X_j^{-1} = t$$

The Picard-Fuchs equation of the A_n-family of Calabi-Yau Varieties

①

(work in progress)

Definition

The A_n-family of Calabi-Yaus (also called "Darboux varieties" (Borcea))

Def 1: $\mathfrak{X}^{a_1, a_2, \ldots, a_{n+2}}$ is the resolution of:

$$\boxed{(X_1 + X_2 + \cdots + X_{n+1})\left(\frac{a_1}{X_1} + \cdots + \frac{a_{n+1}}{X_{n+1}}\right) = a_{n+2}}$$

where $X_1 \ldots X_{n+1}$ co-ords of $\mathbb{P}^n$
parameters $a_1 \ldots a_{n+2}$

③

note, the toric variety is obtained from Cremona blow up of $\mathbb{P}^n$
– blow up all hyperplanes, lines, points, with zero coordinates.

Def^n 3
$\mathfrak{X}^{a_1 \ldots a_{n+2}}$ is the resolution of

$$X_1 + \cdots + X_{n+2} = 0$$

$$\frac{a_1}{X_1} + \cdots + \frac{a_{n+2}}{X_{n+2}} = 0$$

(Darboux: $n=2$; Borcea more generally)
Borcea: Toric complete intersection – can find mirror

The Picard-Fuchs equation of the A_n-family of Calabi-Yau Varieties

①

(work in progress)

Definition

The A_n-family of Calabi-Yaus (also called "Darboux varieties" (Borcea))

Def 1: $\mathfrak{X}^{a_1, a_2, \ldots, a_{n+2}}$ is the resolution of:

$$\boxed{(X_1 + X_2 + \cdots + X_{n+1})\left(\frac{a_1}{X_1} + \cdots + \frac{a_{n+1}}{X_{n+1}}\right) = a_{n+2}}$$

where $X_1 \ldots X_{n+1}$ co-ords of $\mathbb{P}^n$

parameters $a_1 \ldots a_{n+2}$

③

note, the toric variety is obtained from Cremona blow up of $\mathbb{P}^n$

– blow up all hyperplanes, lines, points, with zero coordinates.

Def^n 3

$\mathfrak{X}^{a_1 \ldots a_{n+2}}$ is the resolution of

$$X_1 + \cdots + X_{n+2} = 0$$

$$\frac{a_1}{X_1} + \cdots + \frac{a_{n+2}}{X_{n+2}} = 0$$

(Darboux: $n=2$; Borcea more generally)

Borcea: Toric complete intersection – can find mirror

The Picard-Fuchs equation of the A_n-family of Calabi-Yau Varieties ①

(work in progress)

Definition

The A_n-family of Calabi-Yaus (also called "Darboux varieties" (Borcea))

Def 1: $\mathcal{X}^{a_1,a_2,\dots,a_{n+2}}$ is the resolution of:

$$\boxed{(X_1 + X_2 + \cdots + X_{n+1})\left(\frac{a_1}{X_1} + \cdots + \frac{a_{n+1}}{X_{n+1}}\right) = a_{n+2}}$$

where $X_1 \dots X_{n+1}$ co-ords of $\mathbb{P}^n$
parameters $a_1 \dots a_{n+2}$

③

note, the toric variety is obtained from Cremona blow up of $\mathbb{P}^n$
– blow up all hyperplanes, lines, points, with zero coordinates.

Def^n 3

$\mathcal{X}^{a_1 \dots a_{n+2}}$ is the resolution of

$$X_1 + \cdots + X_{n+2} = 0$$

$$\frac{a_1}{X_1} + \cdots + \frac{a_{n+2}}{X_{n+2}} = 0$$

(Darboux: $n=2$; Borcea more generally)
Borcea: Toric complete intersection – can find mirror

③

note, the toric variety is obtained from Cremona blow up of $\mathbb{P}^n$

– blow up all hyperplanes, lines, points, with zero coordinates.

Def^n 3

$\mathfrak{X}^{a_1 \dots a_{n+2}}$ is the resolution of

$$X_1 + \dots + X_{n+2} = 0$$

$$\frac{a_1}{X_1} + \dots + \frac{a_{n+2}}{X_{n+2}} = 0$$

(Darboux : $n=2$; Borcea more generally)

Borcea: Toric complete intersection – can find mirror

④

Reasons to study $\mathfrak{X}^{a_1 \dots a_n}$:

1) Source of examples of rigid and non-rigid Calabi-Yaus which are modular

(joint work with Klaus Hulek for $n=4$)

(Livné – other cases)

2) Examples of Picard-Fuchs equations to extend Almkvist – Zudilin list.

3) Borcea uses these to give explicit examples of Special Lagrangians.

3

note, the toric variety is obtained from Cremona blow up of $\mathbb{P}^n$

– blow up all hyperplanes, lines, points, with zero coordinates.

Def^n 3

$\mathfrak{X}^{a_1 \ldots a_{n+2}}$ is the resolution of

$$X_1 + \cdots + X_{n+2} = 0$$

$$\frac{a_1}{X_1} + \cdots + \frac{a_{n+2}}{X_{n+2}} = 0$$

(Darboux $n=2$; Borcea more generally)

Borcea: Toric complete intersection – can find mirror

4

Reasons to study $\mathfrak{X}^{a_1 \ldots a_n}$:

1) Source of examples of rigid and non-rigid Calabi-Yaus which are modular

(joint work with Klaus Hulek for $n=4$)

(Livné – other cases)

2) Examples of Picard-Fuchs equations to extend Almkvist – Zudilin list.

3) Borcea uses these to give explicit examples of Special Lagrangians.

Reasons to study $\mathfrak{X}^{a_1 \dots a_n}$:

1) Source of examples of rigid and non-rigid Calabi-Yaus which are modular
(joint work with Klaus Hulek for n=4)
(Lionie - other cases)

2) Examples of Picard-Fuchs equations to extend Almkvist - Zudilin list.

3) Borcea uses these to give explicit examples of Special Lagrangians.

④

Heirachical Structure:

⑤

Relation of A_n family to lower dimensional families

$$\underbrace{\mathfrak{X}^{a_1 \dots a_n}}_{\substack{n-3 \text{ dimensional} \\ \text{C.Y.}}} \doteq \underbrace{\mathfrak{X}^{a_1, \dots a_j, t}}_{\substack{\text{family of} \\ j-2 \text{ dim}^l \text{ CYs}}} \times_{\mathbb{P}^1} \underbrace{\mathfrak{X}^{a_{j+1}, \dots, a_n, t}}_{\substack{\text{family of} \\ n-j-2 \text{ dim}^l \text{ CYs}}}$$

parameter t

(A_{n-1} CY)

$\underbrace{(A_j \text{ family}) \qquad (A_{n-j} \text{ family})}_{\text{take fibre product}}$

Heirachical structure:

⑤

Relation of A_n family to lower dimensional families

$$\underbrace{\mathfrak{X}^{a_1 \cdots a_n}}_{\substack{n-3 \text{ dimensional} \\ \text{C.Y.}}} \doteq \underbrace{\mathfrak{X}^{a_1, \cdots a_j, t}}_{\substack{\text{family of} \\ j-2 \text{ dim}^{l} \text{ CYs}}} \times_{\mathbb{P}^1} \underbrace{\mathfrak{X}^{a_{j+1}, \cdots, a_n, t}}_{\substack{\text{family of} \\ n-j-2 \text{ dim}^{l} \text{ CYs}}}$$

parameter t

$(A_{n-1}$ CY$)$

$\underbrace{(A_j \text{ family}) \quad (A_{n-j} \text{ family})}_{\text{take fibre product}}$

⑥

this comes from writing

$$\mathfrak{X}^{a_1 \cdots a_n} : \quad X_1 + \cdots + X_n = 0 , \quad \frac{a_1}{X_1} + \frac{a_2}{X_2} + \cdots + \frac{a_n}{X_n}$$

as

$$X_1 + \cdots + X_j = -(X_{j+1} + \cdots + X_n)$$

$$\frac{a_1}{X_1} + \cdots + \frac{a_j}{X_j} = -\left(\frac{a_{j+1}}{X_{j+1}} + \cdots + \frac{a_n}{X_n}\right)$$

so

$$\sum_{i=1}^{j} X_i \sum_{i=1}^{j} \frac{a_i}{X_i} = \sum_{i=j+1}^{n} X_i \sum_{i=j+1}^{n} \frac{a_i}{X_i} = t$$

gives

$$\mathfrak{X}^{a_1 \cdots a_j, t} \times_{\mathbb{P}^1} \mathfrak{X}^{a_{j+1} \cdots a_n, t}$$

Heirachical structure: (5)

Relation of A_n family to lower dimensional families

$$\underbrace{\mathfrak{X}^{a_1 \cdots a_n}}_{\substack{n-3 \text{ dimensional} \\ \text{C.Y.}}} \doteq \underbrace{\mathfrak{X}^{a_1, \cdots a_j, t}}_{\substack{\text{family of} \\ j-2 \text{ dim}^{\underline{l}} \text{ CYs}}} \times_{\mathbb{P}^1} \underbrace{\mathfrak{X}^{a_{j+1}, \cdots, a_n, t}}_{\substack{\text{family of} \\ n-j-2 \text{ dim}^{\underline{l}} \text{ CYs}}}$$

parameter t

$(A_{n-1}$ CY$)$

$$\underbrace{(A_j \text{ family}) \qquad (A_{n-j} \text{ family})}_{\text{take fibre product}}$$

take <u>fibre product</u>

(6)

this comes from writing

$$\mathfrak{X}^{a_1 \cdots a_n} : \quad X_1 + \cdots + X_n = 0 \,, \quad \frac{a_1}{X_1} + \frac{a_2}{X_2} + \cdots + \frac{a_n}{X_n}$$

as

$$X_1 + \cdots + X_j = -(X_{j+1} + \cdots + X_n)$$

$$\frac{a_1}{X_1} + \cdots + \frac{a_j}{X_j} = -\left(\frac{a_{j+1}}{X_{j+1}} + \cdots + \frac{a_n}{X_n}\right)$$

so

$$\sum_{i=1}^{j} X_i \sum_{i=1}^{j} \frac{a_i}{X_i} = \sum_{i=j+1}^{n} X_i \sum_{i=j+1}^{n} \frac{a_i}{X_i} = t$$

gives $\quad \mathfrak{X}^{a_1 \cdots a_j, t} \times_{\mathbb{P}^1} \mathfrak{X}^{a_{j+1} \cdots a_n, t}$

Heirachical structure: (5)

Relation of A_n family to lower dimensional families

$$\underbrace{\mathfrak{X}^{a_1 \cdots a_n}}_{\substack{n-3 \text{ dimensional} \\ \text{C.Y.}}} \doteq \underbrace{\mathfrak{X}^{a_1, \cdots a_j, t}}_{\substack{\text{family of} \\ j-2 \text{ dim}^{\underline{l}} \text{ CYs}}} \times_{\mathbb{P}^1} \underbrace{\mathfrak{X}^{a_{j+1}, \cdots, a_n, t}}_{\substack{\text{family of} \\ n-j-2 \text{ dim}^{\underline{l}} \text{ CYs}}}$$

parameter t

$(A_{n-1}$ CY$)$

$$\underbrace{(A_j \text{ family}) \quad (A_{n-j} \text{ family})}_{\text{take fibre product}}$$

(6)

this comes from writing

$$\mathfrak{X}^{a_1 \cdots a_n} : \quad X_1 + \cdots + X_n = 0 , \quad \frac{a_1}{X_1} + \frac{a_2}{X_2} + \cdots + \frac{a_n}{X_n}$$

as

$$X_1 + \cdots + X_j = -(X_{j+1} + \cdots + X_n)$$

$$\frac{a_1}{X_1} + \cdots + \frac{a_j}{X_j} = -\left(\frac{a_{j+1}}{X_{j+1}} + \cdots + \frac{a_n}{X_n}\right)$$

so

$$\sum_{i=1}^{j} X_i \sum_{i=1}^{j} \frac{a_i}{X_i} = \sum_{i=j+1}^{n} X_i \sum_{i=j+1}^{n} \frac{a_i}{X_i} = t$$

gives

$$\mathfrak{X}^{a_1 \cdots a_j, t} \times_{\mathbb{P}^1} \mathfrak{X}^{a_{j+1} \cdots a_n, t}$$

Heirachical structure:

Relation of A_n family to lower dimensional families

$$\underbrace{\mathfrak{X}^{a_1 \cdots a_n}}_{\substack{n-3 \text{ dimensional} \\ \text{C.Y.}}} \doteq \underbrace{\mathfrak{X}^{a_1, \cdots a_j, t}}_{\substack{\text{family of} \\ j-2 \text{ dim}^{\text{l}} \text{ CYs}}} \times_{\mathbb{P}^1} \underbrace{\mathfrak{X}^{a_{j+1}, \cdots, a_n, t}}_{\substack{\text{family of} \\ n-j-2 \text{ dim}^{\text{l}} \text{ CYs}}}$$

parameter t

$$(A_{n-1} \text{ CY}) \qquad \underbrace{(A_j \text{ family}) \qquad (A_{n-j} \text{ family})}_{\text{take fibre product}}$$

So e.g. the Calabi-Yau 3fold $\mathfrak{X}^{a_1 \cdots a_6}$

has a K3 fibration:

$$(X_1 + X_2 + X_3 + X_4)\left(\frac{a_1}{X_1} + \frac{a_2}{X_2} + \frac{a_3}{X_3} + \frac{a_4}{X_4}\right) = t$$

$$(\lambda + 1)\left(\frac{a_5}{\lambda} + a_6\right) = t$$

- there are $\binom{6}{2} = 15$ such fibrations
- this is a double cover of a rational K3 fibration.
- can view as double cover of $\mathbb{P}^2$ branched along octic.

Heirachical structure:

(5)

Relation of A_n family to lower dimensional families

$$\underbrace{\mathfrak{X}^{a_1 \cdots a_n}}_{\substack{n-3 \text{ dimensional} \\ \text{C.Y.}}} \doteq \underbrace{\mathfrak{X}^{a_1, \cdots a_j, t}}_{\substack{\text{family of} \\ j-2 \text{ dim}^{\text{c}} \text{ CYs}}} \times_{\mathbb{P}^1} \underbrace{\mathfrak{X}^{a_{j+1}, \cdots, a_n, t}}_{\substack{\text{family of} \\ n-j-2 \text{ dim}^{\text{c}} \text{ CYs}}}$$

parameter t

$(A_{n-1}$ CY$)$

$(A_j$ family$)$ $(A_{n-j}$ family$)$

take fibre product

(7)

So e.g. the Calabi-Yau 3fold $\mathfrak{X}^{a_1 \cdots a_6}$

has a K3 fibration:

$$(X_1 + X_2 + X_3 + X_4)\left(\frac{a_1}{X_1} + \frac{a_2}{X_2} + \frac{a_3}{X_3} + \frac{a_4}{X_4}\right) = t$$

$$(\lambda + 1)\left(\frac{a_5}{\lambda} + a_6\right) = t$$

- there are $\binom{6}{2} = 15$ such fibrations
- this is a double cover of a rational K3 fibration.
- can view as double cover of $\mathbb{P}^2$ branched along octic.

Heirachical structure: ⑤

Relation of A_n family to lower dimensional families

$$\underbrace{\mathfrak{X}^{a_1 \cdots a_n}}_{\substack{n-3 \text{ dimensional} \\ \text{C.Y.}}} \doteq \underbrace{\mathfrak{X}^{a_1, \cdots a_j, t}}_{\substack{\text{family of} \\ j-2 \text{ dim}^{\underline{e}} \text{ CYs}}} \times_{\mathbb{P}^1} \underbrace{\mathfrak{X}^{a_{j+1}, \cdots, a_n, t}}_{\substack{\text{family of} \\ n-j-2 \text{ dim}^{\underline{e}} \text{ CYs}}}$$

parameter t

(A_{n-1} CY) (A_j family) (A_{n-j} family)

take fibre product

---

⑦

So e.g. the Calabi-Yau 3fold $\mathfrak{X}^{a_1 \cdots a_6}$

has a K3 fibration:

$$\left(X_1 + X_2 + X_3 + X_4\right)\left(\frac{a_1}{X_1} + \frac{a_2}{X_2} + \frac{a_3}{X_3} + \frac{a_4}{X_4}\right) = t$$

$$(\lambda + 1)\left(\frac{a_5}{\lambda} + a_6\right) = t$$

- there are $\binom{6}{2} = 15$ such fibrations
- this is a double cover of a rational K3 fibration.
- can view as double cover of $\mathbb{P}^2$ branched along octic.

7

So eg. the Calabi-Yau 3fold $\mathfrak{X}^{a_1 \cdots a_6}$

has a K3 fibration:

$$\left(X_1 + X_2 + X_3 + X_4\right)\left(\frac{a_1}{X_1} + \frac{a_2}{X_2} + \frac{a_3}{X_3} + \frac{a_4}{X_4}\right) = t$$

$$(\lambda+1)\left(\frac{a_5}{\lambda} + a_6\right) = t$$

- there are $\binom{6}{2} = 15$ such fibrations
- this is a double cover of a rational K3 fibration.
- can view as double cover of $\mathbb{P}^2$ branched along octic.

So $^{eg.}$ the Calabi-Yau 3fold $\mathfrak{X}^{a_1 \dots a_6}$ ⑦

has a K3 fibration:

$$(X_1 + X_2 + X_3 + X_4)\left(\frac{a_1}{X_1} + \frac{a_2}{X_2} + \frac{a_3}{X_3} + \frac{a_4}{X_4}\right) = t$$

$$(\lambda + 1)\left(\frac{a_5}{\lambda} + a_6\right) = t$$

- there are $\binom{6}{2} = 15$ such fibrations
- this is a double cover of a rational K3 fibration.
- can view as double cover of $\mathbb{P}^3$ branched along octic.

(7)

So e.g. the Calabi-Yau 3fold $\mathfrak{X}^{a_1 \dots a_6}$

has a K3 fibration:

$$(X_1 + X_2 + X_3 + X_4)\left(\frac{a_1}{X_1} + \frac{a_2}{X_2} + \frac{a_3}{X_3} + \frac{a_4}{X_4}\right) = t$$

$$(\lambda + 1)\left(\frac{a_5}{\lambda} + a_6\right) = t$$

- there are $\binom{6}{2} = 15$ such fibrations
- this is a double cover of a rational K3 fibration.
- can view as double cover of $\mathbb{P}^2$ branched along octic.

(8)

Also $\mathfrak{X}^{a_1 \dots a_6}$ is a fibre product of families of elliptic curves:

$$(X_1 + X_2 + X_3)\left(\frac{a_1}{X_1} + \frac{a_2}{X_2} + \frac{a_3}{X_3}\right) = t$$

$$(X_4 + X_5 + X_6)\left(\frac{a_4}{X_4} + \frac{a_5}{X_5} + \frac{a_6}{X_6}\right) = t$$

- there are $\binom{6}{3} = 20$ such fibrations
- can now apply C. Schoen's results about Calabi-Yau 3 folds given by fibre products of familys of elliptic curves
- this gives many elliptic surfaces inside $\mathfrak{X}^{a_1 \dots a_6}$ - can sometimes be used to prove modularity (jt. with K Hulek)

Also $\mathcal{X}^{a_1 \cdots a_6}$ is a fibre product of families of elliptic curves:

$$(X_1 + X_2 + X_3)\left(\frac{a_1}{X_1} + \frac{a_2}{X_2} + \frac{a_3}{X_3}\right) = t$$

$$(X_4 + X_5 + X_6)\left(\frac{a_4}{X_4} + \frac{a_5}{X_5} + \frac{a_6}{X_6}\right) = t$$

- there are $\binom{6}{3} = 20$ such fibrations
- can now apply C. Schoen's results about Calabi-Yau 3 folds given by fibre products of familys of elliptic curves
- this gives many elliptic surfaces inside $\mathcal{X}^{a_1 \cdots a_6}$ - can sometimes be used to prove modularity (jt. with K Hulek)

(8)

Modularity results for A_4 case (joint with K. Hulek)

(9)

$\mathcal{X}^{a_1, a_2, a_3 \cdots a_6}$ is modular for:

$a_1, a_2 \cdots a_6 =$	h^{12}
1, 1, 1, 1, 1, 1	0
1, 1, 1, 1, 1, 9	0
1, 1, 1, 1, 1, 25	4
1, 1, 1, 1, 4, 4	0
1, 1, 1, 9, 9, 9	2
1, 1, 1, 4, 4, 9	0
1, 1, 4, 4, 4, 16	1

Modularity results for A_4 case

(joint with K. Hulek)

$\mathcal{X}^{a_1, a_2, a_3 \cdots a_6}$ is modular for:

$a_1, a_2 \cdots a_6 =$

$a_1, a_2 \cdots a_6$	h^{12}
1, 1, 1, 1, 1, 1	0
1, 1, 1, 1, 1, 9	0
1, 1, 1, 1, 1, 25	4
1, 1, 1, 1, 4, 4	0
1, 1, 1, 9, 9, 9	2
1, 1, 1, 4, 4, 9	0
1, 1, 4, 4, 4, 16	1

Modularity results for A_4 case

(joint with K. Hulek)

9

$X^{a_1,a_2,a_3\dots a_6}$ is modular for:

$a_1, a_2 \dots a_6 =$	h^{12}
1, 1, 1, 1, 1, 1	0
1, 1, 1, 1, 1, 9	0
1, 1, 1, 1, 1, 25	4
1, 1, 1, 1, 4, 4	0
1, 1, 1, 9, 9, 9	2
1, 1, 1, 4, 4, 9	0
1, 1, 4, 4, 4, 16	1

Useful fact for proving modularity:

10

if $b_1, \dots, b_5$ & $b_1 + \dots + b_5 \neq 0$,
$(b_1 : b_2 : b_3 : b_4 : b_5)$ is a singularity on $X_{b_1^2, b_2^2, \dots, b_5^2, (b_1+\dots+b_5)^2}$

& for $a_1, a_2 \dots a_n \neq 0$, After resolving singularities with $\prod X_i = 0$,
$X^{a_1 \dots a_n}$ has a singularity $\Leftrightarrow \sum_i \pm\sqrt{a_i} = 0$
for some sign choice.

singularities = # subsets of $\{\sqrt{a_1}, \dots, \sqrt{a_n}\}$ summing to 0
– these reduce h^{12}

if there are sufficiently many elliptic surfaces in $X^{a_1 \dots a_6}$
ie, sets a_i, a_j, a_k, with $\pm\sqrt{a_i} \pm \sqrt{a_j} \pm \sqrt{a_k} \neq 0$ for all sign choices,
end up with proof of modularity.

Modularity results for A_4 case

(joint with K. Hulek)

9

$\mathcal{X}^{a_1, a_2, a_3 \cdots a_6}$ is modular for:

$a_1, a_2 \cdots a_6 =$

	h^{12}
1, 1, 1, 1, 1, 1	0
1, 1, 1, 1, 1, 9	0
1, 1, 1, 1, 1, 25	4
1, 1, 1, 1, 4, 4	0
1, 1, 1, 9, 9, 9	2
1, 1, 1, 4, 4, 9	0
1, 1, 4, 4, 4, 16	1

10

Useful fact for proving modularity:

if $b_1, \ldots, b_5$ & $b_1 + \cdots + b_5 \neq 0$,

$(b_1 : b_2 : b_3 : b_4 : b_5)$ is a singularity on $\mathcal{X}_{b_1^2, b_2^2 \cdots, b_5^2, (b_1 + \cdots + b_5)^2}$

& for $a_1, a_2 \cdots a_n \neq 0$, After resolving singularities with $\prod X_i = 0$,

$\mathcal{X}^{a_1 \cdots a_n}$ has a singularity $\Leftrightarrow \sum_{i} \pm\sqrt{a_i} = 0$

for some sign choice.

singularities = # subsets of $\{\sqrt{a_1}, \ldots, \sqrt{a_n}\}$ summing to 0

– these reduce h^{12}

if there are sufficiently many elliptic surfaces in $\mathcal{X}^{a_1 \cdots a_6}$

ie, sets a_i, a_j, a_k, with $\pm\sqrt{a_i} \pm \sqrt{a_j} \pm \sqrt{a_k} \neq 0$ for all sign choices,

end up with proof of modularity.

Also $\mathfrak{X}^{a_1 \dots a_6}$ is a fibre product of families of elliptic curves:

$$\left(X_1 + X_2 + X_3\right)\left(\frac{a_1}{X_1} + \frac{a_2}{X_2} + \frac{a_3}{X_3}\right) = t$$

$$\left(X_4 + X_5 + X_6\right)\left(\frac{a_4}{X_4} + \frac{a_5}{X_5} + \frac{a_6}{X_6}\right) = t$$

8

- there are $\binom{6}{3} = 20$ such fibrations
- can now apply C. Schoen's results about Calabi-Yau 3 folds given by fibre products of familys of elliptic curves
- this gives many elliptic surfaces inside $\mathfrak{X}^{a_1 \dots a_6}$ - can sometimes be used to prove modularity (j.t. with K Hulek)

10

Useful fact for proving modularity:

if $b_1, \dots, b_5$ & $b_1 + \dots + b_5 \neq 0$, $(b_1 : b_2 : b_3 : b_4 : b_5)$ is a singularity on $\mathfrak{X}_{b_1^2, b_2^2, \dots, b_5^2, (b_1 + \dots + b_5)^2}$

for $a_1, a_2 \dots a_n \neq 0$, After resolving singularities with $\prod X_i = 0$, $\mathfrak{X}^{a_1 \dots a_n}$ has a singularity $\Leftrightarrow \sum_i \pm\sqrt{a_i} = 0$ for some sign choice.

singularities = # subsets of $\{\sqrt{a_1}, \dots \sqrt{a_n}\}$ summing to 0
- these reduce h^{12}

If there are sufficiently many elliptic surfaces in $\mathfrak{X}^{a_1 \dots a_6}$ ie, sets a_i, a_j, a_k, with $\pm\sqrt{a_i} \pm \sqrt{a_j} \pm \sqrt{a_k} \neq 0$ for all sign choices, end up with proof of modularity.

10

Useful fact for proving modularity:

if $b_1, \dots, b_5$ & $b_1 + \dots + b_5 \neq 0$,

$(b_1 : b_2 : b_3 : b_4 : b_5)$ is a singularity on $\mathcal{X}_{b_1^2, b_2^2, \dots, b_5^2, (b_1 + \dots + b_5)^2}$

& for $a_1, a_2 \dots a_n \neq 0$, After resolving singularities with $\prod X_i = 0$,

$\underline{\mathcal{X}^{a_1 \dots a_n} \text{ has a singularity} \Leftrightarrow \sum_{i} \pm\sqrt{a_i} = 0}$

for some sign choice.

singularities = # subsets of $\{\sqrt{a_1}, \dots, \sqrt{a_n}\}$ summing to 0

– these reduce h^{12}

if there are sufficiently many elliptic surfaces in $\mathcal{X}^{a_1 \dots a_n}$

ie, sets a_i, a_j, a_k, with $\pm\sqrt{a_i} \pm \sqrt{a_j} \pm \sqrt{a_k} \neq 0$ for all sign choices,

end up with proof of modularity.

Modularity results for A4 case

9

(joint with K. Hulek)

$\mathcal{X}^{a_1,a_2,a_3\dots a_6}$ is modular for:

$a_1, a_2 \dots a_6 =$	h^{12}
1, 1, 1, 1, 1, 1	0
1, 1, 1, 1, 1, 9	0
1, 1, 1, 1, 1, 25	4
1, 1, 1, 1, 4, 4	0
1, 1, 1, 9, 9, 9	2
1, 1, 1, 4, 4, 9	0
1, 1, 4, 4, 4, 16	1

10

Useful fact for proving modularity:

if $b_1,\dots,b_5$ & $b_1+\dots+b_5 \neq 0$,
$(b_1 : b_2 : b_3 : b_4 : b_5)$ is a singularity on $\mathcal{X}_{b_1^2, b_2^2, \dots, b_5^2, (b_1+\dots+b_5)^2}$

& for $a_1, a_2 \dots a_n \neq 0$, After resolving singularities with $\prod X_i = 0$,
$\mathcal{X}^{a_1 \dots a_n}$ has a singularity $\Leftrightarrow \sum \pm\sqrt{a_i} = 0$
for some sign choice.

\# singularities = \# subsets of $\{\sqrt{a_1}, \dots, \sqrt{a_n}\}$ summing to 0
– these reduce h^{12}

if there are sufficiently many elliptic surfaces in $\mathcal{X}^{a_1 \dots a_6}$
ie, sets a_i, a_j, a_k, with $\pm\sqrt{a_i} \pm \sqrt{a_j} \pm \sqrt{a_k} \neq 0$ for all sign choices,
end up with proof of modularity.

Modularity results for A4 case

(joint with K. Hulek)

$\mathfrak{X}^{a_1, a_2, a_3 \dots a_6}$ is modular for:

$a_1, a_2 \dots a_6 =$

$a_1, \dots, a_6$	h^{12}
1, 1, 1, 1, 1, 1	0
1, 1, 1, 1, 1, 9	0
1, 1, 1, 1, 1, 25	4
1, 1, 1, 1, 4, 4	0
1, 1, 1, 9, 9, 9	2
1, 1, 1, 4, 4, 9	0
1, 1, 4, 4, 4, 16	1

(9)

Further conjecturally modular $\mathfrak{X}^{a_1 \dots a_6}$

(11)

C. Meyer conjectures:

for $a_1, \dots a_6 =$

$a_1, \dots, a_6$	h^{12}
1, 1, 1, 1, 1, −7	5
1, 1, −1, −1, 4, −4	3
1, 1, 1, −1, −1, −1	5
1, 1, 9, 9, 9, 81	3
1, 9, 9, 9, 9, 25	1

$\mathfrak{X}^{a_1, \dots, a_6}$ is modular

but after extracting elliptic surface contributions from H^3,

still have dim 4 part left. modular forms correspond

Further conjecturally modular $\mathcal{X}^{a_1 \ldots a_6}$ (11)

C. Meyer conjectures:

for $a_1, \ldots a_6 =$

	h^{12}
1, 1, 1, 1, 1, −7	5
1, 1, −1, −1, 4, −4	3
1, 1, 1, −1, −1, −1	5
1, 1, 9, 9, 9, 81	3
1, 9, 9, 9, 9, 25	1

$\mathcal{X}^{a_1 \ldots a_6}$ is modular

but after extracting elliptic surface contributions from H^3, still have dim 4 part left. modular forms correspond

to 2 dimensional subspaces of H^3. (12)

So we don't yet have a proof of modularity.

Alternative approach:

try to prove modularity of some variation $\mathcal{X}^{a_1, a_2, a_3, a_4, t}$ of family of K3 surfaces,

use this to prove modularity of double cover

$$(\lambda+1)(a_5/\lambda + a_6) = t$$

& Study relationship between Coefficients of L-series for $\mathcal{X}^{a_1 \ldots a_4, a_5, a_6}$ & expansion of sol^n of Picard-Fuchs eq^n.

— hope to use congruence results of Steinstra, etc.

Further conjecturally modular $\mathcal{X}^{a_1 \dots a_6}$ (11)

C. Meyer conjectures:

for $a_1, \dots a_6 =$

	h^{12}
1, 1, 1, 1, 1, −7	5
1, 1, −1, −1, 4, −4	3
1, 1, 1, −1, −1, −1	5
1, 1, 9, 9, 9, 81	3
1, 9, 9, 9, 9, 25	1

$\mathcal{X}^{a_1, \dots, a_6}$ is modular

but after extracting elliptic surface contributions from H^3, still have dim 4 part left. modular forms correspond

to 2 dimensional subspaces of H^3. (12)

So we don't yet have a proof of modularity.

Alternative approach:

try to prove modularity of some variation $\mathcal{X}^{a_1, a_2, a_3, a_4, t}$ of family of K3 surfaces,

use this to prove modularity of double cover

$$(\lambda + 1)(a_5/\lambda + a_6) = t$$

& Study relationship between Coefficients of L-series for $\mathcal{X}^{a_1, \dots a_4, a_5, a_6}$ & expansion of sol^n of Picard-Fuchs eq^n.

— hope to use congruence results of Steinstra, etc.

Further conjecturally modular $\mathcal{X}^{a_1 \dots a_6}$ (11)

C. Meyer conjectures:

for $a_1, \dots a_6 =$

$a_1, \dots, a_6$	h^{12}
1, 1, 1, 1, 1, −7	5
1, 1, −1, −1, 4, −4	3
1, 1, 1, −1, −1, −1	5
1, 1, 9, 9, 9, 81	3
1, 9, 9, 9, 9, 25	1

$\mathcal{X}^{a_1, \dots, a_6}$ is modular

but after extracting elliptic surface contributions from H^3,

still have dim 4 part left. modular forms correspond

to 2 dimensional subspaces of H^3. (12)

So we don't yet have a proof of modularity.

Alternative approach:

try to prove modularity of some variation

$\mathcal{X}^{a_1, a_2, a_3, a_4, t}$ of family of K3 surfaces,

use this to prove modularity of double cover

$$(\lambda+1)(a_5/\lambda + a_6) = t$$

& Study relationship between Coefficients of

L-series for $\mathcal{X}^{a_1, \dots, a_4, a_5, a_6}$ & expansion of sol^n of Picard-Fuchs eq^n.

— hope to use congruence results of Stienstra, etc.

to 2 dimensional subspaces of H^2. (12)

So we don't yet have a proof of modularity.

Alternative approach:

try to prove modularity of some variation

$\mathcal{X}^{a_1,a_2,a_3,a_4,t}$ of family of K3 surfaces,

use this to prove modularity of double cover

$(\lambda+1)(a_5/\lambda + a_6) = t$

& Study relationship between coefficients of L-series for $\mathcal{X}^{a_1,\dots a_4,a_5,a_6}$ & expansion of sol^n of Picard-Fuchs eq^n.

— hope to use congruence results of Stienstra, etc.

(13)

example of this phenomena:

for A_3 family $(X_1+X_2+X_3+X_4)\left(\frac{1}{X_1}+\frac{1}{X_2}+\frac{1}{X_3}+\frac{1}{X_4}\right)=\lambda$

the PF eq^n has solution

$$f=\sum b_n \lambda^n \quad \text{with} \quad f=\frac{(\eta(2\tau)\eta(6\tau))^4}{(\eta(\tau)\eta(3\tau))^2}$$

$$\left(\begin{array}{l}\eta(\tau)=q^{1/24}\prod_{n\geq 1}(1-q^n)\\ q=e^{2\pi i\tau}\end{array}\right) \quad \& \quad t=\left(\frac{\eta(\tau)\eta(3\tau)}{\eta(2\tau)\eta(6\tau)}\right)^6$$

modular forms for $\Gamma_0(6)+3$

[this is a fibration of a toric variety – L-series can't be modular)

take a double cover.....

(even if the P-F solution is modular, need modularity of double cover)

to 2 dimensional subspaces of H^2. (12)
So we don't yet have a proof of modularity.

Alternative approach:

try to prove modularity of some variation $\mathcal{X}^{a_1,a_2,a_3,a_4,t}$ of family of K3 surfaces,

use this to prove modularity of double cover
$$(\lambda+1)(a_5/\lambda + a_6) = t$$

& Study relationship between Coefficients of L-series for $\mathcal{X}^{a_1,\dots,a_4,a_5,a_6}$ & expansion of sol^n of Picard-Fuchs eq^n.

— hope to use congruence results of Stienstra, etc.

(13)

example of this phenomena:

for A_3 family $(X_1+X_2+X_3+X_4)\left(\frac{1}{X_1}+\frac{1}{X_2}+\frac{1}{X_3}+\frac{1}{X_4}\right)=\lambda$

the PF eq^n has solution

$$f=\sum b_n \lambda^n \quad \text{with} \quad f=\frac{(\eta(2\tau)\eta(6\tau))^4}{(\eta(\tau)\eta(3\tau))^2}$$

$$\left(\begin{array}{l}\eta(\tau)=q^{1/24}\prod_{n\geq 1}(1-q^n)\\ q=e^{2\pi i\tau}\end{array}\right) \qquad \& \quad t=\left(\frac{\eta(\tau)\eta(3\tau)}{\eta(2\tau)\eta(6\tau)}\right)^6$$

modular forms for $\Gamma_0(6)+3$

[this is a fibration of a toric variety - L-series can't be modular)
take a double cover.....
(even if the P-F solution is modular, need modularity of double cover)

to 2 dimensional subspaces of H^2. (12)

So we don't yet have a proof of modularity.

Alternative approach:

try to prove modularity of some variation

$\mathcal{X}^{a_1,a_2,a_3,a_4,t}$ of family of K3 surfaces,

use this to prove modularity of double cover

$(\lambda+1)(a_5/\lambda + a_6) = t$

& Study relationship between Coefficients of L-series for $\mathcal{X}^{a_1,\ldots,a_4,a_5,a_6}$ & expansion of sol^n of Picard-Fuchs eq^n.

— hope to use congruence results of Stienstra, etc.

(13)

example of this phenomena:

for A_3 family $(X_1+X_2+X_3+X_4)\left(\frac{1}{X_1}+\frac{1}{X_2}+\frac{1}{X_3}+\frac{1}{X_4}\right)=\lambda$

the PF eq^n has solution

$$f = \sum b_n \lambda^n \quad \text{with} \quad f = \frac{(\eta(2\tau)\eta(6\tau))^4}{(\eta(\tau)\eta(3\tau))^2}$$

$$\& \quad \lambda = \left(\frac{\eta(\tau)\eta(3\tau)}{\eta(2\tau)\eta(6\tau)}\right)^6$$

modular forms for $\Gamma_0(6)+3$

$\left(\eta(\tau) = q^{1/24}\prod_{n\geq 1}(1-q^n), \quad q = e^{2\pi i\tau}\right)$

[this is a fibration of a toric variety - L-series can't be modular)

take a double cover.....

(even if the P-F solution is modular, need modularity of double cover)

to 2 dimensional subspaces of H^2.

So we don't yet have a proof of modularity. (12)

Alternative approach:

try to prove modularity of some variation $\mathcal{X}^{a_1,a_2,a_3,a_4,t}$ of family of K3 surfaces,

use this to prove modularity of double cover
$$(\lambda+1)(a_5/\lambda + a_6) = t$$

& Study relationship between Coefficients of L-series for $\mathcal{X}^{a_1,\dots,a_4,a_5,a_6}$ & expansion of sol^n of Picard-Fuchs eq^n.

— hope to use congruence results of Stienstra, etc.

(13)

example of this phenomena:

for A_3 family $(X_1+X_2+X_3+X_4)\left(\frac{1}{X_1}+\frac{1}{X_2}+\frac{1}{X_3}+\frac{1}{X_4}\right)=\lambda$

the PF eq^n has solution

$$f=\sum b_n\lambda^n \quad \text{with} \quad f=\frac{(\eta(2\tau)\eta(6\tau))^4}{(\eta(\tau)\eta(3\tau))^2}$$

$$\left(\begin{array}{l}\eta(\tau)=q^{1/24}\prod_{q\geq 1}(1-q^n)\\ q=e^{2\pi i\tau}\end{array}\right) \quad \& \quad \frac{t}{\lambda}=\left(\frac{\eta(\tau)\eta(3\tau)}{\eta(2\tau)\eta(6\tau)}\right)^6$$

modular forms for $\Gamma_0(6)+3$

[this is a fibration of a toric variety - L-series can't be modular)
take a double cover.....

(even if the P-F solution is modular, need modularity of double cover)

example of this phenomena:

(13)

for A_3 family $(X_1+X_2+X_3+X_4)\left(\frac{1}{X_1}+\frac{1}{X_2}+\frac{1}{X_3}+\frac{1}{X_4}\right)=\lambda$

the PF eq^n has solution

$$f=\sum b_n \lambda^n \quad \text{with} \quad f=\frac{(\eta(2\tau)\,\eta(6\tau))^4}{(\eta(\tau)\,\eta(3\tau))^2}$$

$$\left(\begin{array}{l}\eta(\tau)=q^{1/24}\prod_{n\geq 1}(1-q^n)\\ q=e^{2\pi i\tau}\end{array}\right) \qquad \& \quad \frac{t}{\lambda}=\left(\frac{\eta(\tau)\,\eta(3\tau)}{\eta(2\tau)\,\eta(6\tau)}\right)^6$$

modular forms for $\underline{\Gamma_0(6)+3}$

[this is a fibration of a toric variety - L-series can't be modular)

take a double cover.....

(even if the P-F solution is modular, need modularity of double cover)

example of this phenomena: (13)

for A_3 family $(X_1+X_2+X_3+X_4)\left(\frac{1}{X_1}+\frac{1}{X_2}+\frac{1}{X_3}+\frac{1}{X_4}\right)=\lambda$

the PF eqn has solution

$$f=\sum b_n\lambda^n \quad \text{with} \quad f=\frac{(\eta(2\tau)\eta(6\tau))^4}{(\eta(\tau)\eta(3\tau))^2}$$

$$\left(\eta(\tau)=q^{1/24}\prod_{n\geq 1}(1-q^n),\ q=e^{2\pi i\tau}\right) \quad \& \quad \frac{1}{\lambda}=\left(\frac{\eta(\tau)\eta(3\tau)}{\eta(2\tau)\eta(6\tau)}\right)^6$$

modular forms for $\underline{\Gamma_0(6)+3}$

[this is a fibration of a toric variety - L-series can't be modular)
take a double cover.....

(even if the P-F solution is modular, need modularity of double cover)

for the double cover, $\mathfrak{X}^{1,1,1,1,1,1}$, a rigid CY 3fold (13A)

$$(X_1+X_2+X_3+X_4)\left(\frac{1}{X_1}+\frac{1}{X_2}+\frac{1}{X_3}+\frac{1}{X_4}\right)=(t+1)\left(\frac{1}{t}+1\right)$$

the double cover corresponds to the subgroup

$$\Gamma_0(12)+12 \underset{\text{index } 2}{\subseteq} \Gamma_0(6)+3$$

$$t=\frac{\eta(3\tau)^4\eta(12\tau)^8\eta(2\tau)^{12}}{\eta(\tau)^4\eta(4\tau)^8\eta(6\tau)^{12}}$$

we have $f=\sum b_n t^n$
and $b_p\equiv a_p \bmod p$ (for $p\neq 2,3$) where $\sum a_n q^n=(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2$
is the Mellin transform of the L-series of middle cohomology of $\mathfrak{X}^{1,1,1,1,1,1}$

example of this phenomena: (13)

for A_3 family $(X_1+X_2+X_3+X_4)\left(\frac{1}{X_1}+\frac{1}{X_2}+\frac{1}{X_3}+\frac{1}{X_4}\right)=\lambda$

the PF eqn has solution

$$f=\sum b_n \lambda^n \quad \text{with} \quad f=\frac{(\eta(2\tau)\eta(6\tau))^4}{(\eta(\tau)\eta(3\tau))^2}$$

$$\left(\begin{array}{l}\eta(\tau)=q^{1/24}\prod_{n\geq 1}(1-q^n)\\ q=e^{2\pi i\tau}\end{array}\right) \quad \& \quad \frac{t}{\lambda}=\left(\frac{\eta(\tau)\eta(3\tau)}{\eta(2\tau)\eta(6\tau)}\right)^6$$

modular forms for $\underline{\Gamma_0(6)+3}$

[this is a fibration of a toric variety - L-series can't be modular)

take a double cover.....

(even if the P-F solution is modular, need modularity of double cover)

for the double cover, $\mathfrak{X}^{1,1,1,1,1,1}$, a rigid CY 3fold (13A)

$$(X_1+X_2+X_3+X_4)\left(\frac{1}{X_1}+\frac{1}{X_2}+\frac{1}{X_3}+\frac{1}{X_4}\right)=(t+1)\left(\frac{1}{t}+1\right)$$

the double cover corresponds to the subgroup

$$\Gamma_0(12)+12 \underset{\text{index } 2}{\subseteq} \Gamma_0(6)+3$$

$$t=\frac{\eta(3\tau)^4\eta(12\tau)^8\eta(2\tau)^{12}}{\eta(\tau)^4\eta(4\tau)^8\eta(6\tau)^{12}}$$

we have $f=\sum b_n t^n$

and $b_p\equiv a_p \mod p$ (for $p\neq 2,3$) where $\sum a_n q^n=(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2$

is the Mellin transform of the L-series of middle cohomology of $\mathfrak{X}^{1,1,1,1,1,1}$

example of this phenomena: (13)

for A_3 family $(X_1+X_2+X_3+X_4)\left(\frac{1}{X_1}+\frac{1}{X_2}+\frac{1}{X_3}+\frac{1}{X_4}\right)=\lambda$

the PF eqn has solution

$$f=\sum b_n\lambda^n \quad \text{with} \quad f=\frac{(\eta(2\tau)\eta(6\tau))^4}{(\eta(\tau)\eta(3\tau))^2}$$

$\left(\begin{array}{l}\eta(\tau)=q^{1/24}\prod_{n\geq 1}(1-q^n)\\ q=e^{2\pi i\tau}\end{array}\right)$ & $\frac{t}{\lambda}=\left(\frac{\eta(\tau)\eta(3\tau)}{\eta(2\tau)\eta(6\tau)}\right)^6$ modular forms for $\Gamma_0(6)+3$

[this is a fibration of a toric variety - L-series can't be modular)
take a double cover.....
(even if the P-F solution is modular, need modularity of double cover)

for the double cover, $\mathfrak{X}^{1,1,1,1,1,1}$, a rigid CY 3fold (13A)

$$(X_1+X_2+X_3+X_4)\left(\frac{1}{X_1}+\frac{1}{X_2}+\frac{1}{X_3}+\frac{1}{X_4}\right)=(t+1)\left(\frac{1}{t}+1\right)$$

the double cover corresponds to the subgroup

$$\Gamma_0(12)+12 \underset{\text{index } 2}{\subseteq} \Gamma_0(6)+3$$

$$t=\frac{\eta(3\tau)^4\eta(12\tau)^8\eta(2\tau)^{12}}{\eta(\tau)^4\eta(4\tau)^8\eta(6\tau)^{12}}$$

we have $f=\sum b_n t^n$

and $b_p\equiv a_p \mod p$ (for $p\neq 2,3$) where $\sum a_n q^n=(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2$ is the Mellin transform of the L-series of middle cohomology of $\mathfrak{X}^{1,1,1,1,1,1}$

for the double cover, $\mathfrak{X}^{1,1,1,1,1,1}$, a rigid CY 3fold (13A)

$$(X_1 + X_2 + X_3 + X_4)\left(\frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{X_3} + \frac{1}{X_4}\right) = (t+1)\left(\frac{1}{t}+1\right)$$

the double cover corresponds to the subgroup

$$\Gamma_0(12) + 12 \underset{\text{index } 2}{\subseteq} \Gamma_0(6) + 3$$

$$t = \frac{\eta(3\tau)^4 \eta(12\tau)^8 \eta(2\tau)^{12}}{\eta(\tau)^4 \eta(4\tau)^6 \eta(6\tau)^{12}}$$

we have $f = \sum b_n t^n$

and $b_p \equiv a_p \mod p$ (for $p \neq 2,3$) where $\sum a_n q^n = \left(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)\right)^2$

is the Mellin transform of the L-series of middle cohomology of $\mathfrak{X}^{1,1,1,1,1,1}$

for the double cover, $\mathfrak{X}^{1,1,1,1,1,1}$, a rigid CY 3fold (13A)

$$(X_1 + X_2 + X_3 + X_4)\left(\frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{X_3} + \frac{1}{X_4}\right) = (t+1)\left(\frac{1}{t}+1\right)$$

the double cover corresponds to the subgroup

$$\Gamma_0(12) + 12 \underset{\substack{\uparrow \\ \text{index } 2}}{\subseteq} \Gamma_0(6) + 3$$

$$t = \frac{\eta(3\tau)^4 \eta(12\tau)^8 \eta(2\tau)^{12}}{\eta(\tau)^4 \eta(4\tau)^6 \eta(6\tau)^{12}}$$

we have $f = \sum b_n t^n$

and $b_p \equiv a_p \mod p$ (for $p \neq 2,3$) where $\sum a_n q^n = \left(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)\right)^2$

is the Mellin transform of the L-series of middle cohomology of $\mathfrak{X}^{1,1,1,1,1,1}$

for the double cover, $\mathfrak{X}^{1,1,1,1,1,1}$, a rigid CY 3fold (13A)

$$(X_1+X_2+X_3+X_4)\left(\frac{1}{X_1}+\frac{1}{X_2}+\frac{1}{X_3}+\frac{1}{X_4}\right)=(t+1)\left(\frac{1}{t}+1\right)$$

the double cover corresponds to the subgroup

$$\Gamma_0(12)+12 \underset{\text{index }2}{\subseteq} \Gamma_0(6)+3$$

$$t=\frac{\eta(3\tau)^4\eta(12\tau)^8\eta(2\tau)^{12}}{\eta(\tau)^4\eta(4\tau)^8\eta(6\tau)^{12}}$$

we have $f=\sum b_n t^n$

and $b_p \equiv a_p \mod p$ (for $p\neq 2,3$) where $\sum a_n q^n = \{\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)\}^2$

is the Mellin transform of the L-series of middle cohomology of $\mathfrak{X}^{1,1,1,1,1,1}$

Other examples of relationship between (13B)

Coefficients of L-series & coefficients of expansion of solution of Picard - Fuchs equation

- for the self fibre products of Beauvilles families of families of elliptic curves, (semi stable, rational, 4 singular fibres)

(fibre products modular by Yui - Saito)

eg $\Gamma_1(6)$: $(x+y+z)(xy+yz+zx)=txyz$

fibre product has L-series $\mathcal{L}=\sum a_n q^n$,

P.F. eqn has soln $f=\sum b_n t^n$

f wt 2 form for $\Gamma_0(6)$

t wt 0 function for $\Gamma_0(6)$

$\mathcal{L}$ wt 4 form for $\Gamma_0(6)$

& $\boxed{\underbrace{\mathcal{L}(q)}_{\text{L-series}} = \underbrace{f(q)}_{\text{monodromy}}\,\frac{q\frac{dt(q)}{dq}}{t(q)}}$

Other examples of relationship between coefficients of L-series & coefficients of expansion of solution of Picard - Fuchs equation (138)

- for the self fibre products of Beauvilles families of families of elliptic curves, (semi stable, rational, 4 singular fibres)
(fibre products modular by Yui-Saito)

eg $\Gamma_1(6)$: $(x+y+z)(xy+yz+zx) = txyz$

fibre product has L-series $L = \sum a_n q^n$,

P.F. eqn has soln $f = \sum b_n t^n$

f wt 2 form for $\Gamma_0(6)$
t wt 0 function for $\Gamma_0(6)$
L wt 4 form for $\Gamma_0(6)$

& $$\boxed{\underbrace{L(q)}_{\text{L-series}} = \underbrace{f(q)}_{\text{monodromy}} \frac{q\frac{dt(q)}{dq}}{t(q)}}$$

Other examples of relationship between (13B)

Coefficients of L-series & coefficients of expansion of solution of Picard - Fuchs equation

- for the self fibre products of Beauvilles families of families of elliptic curves, (semi stable, rational, 4 singular fibres)

(fibre products modular by Yui - Saito)

eg $\Gamma_1(6)$: $(x+y+z)(xy+yz+zx) = t\,xyz$

fibre product has L-series $\mathcal{L} = \sum a_n q^n$,

P.F. eqn has soln $f = \sum b_n t^n$

f wt 2 form for $\Gamma_0(6)$

t wt 0 functn for $\Gamma_0(6)$

$\mathcal{L}$ wt 4 form for $\Gamma_0(6)$

& $$\boxed{\underbrace{\mathcal{L}(q)}_{\text{L-series}} = \underbrace{f(q)}_{\text{monodromy}} \frac{q\frac{dt(q)}{dq}}{t(q)}}$$

Generally expect some relation like (13C)

$$b_{mp^r} - a_p b_{mp^{r-1}} + p^3 b_{mp^{r-2}} \equiv 0 \mod p^r$$

$(\Rightarrow b_p \equiv a_p \mod p)$

- use results of Stienstra ...

may lead to Atkin - Swinnerton - Dyer congruences between congruent and non congruent modular forms.

for the double cover, $\mathfrak{X}^{1,1,1,1,1,1}$, a rigid CY 3fold (13A)

$$(X_1 + X_2 + X_3 + X_4)\left(\frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{X_3} + \frac{1}{X_4}\right) = (t+1)\left(\frac{1}{t}+1\right)$$

the double cover corresponds to the subgroup

$$\Gamma_0(12) + 12 \underset{\text{index } 2}{\subseteq} \Gamma_0(6) + 3$$

$$t = \frac{\eta(3\tau)^4 \eta(12\tau)^8 \eta(2\tau)^{12}}{\eta(\tau)^4 \eta(4\tau)^6 \eta(6\tau)^{12}}$$

we have $f = \sum b_n t^n$

and $b_p \equiv a_p \mod p$ (for $p \neq 2,3$) where $\sum a_n q^n = \{\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)\}^2$ is the Mellin transform of the L-series of middle cohomology of $\mathfrak{X}^{1,1,1,1,1,1}$

Generally expect some relation like (13C)

$$b_{mp^r} - a_p b_{mp^{r-1}} + p^3 b_{mp^{r-2}} \equiv 0 \mod p^r$$

$(\Rightarrow b_p \equiv a_p \mod p)$

– use results of Stienstra . . .

may lead to Atkin-Swinnerton-Dyer congruences between congruent and non congruent modular forms.

for the double cover, $\mathfrak{X}^{1,1,1,1,1,1}$, a rigid CY 3fold (13A)

$$(X_1+X_2+X_3+X_4)\left(\frac{1}{X_1}+\frac{1}{X_2}+\frac{1}{X_3}+\frac{1}{X_4}\right)=(t+1)\left(\frac{1}{t}+1\right)$$

the double cover corresponds to the subgroup

$$\Gamma_0(12)+12 \underset{\text{index } 2}{\subseteq} \Gamma_0(6)+3$$

$$t=\frac{\eta(3\tau)^4\eta(12\tau)^8\eta(2\tau)^{12}}{\eta(\tau)^4\eta(4\tau)^8\eta(6\tau)^{12}}$$

we have $f=\sum b_n t^n$

and $b_p \equiv a_p \mod p$ (for $p\neq 2,3$) where $\sum a_n q^n = \{\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)\}^2$

is the Mellin transform of the L-series of middle cohomology of $\mathfrak{X}^{1,1,1,1,1,1}$

Further conjecturally modular $\mathfrak{X}^{a_1 \dots a_6}$ (11)

C. Meyer conjectures:

for $a_1, \dots a_6 =$

$a_1,\dots,a_6$	h^{12}
1, 1, 1, 1, 1, −7	5
1, 1, −1, −1, 4, −4	3
1, 1, 1, −1, −1, −1	5
1, 1, 9, 9, 9, 81	3
1, 9, 9, 9, 9, 25	1

$\mathfrak{X}^{a_1,\dots,a_6}$ is modular

but after extracting elliptic surface contributions from H^3, still have dim 4 part left. modular forms correspond

13c

Generally expect some relation like

$$b_{mp^r} - a_p b_{mp^{r-1}} + p^3 b_{mp^{r-2}} \equiv 0 \mod p^r$$

$(\Rightarrow b_p \equiv a_p \mod p)$

– use results of Stienstra . . .

may lead to Atkin-Swinnerton-Dyer congruences between congruent and non congruent modular forms.

(13c)

Generally expect some relation like

$$b_{mp^r} - a_p b_{mp^{r-1}} + p^2 b_{mp^{r-2}} \equiv 0 \mod p^r$$

($\Rightarrow b_p \equiv a_p \mod p$)

– use results of Stienstra . . .

may lead to Atkin–Swinnerton–Dyer congruences between congruent and non congruent modular forms.

How to find the Picard – Fuchs equation?

(14)

this is a differential equation for periods of $x^{a_1 \cdots a_n \frac{1}{t}}$

One is given by

$$\int_\gamma \mathrm{Res}\left(\frac{dX_1 \wedge dX_2 \cdots \wedge dX_n}{\left((X_1 + \cdots + X_n)\left(\frac{a_1}{X_1} + \cdots + \frac{a_n}{X_n}\right) t - 1\right) X_1 \cdots X_n}\right)$$

$$= \text{const.} \sum \alpha_m t^m$$

$$\alpha_m = \text{constant term in} \left(\left((X_1 + \cdots + X_n)\left(\frac{a_1}{X_1} + \cdots + \frac{a_n}{X_n}\right)\right)^m\right) = \sum_{p_1 + \cdots + p_n = m} a_1^{p_1} \cdots a_n^{p_n} \binom{m}{p_1, \ldots, p_n}^2$$

multinomial coefficient

How to find the Picard–Fuchs equation? (14)

this is a differential equation for periods of $\mathcal{X}^{a_1 \cdots a_n, t}$

One is given by

$$\int_\gamma \mathrm{Res}\left(\frac{dX_1\wedge dX_2 \cdots \wedge dX_n}{\left((X_1+\cdots+X_n)\left(\frac{a_1}{X_1}+\cdots+\frac{a_n}{X_n}\right)t-1\right)X_1\cdots X_n}\right)$$

$$= \text{const.} \sum \alpha_m t^m$$

$$\alpha_m = \text{constant term in}\left(\left((X_1+\cdots+X_n)\left(\frac{a_1}{X_1}+\cdots+\frac{a_n}{X_n}\right)\right)^m\right) = \sum_{p_1+\cdots+p_n=m} a_1^{p_1}\cdots a_n^{p_n}\binom{m}{p_1,\dots,p_n}^2$$

multinomial coefficient

(15)

To find the P-F equation,

need to find recurrence relation for

$$\alpha_m^{a_1\cdots a_n} = \sum_{\Sigma p_i = m} a_1^{p_1}\cdots a_n^{p_n}\binom{m}{p_1\cdots p_n}^2 .$$

For any given $a_1 \dots a_n$, can use methods of finite D-series to find a recurrence relation.

In practice, if $a_1 \dots a_n$ are not fixed, this is very slow.

How to find the Picard–Fuchs equation? (14)

this is a differential equation for periods of $X^{a_1 \cdots a_n, \frac{1}{t}}$

One is given by

$$\int_\gamma \operatorname{Res}\left(\frac{dX_1 \wedge dX_2 \cdots \wedge dX_n}{\left((X_1 + \cdots + X_n)\left(\frac{a_1}{X_1} + \cdots + \frac{a_n}{X_n}\right)t - 1\right) X_1 \cdots X_n}\right)$$

$$= \text{const} \times \sum \alpha_m t^m$$

$$\alpha_m = \text{constant term in} \left(\left((X_1 + \cdots + X_n)\left(\frac{a_1}{X_1} + \cdots + \frac{a_n}{X_n}\right)\right)^m\right) = \sum_{p_1 + \cdots + p_n = m} a_1^{p_1} \cdots a_n^{p_n} \binom{m}{p_1, \cdots, p_n}^2$$

multinomial coefficient

To find the P-F equation, (15)

need to find recurrence relation for

$$\alpha_m^{a_1, \cdots a_n} = \sum_{\Sigma p_i = m} a_1^{p_1} \cdots a_n^{p_n} \binom{m}{p_1 \cdots p_n}^2 .$$

For any given $a_1 \ldots a_n$, can use methods of finite D-series to find a recurrence relation.

In practice, if $a_1 \cdots a_n$ are not fixed, this is <u>very</u> slow.

(15)

To find the P-F equation,

need to find recurrence relation for

$$\alpha_m^{a_1 \dots a_n} = \sum_{\sum p_i = m} a_1^{p_1} \cdots a_n^{p_n} \binom{m}{p_1 \cdots p_n}^2 .$$

For any given $a_1 \dots a_n$, can use methods of finite D-series to find a recurrence relation.

In practice, if $a_1 \dots a_n$ are not fixed, this is very slow.

(16)

example (using method related to D-series method, but devised specially for this situation for speed)
(n=2)
(elliptic curves)

To find recurrence relation for $\alpha_m^{a,b,c}$

is equivalent to finding the kernel of the matrix

$$\begin{pmatrix} \dfrac{\left[\begin{array}{l}\text{polynomial,}\\ \text{degree 4 in } a,b,c\\ \text{degree 5 in } m\end{array}\right]}{n^2(n-1)^2(n-2)} & * & (a+b+c)^2 + \frac{4(n-3)}{(n-2)}(ab+bc+ca) & (a+b+c) & 1 \\ \left[\begin{array}{l}\text{Similar kind of}\\ \text{rational function}\end{array}\right] & * & * & \frac{2}{(n-3)^2} & 0 \\ * & * & * & \frac{2}{(n-3)^2} & 0 \\ * & * & * & \frac{2}{(n-3)^2} & 0 \end{pmatrix}$$

To find the P-Fequation, (15)

need to find recurrence relation for

$$\alpha_m^{a_1 \dots a_n} = \sum_{\sum p_i = m} a_1^{p_1} \cdots a_n^{p_n} \binom{m}{p_1 \cdots p_n}^2 .$$

For any given $a_1 \dots a_n$, can use methods of finite D-series to find a recurrence relation.

In practice, if $a_1 \dots a_n$ are not fixed, this is <u>very</u> slow.

example (using method related to D-series method, (16).
(n=2) but devised specially for this situation for speed)
(elliptic curves)

To find recurrence relation for $\alpha_n^{a,b,c}$

is equivalent to <u>finding the kernel of the matrix</u>

$$\begin{pmatrix} \dfrac{\left[\begin{array}{l}\text{polynomial,} \\ \text{degree 4 in } a,b,c \\ \text{degree 5 in } n\end{array}\right]}{n^2(n-1)^2(n-2)} & * & (a+b+c)^2 + \frac{4(n-3)}{(n-2)}(ab+bc+ca) & (a+b+c) & 1 \\ \left[\begin{array}{l}\text{Similar kind of} \\ \text{rational functn}\end{array}\right] & * & * & \frac{2}{(n-3)^2} & 0 \\ * & * & * & \frac{2}{(n-3)^2} & 0 \\ * & * & * & \frac{2}{(n-3)^2} & 0 \end{pmatrix}$$

example (using method related to D-series method, (16)
(n=2) (elliptic curves) but devised specially for this situation for speed)

To find recurrence relation for $\alpha^{a,b,c}_{n}$

is equivalent to finding the kernel of the matrix

$$\begin{pmatrix} \dfrac{\left[\begin{matrix}\text{polynomial,}\\ \text{degree 4 in } a,b,c\\ \text{degree 5 in } n\end{matrix}\right]}{n^2(n-1)^2(n-2)} & * & (a+b+c)^2+\frac{4(n-3)}{(n-2)}(ab+bc+ca) & (a+b+c) & 1 \\ \left[\begin{matrix}\text{Similar kind of}\\ \text{rational function}\end{matrix}\right] & * & * & \frac{2}{(n-3)^2} & 0 \\ * & * & * & \frac{2}{(n-3)^2} & 0 \\ * & * & * & \frac{2}{(n-3)^2} & 0 \end{pmatrix}$$

eg Picard Fuchs equation for n=2 case (17)

$$\mathcal{E}_{a,b,c,t}: \quad (X_1+X_2+X_3)\left(\frac{a}{X_1}+\frac{b}{X_2}+\frac{c}{X_3}\right)=t$$

$$j(\mathcal{E}_{a,b,c,t}) = \frac{(P+16abct)^3}{(abct)^2 P}$$

where $P = \prod_{\substack{\text{over}\\ \text{4 choices of}\\ \text{signs}}} \left((\sqrt{a} \pm \sqrt{b} \pm \sqrt{c})^2 - t \right)$

example (using method related to D-series method, but devised specially for this situation for speed) (16)

($n=2$)
(elliptic curves)

To find recurrence relation for $\alpha_{n}^{a,b,c}$

is equivalent to finding the kernel of the matrix

$$\begin{pmatrix} \dfrac{\left[\begin{matrix}\text{polynomial,} \\ \text{degree 4 in } a,b,c \\ \text{degree 5 in } n\end{matrix}\right]}{n^2(n-1)^2(n-2)} & * & (a+b+c)^2 + \frac{4(n-3)}{(n-2)}(ab+bc+ca) & (a+b+c) & 1 \\ \left[\begin{matrix}\text{Similar kind of} \\ \text{rational function}\end{matrix}\right] & * & * & \frac{2}{(n-3)^2} & 0 \\ * & * & * & \frac{2}{(n-3)^2} & 0 \\ * & * & * & \frac{2}{(n-3)^2} & 0 \end{pmatrix}$$

eg Picard Fuchs equation for $n=2$ case (17)

$$\mathcal{E}_{a,b,c,t} : \quad (X_1 + X_2 + X_3)\left(\frac{a}{X_1} + \frac{b}{X_2} + \frac{c}{X_3}\right) = t$$

$$j(\mathcal{E}_{a,b,c,t}) = \frac{(P + 16abct)^3}{(abct)^2 P}$$

where $P = \prod_{\substack{\text{over} \\ \text{4 choices of} \\ \text{signs}}} \left((\sqrt{a} \pm \sqrt{b} \pm \sqrt{c})^2 - t \right)$

eg Picard Fuchs equation for $\underline{n=2 \text{ case}}$. (17)

$$\mathcal{E}_{a,b,c,t} : \quad (X_1 + X_2 + X_3)\left(\frac{a}{X_1} + \frac{b}{X_2} + \frac{c}{X_3}\right) = t$$

$$j(\mathcal{E}_{a,b,c,t}) = \frac{(P + 16abct)^3}{(abct)^2 P}$$

where $P = \prod\limits_{\substack{\text{over} \\ \text{4 choices of} \\ \text{signs}}} \left((\sqrt{a} \pm \sqrt{b} \pm \sqrt{c})^2 - t \right)$

eg Picard Fuchs equation for n=2 case. (17)

$$\mathcal{E}_{a,b,c,t} : \quad (X_1 + X_2 + X_3)\left(\frac{a}{X_1} + \frac{b}{X_2} + \frac{c}{X_3}\right) = t$$

$$j(\mathcal{E}_{a,b,c,t}) = \frac{(P + 16abct)^3}{(abct)^2 P}$$

where $P = \prod_{\substack{\text{over} \\ \text{4 choices of} \\ \text{signs}}} \left((\sqrt{a} \pm \sqrt{b} \pm \sqrt{c})^2 - t \right)$

The P-F equation is: (18)

$$\boxed{\Theta^2 + \tau \sum_{i=1}^{6} \frac{\varepsilon_i}{(\tau - u_i)} \Theta + \tau^2 \sum_{i=0}^{6} \frac{\beta_i}{\tau - u_i}}$$

$(\tau = 1/t \ , \Theta = \tau d/d\tau)$

where $u_0 = 0$

u_1, u_2, u_3, u_4 = values of $1/(\sqrt{a} \pm \sqrt{b} \pm \sqrt{c})^2$ (real singularities)

u_5, u_6 = roots of

$$-3 + 2(a+b+c)\tau + (a^2 + b^2 + c^2 - 2ab - 2bc - 2ac)\tau^2 = 0$$

$\varepsilon_1 = \cdots = \varepsilon_4 = 1$

$\varepsilon_5 = \varepsilon_6 = -1$

eg Picard Fuchs equation for $\underline{n=2 \text{ case}}$. (17)

$$\mathcal{E}_{a,b,c,t} : \quad (X_1 + X_2 + X_3)\left(\frac{a}{X_1} + \frac{b}{X_2} + \frac{c}{X_3}\right) = t$$

$$j(\mathcal{E}_{a,b,c,t}) = \frac{(P + 16abct)^3}{(abct)^2 P}$$

where $P = \prod\limits_{\substack{\text{over} \\ \text{4 choices of} \\ \text{signs}}} \left((\sqrt{a} \pm \sqrt{b} \pm \sqrt{c})^2 - t\right)$

The P-F equation is: (18)

$$\boxed{\Theta^2 + \tau \sum_{i=1}^{6} \frac{\varepsilon_i}{(\tau - u_i)} \Theta + \tau^2 \sum_{i=0}^{6} \frac{\beta_i}{\tau - u_i}}$$

$(\tau = 1/t \ , \Theta = \tau\, d/d\tau)$

where $u_0 = 0$

u_1, u_2, u_3, u_4 = values of $1/(\sqrt{a} \pm \sqrt{b} \pm \sqrt{c})^2$ (real singularities)

u_5, u_6 = roots of

$$-3 + 2(a+b+c)\tau + (a^2 + b^2 + c^2 - 2ab - 2bc - 2ac)\tau^2 = 0$$

$\varepsilon_1 = \cdots = \varepsilon_4 = 1$

$\varepsilon_5 = \varepsilon_6 = -1$

The P-F equation is: (18)

$$\boxed{\Theta^2 + r\sum_{i=1}^{6}\frac{\varepsilon_i}{(r-u_i)}\Theta + r^2\sum_{i=0}^{6}\frac{\beta_i}{r-u_i}}$$

$\left(r = 1/t \; , \Theta = r\, d/dr\right)$

where $u_0 = 0$

u_1, u_2, u_3, u_4 = values of $1/(\sqrt{a} \pm \sqrt{b} \pm \sqrt{c})^2$ (real singularities)

u_5, u_6 = roots of

$$-3 + 2(a+b+c)r + (a^2+b^2+c^2-2ab-2bc-2ac)r^2 = 0$$

$\varepsilon_1 = \cdots = \varepsilon_4 = 1$

$\varepsilon_5 = \varepsilon_6 = -1$

(19)

$\beta_0 = -a-b-c$

$\beta_5 = -3/4u_5 \qquad \beta_6 = -3/4u_6$

$$\beta_i = \frac{1}{32u_i}\left(\varepsilon_i^1\varepsilon_i^2\sqrt{\frac{1}{abcu_i}}\left(a+b+c-\frac{1}{u_i}\right)+26\right)$$

for $i = 1,2,3,4$

where $u_i = 1/(\sqrt{a} + \varepsilon_i^1 + \varepsilon_i^2\sqrt{c})^2$ (ie $\varepsilon_i^1, \varepsilon_i^2$ are sign choice for u_i)

The P-F equation is:

$$\boxed{\Theta^2 + \tau \sum_{i=1}^{6} \frac{\varepsilon_i}{(\tau - u_i)} \Theta + \tau^2 \sum_{i=0}^{6} \frac{\beta_i}{\tau - u_i}}$$

$\left(\tau = 1/t \;,\; \Theta = \tau \, d/d\tau\right)$

where $u_0 = 0$

u_1, u_2, u_3, u_4 = values of $1/(\sqrt{a} \pm \sqrt{b} \pm \sqrt{c})^2$ (real singularities)

u_5, u_6 = roots of

$$-3 + 2(a+b+c)\tau + (a^2+b^2+c^2 - 2ab - 2bc - 2ac)\tau^2 = 0$$

$\varepsilon_1 = \cdots = \varepsilon_4 = 1$

$\varepsilon_5 = \varepsilon_6 = -1$

$\beta_0 = -a-b-c$

$\beta_5 = -3/4u_5 \qquad \beta_6 = -3/4u_6$

$$\beta_i = \frac{1}{32 u_i}\left(\varepsilon_i^1 \varepsilon_i^2 \sqrt{\frac{1}{abc\, u_i}} \left(a+b+c - \frac{1}{u_i}\right) + 26\right)$$

for $i = 1,2,3,4$

where $u_i = 1/(\sqrt{a} + \varepsilon_i^1 + \varepsilon_i^2 \sqrt{c})^2$ (ie $\varepsilon_i^1, \varepsilon_i^2$ are sign choice for u_i)

The P-F equation is: (18)

$$\boxed{\Theta^2 + \tau \sum_{i=1}^{6} \frac{\varepsilon_i}{(\tau - u_i)} \Theta + \tau^2 \sum_{i=0}^{6} \frac{\beta_i}{\tau - u_i}}$$

$\left(\tau = 1/t \;, \Theta = \tau d/d\tau\right)$

where $u_0 = 0$

u_1, u_2, u_3, u_4 = values of $1/(\sqrt{a} \pm \sqrt{b} \pm \sqrt{c})^2$ (real singularities)

u_5, u_6 = roots of

$$-3 + 2(a+b+c)\tau + (a^2+b^2+c^2 - 2ab - 2bc - 2ac)\tau^2 = 0$$

$\varepsilon_1 = \cdots = \varepsilon_4 = 1$

$\varepsilon_5 = \varepsilon_6 = -1$

to get a general formula for P-F eq^n for $\mathcal{X}^{a,b,c,d,t}$ (20)
takes too much computing time
(need to find the Kernel of 8×9 matrix, coeffs rational functions up to degree 8 in a, b, c, d, & 9 in t)
But, there is a formula for all cases of the form
$\mathcal{X}^{1,1,1,1,1,1,\ldots 1,t}$
Since there is a recurrence relation for

$$a_n^N = \sum_{p_1+\cdots+p_N = n} \binom{n}{p_1, \ldots, p_N}^2$$

(N fixed)

20

to get a general formula for P-F eq^n for $\mathcal{X}^{a,b,c,d,t}$
takes too much computing time
(need to find the Kernel of 8×9 matrix, coeffs
rational functions up to degree 8 in a, b, c, d, & 9 in t)

But, there is a formula for all cases of the form
$\mathcal{X}^{1,1,1,1,1,1,\ldots 1,t}$

Since there is a recurrence relation for

$$a_n^N = \sum_{p_1+\cdots+p_N=n} \binom{n}{p_1, \ldots, p_N}^2$$

(N fixed)

(20)

to get a general formula for P-F eq^n for $\mathcal{X}^{a,b,c,d,t}$

takes too much computing time

(need to find the kernel of 8×9 matrix, coeffs rational functions up to degree 8 in a,b,c,d, & 9 in t)

But, there is a formula for all cases of the form

$\mathcal{X}^{1,1,1,1,1,1,\ldots 1,t}$

Since there is a recurrence relation for

$$a_n^N = \sum_{p_1+\cdots+p_N=n} \binom{n}{p_1, \ldots, p_N}^2$$

(N fixed)

(21)

namely:

$$\boxed{\sum_{K\geq 0} \left(n^{N+1} \sum_{\substack{1\leq i\leq K \\ \alpha_i+\beta_i=N+1, \\ \alpha_i\in\mathbb{N},\ 1<\alpha_{i+1}+1<\alpha_i\in\mathbb{N}}} \prod_{i=1}^{K} -\alpha_i\beta_i \left(\frac{n-i}{n-i-1}\right)^{\alpha_i-1} \right) a_{n-K}^N = 0}$$

(finite sum, polynomial coefficients)

eg:

$$0 = n a_n^2 - 2(2n-1) a_{n-1}^2$$

$$0 = n^2 a_n^3 - (10n^2-10n+3) a_{n-1}^3 + 9(n-1)^2 a_{n-2}^3$$

$$0 = n^3 a_n^4 - 2(2n-1)(5n^2-5n+2) a_{n-1}^4 + 64(n-1)^3 a_{n-2}^4$$

$\vdots$

etc

namely:

$$\boxed{\sum_{k\geq 0}\left(n^{N+1} \sum_{\substack{1\leq i\leq k \\ \alpha_i+\beta_i=N+1, \\ \alpha_i\in\mathbb{N},\ 1<\alpha_{i+1}+1<\alpha_i\leq N}} \prod_{i=1}^{k} -\alpha_i\beta_i \left(\frac{n-i}{n-i-1}\right)^{\alpha_i-1} \right) a_{n-k}^{N} = 0}$$

(21)

(finite sum, polynomial coefficients)

eg:

$$0 = na_n^2 - 2(2n-1)a_{n-1}^2$$

$$0 = n^2a_n^3 - (10n^2-10n+3)a_{n-1}^3 + 9(n-1)^2 a_{n-2}^3$$

$$0 = n^3a_n^4 - 2(2n-1)(5n^2-5n+2)a_{n-1}^4 + 64(n-1)^3 a_{n-2}^4$$

$\vdots$

etc

this gives a differential equation (22)

$$\mathcal{F}_N : \sum_{k\geq 0} t^k \sum_{\substack{1\leq i\leq k \\ \alpha_i+\beta_i=N+1 \\ \alpha_i\in\mathbb{N},\ \alpha_{k+1}=0 \\ 1<\alpha_{i+1}+1<\alpha_i\leq N}} (\Theta+k)^{N+1-\alpha_1} \prod_{i=1}^{k} -\alpha_i\beta_i (\Theta+k-i)^{\alpha_i-\alpha_{i+1}}$$

How To find recurrence relation:

have $\quad a_n^N = \sum \binom{n}{p_1 \cdots p_N}^2$

Set $\quad a_n^{N,j} = \sum_{p_1+\cdots+p_N=n} p_1\cdots p_j \binom{n}{p_1, p_2 \cdots p_N}^2 \quad$ (for $j\geq 0$)

$(=0$ for $j<0)$ $\qquad a_n^N = a_n^{N,0}$

22

this gives a differential equation

$$\mathcal{F}_N : \sum_{k\geq 0} t^k \sum_{\substack{1\leq i\leq k \\ \alpha_i+\beta_i = N+1 \\ \alpha_i \in \mathbb{N},\ \alpha_{k+1}=0 \\ 1< \alpha_{i+1}+1 < \alpha_i \leq N}} (\Theta+k)^{N+1-\alpha_1} \prod_{i=1}^{k} -\alpha_i\beta_i\,(\Theta+k-i)^{\alpha_i-\alpha_{i+1}}$$

How To find recurrence relation:

have $\quad a_n^N = \sum \binom{n}{p_1 \cdots p_N}^2$

Set $\quad a_n^{N,j} = \sum_{p_1+\cdots+p_N=n} p_1\cdots p_j \binom{n}{p_1, p_2 \cdots p_N}^2 \quad$ (for $j\geq 0$)

$(=0$ for $j<0)$ $\qquad a_n^N = a_n^{N,0}$

(22)

this gives a differential equation

$$\mathcal{F}_N : \sum_{k\geq 0} t^k \sum_{\substack{1\leq i\leq k \\ \alpha_i+\beta_i = N+1 \\ \alpha_i \in \mathbb{N},\ \alpha_{k+1}=0 \\ 1<\alpha_{i+1}+1<\alpha_i\leq N}} (\Theta+k)^{N+1-\alpha_1} \prod_{i=1}^{k} -\alpha_i\beta_i (\Theta+k-i)^{\alpha_i-\alpha_{i+1}}$$

<u>How To find recurrence relation:</u>

have $\quad a_n^N = \sum \binom{n}{p_1 \cdots p_N}^2$

Set $\quad a_n^{N,j} = \sum_{p_1+\cdots+p_N=n} p_1\cdots p_j \binom{n}{p_1, p_2 \cdots p_N}^2 \quad$ (for $j\geq 0$)

$(=0$ for $j<0) \qquad a_n^N = a_n^{N,0}$

(23)

<u>So</u>

$$n a_n^{N,j} = \sum (p_1+\cdots+p_N)p_1\cdots p_j \binom{n}{p_1\cdots p_N}^2$$

$$- jn^2 a_{n-1}^{N,j-1} + (N-j)a_n^{j+1}$$

<u>So</u> $\quad (N-j)a_n^{N,j+1} = n a_n^{N,j} - jn^2 a_{n-1}^{N,j-1}$

index decrease $\quad$ index decreased

picture:

$$\boxed{\begin{tikzcd} (N-j)a_n^{N,j+1} \arrow[r, "n"] \arrow[dr, "-jn^2"'] & a_n^{N,j} \\ & a_{n-1}^{N,j-1} \end{tikzcd}}$$

let $g(N,j) = (N-1)\cdots(N-j)$
$= \prod_{i=1}^{j}(N-i)$

$$\begin{tikzcd} g(N,j)a_n^{N,j+1} \arrow[r, "n"] \arrow[dr, "-jn^2(N-j+1)"'] & g(N,j-1)a_n^{N,j} \\ & g(N,j-2)a_{n-1}^{N,j-1} \end{tikzcd}$$

(22)

this gives a differential equation

$$\mathcal{F}_N : \sum_{k\geq 0} t^k \sum_{\substack{1\leq i\leq k \\ \alpha_i+\beta_i = N+1 \\ \alpha_i \in \mathbb{N},\ \alpha_{k+1}=0 \\ 1<\alpha_{i+1}+1<\alpha_i\leq N}} (\Theta+k)^{N+1-\alpha_1} \prod_{i=1}^{k} -\alpha_i\beta_i\,(\Theta+k-i)^{\alpha_i-\alpha_{i+1}}$$

How To find recurrence relation:

have $\quad a_n^N = \sum \binom{n}{p_1 \cdots p_N}^2$

Set $\quad a_n^{N,j} = \sum_{p_1+\cdots+p_N=n} p_1\cdots p_j \binom{n}{p_1, p_2 \cdots p_N}^2 \qquad$ (for $j\geq 0$)

$(=0$ for $j<0)$ $\qquad a_n^N = a_n^{N,0}$

(23)

So

$$n\,a_n^{N,j} = \sum (p_1+\cdots+p_N)\,p_1\cdots p_j \binom{n}{p_1 \cdots p_N}^2$$

$$= jn^2 a_{n-1}^{N,j-1} + (N-j)\,a_n^{j+1}$$

So $\quad (N-j)\,a_n^{N,j+1} = n a_n^{N,j} - jn^2 a_{n-1}^{N,j-1}$

index decrease $\quad$ index decreased

picture:

$$\boxed{\begin{tikzcd} (N-j)\,a_n^{N,j+1} \arrow[r, "n"] \arrow[dr, "-jn^2"'] & a_n^{N,j} \\ & a_{n-1}^{N,j-1} \end{tikzcd}}$$

let $g(N,j) = (N-1)\cdot\cdots\cdot(N-j)$
$= \prod_{i=1}^{j}(N-i)$

$$\begin{tikzcd} g(N,j)\,a_n^{N,j+1} \arrow[r, "n"] \arrow[dr, "-jn^2(N-j+1)"'] & g(N,j-1)\,a_n^{N,j} \\ & g(N,j-2)\,a_{n-1}^{N,j-1} \end{tikzcd}$$

How to find …

have $a_n^N = \sum \binom{n}{p_1 \cdots p_N}^2$

Set $a_n^{N,j} = \sum_{p_1+\cdots+p_N=n} p_1 \cdots p_j \binom{n}{p_1, p_2 \cdots p_N}^2$ (for $j \geq 0$)

$(=0$ for $j<0)$ $\qquad a_n^N = a_n^{N,0}$

(23)

So

$$n a_n^{N,j} = \sum (p_1+\cdots+p_N) p_1 \cdots p_j \binom{n}{p_1 \cdots p_N}^2$$

$$= jn^2 a_{n-1}^{N,j-1} + (N-j) a_n^{j+1}$$

So $\quad (N-j) a_n^{N,j+1} = n a_n^{N,j} - jn^2 a_{n-1}^{N,j-1}$

index decrease $\qquad$ index decreased

picture:

$$\begin{tikzcd} (N-j) a_n^{N,j+1} \arrow[r, "n"] \arrow[dr, "-jn^2"'] & a_n^{N,j} \\ & a_{n-1}^{N,j-1} \end{tikzcd}$$

let $g(N,j) = (N-1) \cdots (N-j)$
$= \prod_{i=1}^{j} (N-i)$

$$\begin{tikzcd} g(N,j) a_n^{N,j+1} \arrow[r, "n"] \arrow[dr, "-jn^2(N-j+1)"'] & g(N,j-1) a_n^{N,j} \\ & g(N,j-2) a_{n-1}^{N,j-1} \end{tikzcd}$$

23

So

$$n a_n^{N,j} = \sum (p_1 + \cdots + p_N) p_1 \cdots p_j \binom{n}{p_1 \cdots p_N}^2$$

$$- j n^2 a_{n-1}^{N,j-1} + (N-j) a_n^{j+1}$$

So $\quad (N-j) a_n^{N,j+1} = n a_n^{N,j} - j n^2 a_{n-1}^{N,j-1}$.

index decrease $\quad$ index decreased

picture:

$$\boxed{\begin{tikzcd}(N-j)a_n^{N,j+1} \arrow[r, "n"] \arrow[dr, "-jn^2"'] & a_n^{N,j} \\ & a_{n-1}^{N,j-1}\end{tikzcd}}$$

let $g(N,j) = (N-1) \cdot \cdots \cdot (N-j)$
$= \prod_{i=1}^{j} (N-i)$

$$\begin{tikzcd}g(N,j)a_n^{N,j+1} \arrow[r, "n"] \arrow[dr, "-jn^2(N-j+1)"'] & g(N,j-1)a_n^{N,j} \\ & g(N,j-2)a_{n-1}^{N,j-1}\end{tikzcd}$$

24

putting pictures together – example

$$\begin{tikzcd}[column sep=small]
g(N,4)a_n^{N,5} \arrow[r, "n"] \arrow[dr, "-4(N-3)n^2"'] & g(N,3)a_n^{N,4} \arrow[r, "n"] \arrow[dr, "-3(N-2)n^2"'] & g(N,2)a_n^{N,3} \arrow[r, "n"] \arrow[dr, "-2(N-1)n^2"'] & g(N,1)a_n^{N,2} \arrow[r, "n"] \arrow[dr, "-Nn^2"'] & g(N,0)a_n^{N,1} \arrow[r, "n"] & a_n^N \\
& g(N,2)a_{n-1}^{N,3} \arrow[r, "n-1"] \arrow[dr, "-2(N-1)(n-1)^2"'] & g(N,1)a_{n-1}^{N,2} \arrow[r, "n-1"] \arrow[dr, "-N(n-1)^2"'] & g(N,0)a_n^{N,1} \arrow[r, "n-1"] & a_{n-1}^N & \\
& & N a_{n-2}^{N,1} \arrow[r, "n-2"] & a_{n-2}^N & &
\end{tikzcd}$$

So, by following all paths from $g(N,4)a_n^{N,5}$ to $a_n^N, a_{n-1}^N, a_{n-2}^N$

get $\quad g(N,4)a_n^N = n^5 a_n^N + (-n^3 N n^2 + \cdots) a_{n-1}^N$
$\quad + (+8(N-3)(N-1)n^2(n-1)^2(n-2) + \cdots) a_{n-2}^N$

Set N=4 to get recurrence relation
for a_n^4

Similarly for all N.

putting pictures together - example

(24)

$$\begin{tikzcd}
g(N,4)\,a_n^{N,5} \arrow[r, "n"] \arrow[dr, "-4(N-3)n^2"'] & g(N,3)\,a_n^{N,4} \arrow[r, "n"] \arrow[dr, "-3(N-2)n^2"] & g(N,2)\,a_n^{N,3} \arrow[r, "n"] \arrow[dr, "-2(N-1)n^2"] & g(N,1)\,a_n^{N,2} \arrow[r, "n"] \arrow[dr, "-Nn^2"] & g(N,0)\,a_n^{N,1} \arrow[r, "n"] & a_n^N \\
 & g(N,2)\,a_{n-1}^{N,3} \arrow[r, "n-1"] \arrow[dr, "-2(N-1)(n-1)^2"'] & g(N,1)\,g_{n-1}^{N,2} \arrow[r, "n-1"] \arrow[dr, "-N(n-1)^2"] & g(N,0)\,a_n^{N,1} \arrow[r, "n-1"] & a_{n-1}^N & \\
 & & N\,a_{n-2}^{N,1} \arrow[r, "n-2"] & a_{n-2}^N & &
\end{tikzcd}$$

So, by following all paths from $g(N,4)\,a_n^{N,5}$ to $a_n^N,\ a_{n-1}^N,\ a_{n-2}^N$

get $$g(N,4)\,a_n^N = n^5 a_n^N + (-n^3Nn^2 + \cdots)\,a_{n-1}^N + (+8(N-3)(N-1)n^2(n-1)^2(n-2) + \cdots)\,a_{n-2}^N$$

Set N=4 to get recurrence relation for a_n^4

Similarly for all N. $\square$

namely:

(21)

$$\boxed{\sum_{\substack{k\geq 0 \\ \alpha_i \in \mathbb{N},\ 1 < \alpha_{i+1}+1 < \alpha_i \in \mathbb{N}}} \left(n^{N+1} \sum_{\substack{1\leq i\leq k \\ \alpha_i+\beta_i = N+1,}} \prod_{i=1}^{k} -\alpha_i\beta_i \left(\frac{n-i}{n-i-1}\right)^{\alpha_i-1} \right) a_{n-k}^N = 0}$$

(finite sum, polynomial coefficients)

eg:

$$0 = n a_n^2 - 2(2n-1)\,a_{n-1}^2$$

$$0 = n^2 a_n^3 - (10n^2-10n+3)\,a_{n-1}^3 + 9(n-1)^2\,a_{n-2}^3$$

$$0 = n^3 a_n^4 - 2(2n-1)(5n^2-5n+2)\,a_{n-1}^4 + 64(n-1)^3\,a_{n-2}^4$$

⋮

etc

putting pictures together - example

(24)

$$\begin{tikzcd}[column sep=small]
g(N,4)\,a_n^{N,5} \arrow[r, "n"] \arrow[dr, "-4(N-3)n^2"'] & g(N,3)\,a_n^{N,4} \arrow[r, "n"] \arrow[dr, "-3(N-2)n^2"'] & g(N,2)\,a_n^{N,3} \arrow[r, "n"] \arrow[dr, "-2(N-1)n^2"'] & g(N,1)\,a_n^{N,2} \arrow[r, "n"] \arrow[dr, "-Nn^2"'] & g(N,0)\,a_n^{N,1} \arrow[r, "n"] & a_n^N \\
 & g(N,2)\,a_{n-1}^{N,3} \arrow[r, "n-1"] \arrow[dr, "-2(N-1)(n-1)^2"'] & g(N,1)\,a_{n-1}^{N,2} \arrow[r, "n-1"] \arrow[dr, "-N(n-1)^2"'] & g(N,0)\,a_n^{N,1} \arrow[r, "n-1"] & a_{n-1}^N & \\
 & & N\,a_{n-2}^{N,1} \arrow[r, "n-2"] & a_{n-2}^N & &
\end{tikzcd}$$

So, by following all paths from $g(N,4)\,a_n^{N,5}$ to $a_n^N, a_{n-1}^N, a_{n-2}^N$

get $$g(N,4)\,a_n^N = n^5 a_n^N + (-n^3 N n^2 + \cdots)\,a_{n-1}^N + (+8(N-3)(N-1)n^2(n-1)^2(n-2) + \cdots)\,a_{n-2}^N$$

Set N=4 to get recurrence relation for a_n^4

Similarly for all N. $\square$

namely:

(21)

$$\boxed{\sum_{R\geq 0} \left(n^{N+1} \sum_{\substack{1\leq i\leq R \\ \alpha_i+\beta_i = N+1, \\ \alpha_i \in \mathbb{N},\ 1<\alpha_{i+1}+1<\alpha_i\in\mathbb{N}}} \prod_{i=1}^{K} -\alpha_i\beta_i \left(\frac{n-i}{n-i-1}\right)^{\alpha_i-1} \right) a_{n-K}^N = 0}$$

(finite sum, polynomial coefficients)

eg:

$$0 = n a_n^2 - 2(2n-1)\,a_{n-1}^2$$

$$0 = n^2 a_n^3 - (10n^2-10n+3)\,a_{n-1}^3 + 9(n-1)^2\,a_{n-2}^3$$

$$0 = n^3 a_n^4 - 2(2n-1)(5n^2-5n+2)\,a_{n-1}^4 + 64(n-1)^3\,a_{n-2}^4$$

$\vdots$

etc

namely:

$$\boxed{\sum_{k\geq 0}\left(n^{N+1} \sum_{\substack{1\leq i\leq k \\ \alpha_i+\beta_i = N+1, \\ \alpha_i \in \mathbb{N},\ 1<\alpha_{i+1}+1<\alpha_i\in\mathbb{N}}} \prod_{i=1}^{k} -\alpha_i\beta_i \left(\frac{n-i}{n-i-1}\right)^{\alpha_i-1}\right) a_{n-k}^{N} = 0}$$

21

(finite sum, polynomial coefficients)

eg:

$$0 = n a_n^2 - 2(2n-1) a_{n-1}^2$$

$$0 = n^2 a_n^3 - (10n^2-10n+3) a_{n-1}^3 + 9(n-1)^2 a_{n-2}^3$$

$$0 = n^3 a_n^4 - 2(2n-1)(5n^2-5n+2) a_{n-1}^4 + 64(n-1)^3 a_{n-2}^4$$

$\vdots$

etc

25

Next step would be to investigate monodromy, write down solutions in terms of hypergeometric functions, Eg done by physicists studying

"Stair case polygons and recurrent lattice walks", in the case $N=3,4$

– Glasser, Montaldi, Prellberg, Guttmann, Essam

with $F_N = \sum_{n\geq 0} a_n^N x^n$, $\quad F_3(x^2) = F(1/4, -1/3; 1,1,1,1; x^2)$

What happens for larger N?

When is the result "modular"?

(monodromy given by subgp of $PSL_2(\mathbb{Z})$)