Title: Homological mirror symmetry for Fano surfaces

Date: Nov 22, 2004 02:00 PM

URL: http://pirsa.org/04110031

Abstract:

Mirror Symmetry

Complex manifolds: (X, J) locally $\simeq (\mathbb{C}^n, i)$

Look at complex analytic cycles + holom. vector bundles, or better: coherent sheaves
Intersection theory = Morphisms and extensions of sheaves.

Symplectic manifolds: (Y, ω) locally $\simeq (\mathbb{R}^{2n}, \sum dx_i \wedge dy_i)$

Look at Lagrangian submanifolds (+ flat unitary bundles):
$L^n \subset Y^{2n}$ with $\omega_{|L} = 0$ (locally $\simeq \mathbb{R}^n \subset \mathbb{R}^{2n}$; in $\dim_{\mathbb{R}} 2$, any embedded curve!)

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(discard intersections that cancel by Hamiltonian isotopy)

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Mirror symmetry:

D-branes = boundary conditions for open strings.

Homological mirror symmetry (Kontsevich): at the level of derived categories.

$$\boxed{\begin{array}{lcl} \text{A-branes} & = & \text{Lagrangian submanifolds.} \\ \text{B-branes} & = & \text{coherent sheaves.} \end{array}}$$

HMS Conjecture: Calabi-Yau case

X, Y Calabi-Yau ($c_1 = 0$) mirror pair $\Rightarrow$ $\boxed{\begin{array}{lcl} D^bCoh(X) & \simeq & D\mathcal{F}(Y) \\ D\mathcal{F}(X) & \simeq & D^bCoh(Y) \end{array}}$

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$Coh(X)$ = category of coherent sheaves on X complex manifold.

D^b = bounded derived category:

Objects = complexes $0 \to \cdots \to \mathcal{E}^i \xrightarrow{d^i} \mathcal{E}^{i+1} \to \cdots \to 0$.

Morphisms = morphisms of complexes (up to homotopy, + inverses of quasi-isoms)

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$\mathcal{F}(Y)$ = Fukaya A_∞-category of (Y, ω). Roughly:

Objects = (some) Lagrangian submanifolds (+ flat unitary bundles)

Morphisms: $\mathrm{Hom}(L, L') = CF^*(L, L') = \mathbb{C}^{|L \cap L'|}$ if $L \pitchfork L'$. (or $\bigoplus \mathrm{Hom}(\mathcal{E}_p, \mathcal{E}'_p)$)

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with: differential $d = m_1$; product m_2 (composition; only associative up to homotopy); and higher products $(m_k)_{k \geq 3}$ (related by A_∞-equations).

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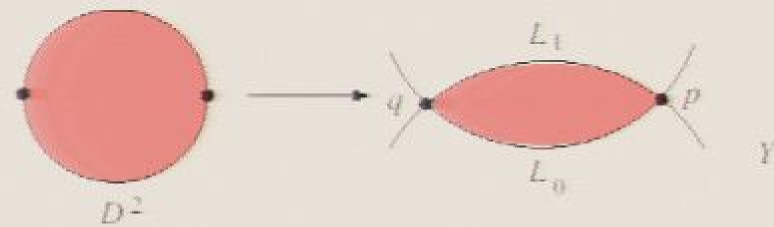
- Differential $d = m_1 : \mathrm{Hom}(L_0, L_1) \to \mathrm{Hom}(L_0, L_1)[1]$

$\langle m_1(p), q\rangle = \sum_{u\in\mathcal{M}(p,q)} \pm\exp(-\int_{D^2} u^*\omega)$

counts pseudo-holomorphic maps

(in $\dim_{\mathbb{R}}$ 2: immersed discs with convex corners)



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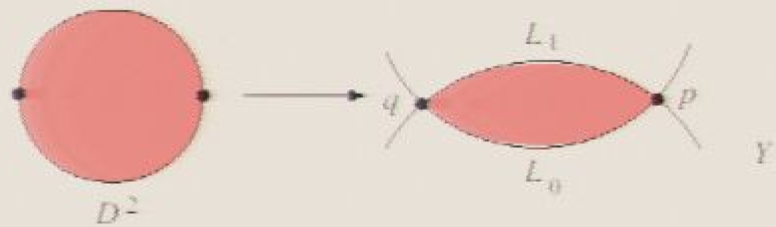

- Product $m_2 : \mathrm{Hom}(L_0, L_1) \otimes \mathrm{Hom}(L_1, L_2) \to \mathrm{Hom}(L_0, L_2)$

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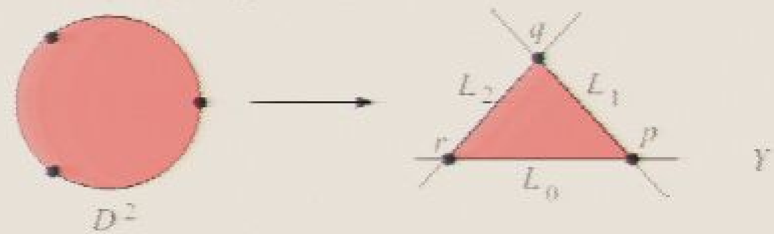

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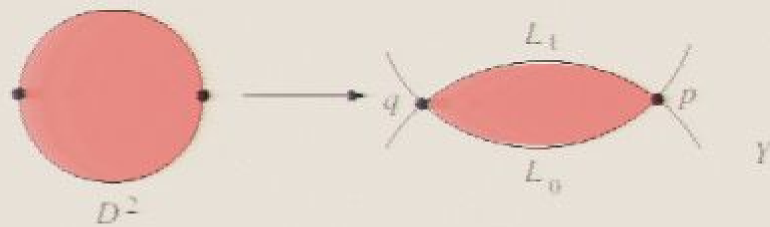

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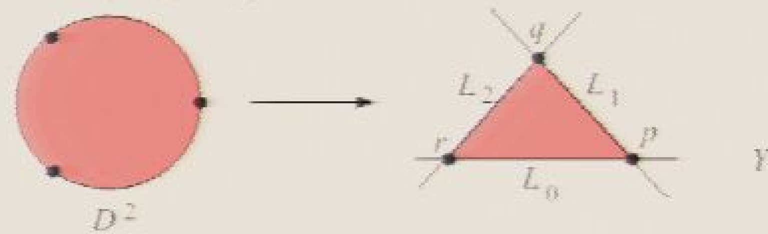

- Higher products $m_k : \mathrm{Hom}(L_0, L_1) \otimes \cdots \otimes \mathrm{Hom}(L_{k-1}, L_k) \to \mathrm{Hom}(L_0, L_k)[2-k]$

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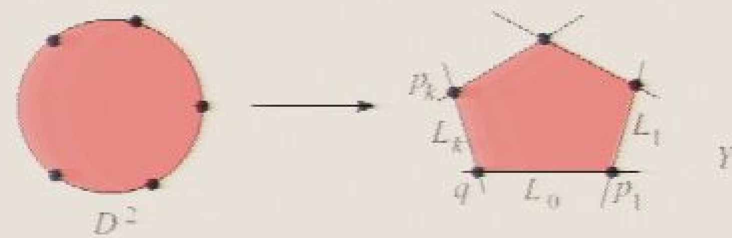

HMS Conjecture: Fano case

X Fano ($c_1(TX) > 0$) $\xleftrightarrow{M.S.}$ "Landau-Ginzburg model" $\begin{cases} Y & \text{(non-compact) manifold} \\ W : Y \to \mathbb{C} & \text{"superpotential"} \end{cases}$

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$$\boxed{\begin{array}{lcl} D^bCoh(X) & \simeq & D\,Lag(W) \\ D\mathcal{F}(X) & \simeq & D^bSing(W) \end{array}}$$

$D\,Lag(W)$ (Lagrangians) and $D^bSing(W)$ (sheaves) =

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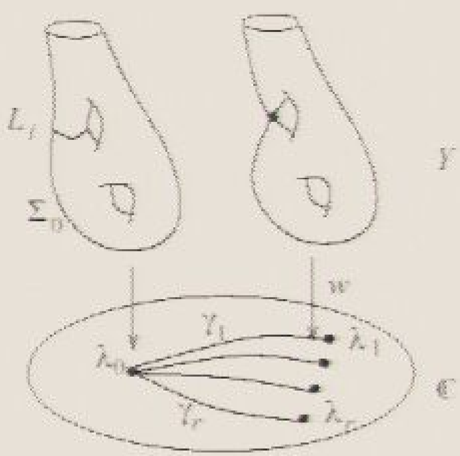
If $W : Y \to \mathbb{C}$ is a Morse function (isolated non-degenerate crit. pts):



$L_i \subset \Sigma_0$ Lagrangian sphere = vanishing cycle associated to γ_i
(collapses to crit. pt. by parallel transport)

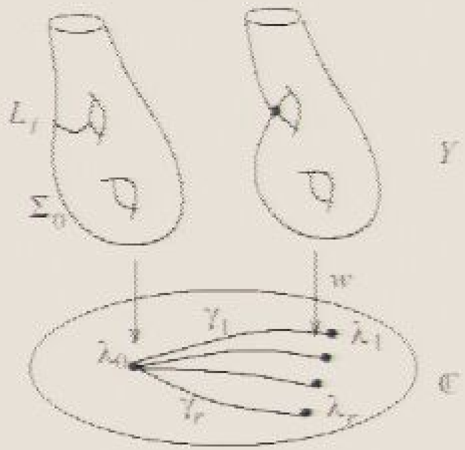
Seidel: $Lag(W, \{\gamma_i\})$ finite, directed A_∞-category.

Objects: $L_1, \ldots, L_r$.

$$\mathrm{Hom}(L_i, L_j) = \begin{cases} CF^*(L_i, L_j) = \mathbb{C}^{|L_i \cap L_j|} & \text{if } i < j \\ \mathbb{C} \cdot \mathrm{Id} & \text{if } i = j \\ 0 & \text{if } i > j \end{cases}$$

Products: $(m_k)_{k \geq 1}$ = Floer theory for Lagrangians $\subset \Sigma_0$.

Categories of Lagrangian vanishing cycles



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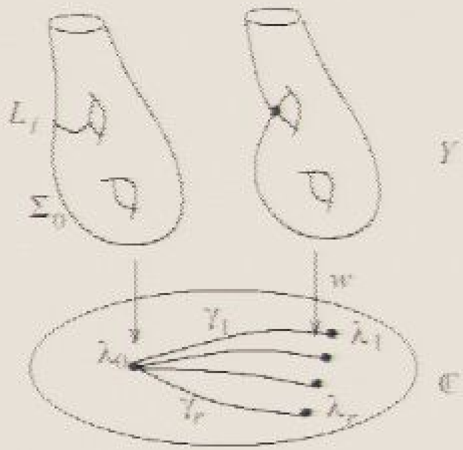

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- m_k counts discs in Σ_0 with boundary in $\bigcup L_i$, with coefficients $\pm \exp(-\int_{D^2} u^*\omega)$.
- in our case $\pi_2(\Sigma_0) = 0$, $\pi_2(\Sigma_0, L_i) = 0$, so no bubbling.

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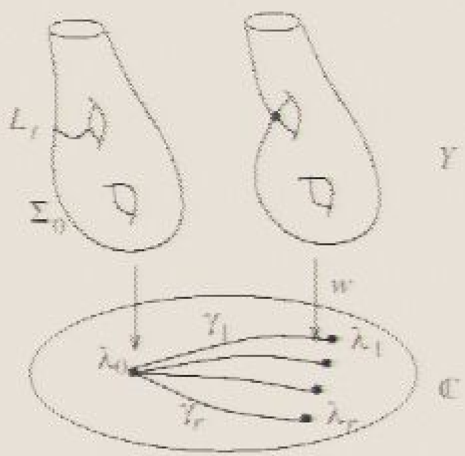

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Remarks:

- $\langle L_1, \dots, L_r \rangle$ = exceptional collection generating $D\,Lag$.
- objects also represent Lefschetz thimbles (Lagrangian discs bounded by L_i, fibering above γ_i)

Theorem. (Seidel) *Changing $\{\gamma_i\}$ affects $Lag(W, \{\gamma_i\})$ by mutations: $D\,Lag(W)$ depends only on $W : (Y, \omega) \to \mathbb{C}$.*

Example 1: weighted projective planes

(Auroux-Katzarkov-Orlov, math.AG/0404281; cf. work of Seidel on $\mathbb{CP}^2$)

$X = \mathbb{CP}^2(a, b, c) = (\mathbb{C}^3 - \{0\})/(x, y, z) \sim (t^a x, t^b y, t^c z)$ (Fano orbifold).

$D^bCoh(X)$ has an exceptional collection $\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(N-1)$ $(N = a + b + c)$

(Homogeneous coords. x, y, z are sections of $\mathcal{O}(a), \mathcal{O}(b), \mathcal{O}(c)$)

$\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j)) \simeq$ deg. $(j - i)$ part of symmetric algebra $\mathbb{C}[x, y, z]$ (degs. a, b, c)

All in degree 0 (no Ext's); composition = obvious.

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Mirror: $Y = \{x^a y^b z^c = 1\} \subset (\mathbb{C}^*)^3$, $W = x+y+z$. $(Y \simeq (\mathbb{C}^*)^2$ if $gcd(a,b,c) = 1)$

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Theorem. $D\,Lag(W) \simeq D^bCoh(X)$.

(this should extend to weighted projective spaces in all dimensions; for technical reasons we only have a partial argument when $\dim_{\mathbb{C}} \geq 3$).

Non-commutative deformations

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Can deform $Lag(W)$ by changing $[\omega]$ (and introducing a B-field).

Choose $t \in \mathbb{C}$, and take $\boxed{\int_{S^1 \times S^1}[B + i\omega] = t}$ $\qquad$ ($S^1 \times S^1 =$ generator of $H_2(Y, \mathbb{Z}) \simeq \mathbb{Z}$)

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This corresponds to a **non-commutative deformation** X_t of X:
deform weighted polynomial algebra $\mathbb{C}[x,y,z]$ to

$$yz = \mu_1\, zy, \quad zx = \mu_2\, xz, \quad xy = \mu_3\, yx, \qquad \text{with } \boxed{\mu_1^a \mu_2^b \mu_3^c = e^{it}}$$

Outline of argument

$Y = \{x^a y^b z^c = 1\} \subset (\mathbb{C}^*)^3$. $W = x + y + z$:

$$\text{crit}\, W = \{\lambda \in \mathbb{C},\ \lambda^{a+b+c} = \frac{(a+b+c)^{a+b+c}}{a^a b^b c^c}\} = \{\lambda_j,\ 0 \le j < N\}$$

Reference fiber: $\Sigma_0 = W^{-1}(0)$; arcs γ_j = straight lines $\Rightarrow$ vanishing cycles $L_j \subset \Sigma_0$.

If ω is $\mathbb{Z}_N$-invariant, then $L_j = \exp(\frac{-2\pi i\, j}{a+b+c}) \cdot L_0$.

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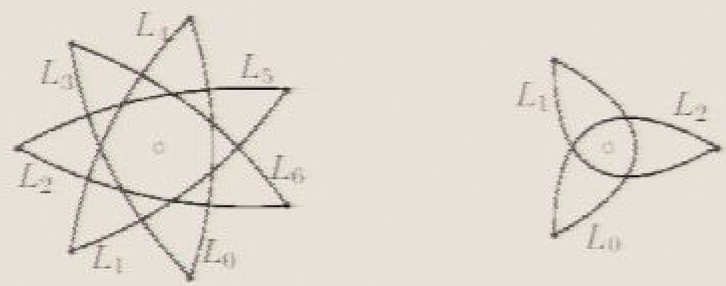
Visualize L_j and intersections via projection $\pi_x : \Sigma_0 \to \mathbb{C}^*$.

($b+c$-fold branched covering, with $a+b+c$ branch points)



$(a,b,c) = (4,2,1)$ $\qquad$ $(a,b,c) = (1,1,1)$

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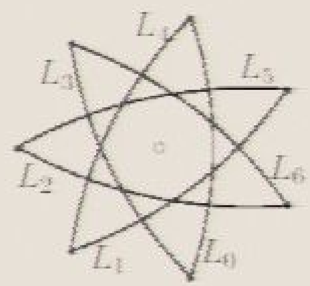

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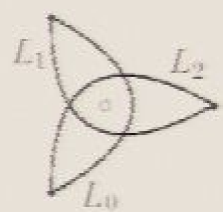

$(a,b,c) = (1,1,1)$

$\Rightarrow$ Description of $Lag(W, \{\gamma_j\})$:

- Objects: L_j, $0 \leq j < N$.
- $\bigoplus_{i<j} CF^*(L_i, L_j)$ = free module of rank $3N$ (=intersections of vanishing cycles)

Outline of argument

- Objects of $Lag(W, \{\gamma_j\})$: L_j, $0 \le j < N$. $(N = a + b + c)$
- $\bigoplus_{i<j} CF^*(L_i, L_j)$ = free module of rank $3N$, generators

$$x_i \in CF^*(L_i, L_{i+a}), \quad y_i \in CF^*(L_i, L_{i+b}), \quad z_i \in CF^*(L_i, L_{i+c}),$$
$$\bar{x}_i \in CF^*(L_i, L_{i+b+c}), \quad \bar{y}_i \in CF^*(L_i, L_{i+a+c}), \quad \bar{z}_i \in CF^*(L_i, L_{i+a+b}).$$

- for suitable graded Lagrangian lifts of L_j,

$$\deg(x_i, y_i, z_i) = 1, \quad \deg(\bar{x}_i, \bar{y}_i, \bar{z}_i) = 2.$$

Outline of argument

- Objects of $Lag(W, \{\gamma_j\})$: L_j, $0 \le j < N$. $(N = a + b + c)$
- $\bigoplus_{i<j} CF^*(L_i, L_j)$ = free module of rank $3N$, generators

$$\begin{array}{lll} x_i \in CF^*(L_i, L_{i+a}), & y_i \in CF^*(L_i, L_{i+b}), & z_i \in CF^*(L_i, L_{i+c}), \\ \bar{x}_i \in CF^*(L_i, L_{i+b+c}), & \bar{y}_i \in CF^*(L_i, L_{i+a+c}), & \bar{z}_i \in CF^*(L_i, L_{i+a+b}). \end{array}$$

- for suitable graded Lagrangian lifts of L_j,

$$\deg(x_i, y_i, z_i) = 1, \quad \deg(\bar{x}_i, \bar{y}_i, \bar{z}_i) = 2.$$

- $m_k = 0$ for $k \ne 2$.
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Outline of argument

$Y = \{x^a y^b z^c = 1\} \subset (\mathbb{C}^*)^3$. $W = x + y + z$:

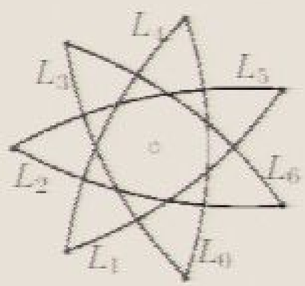
$$\text{crit}\, W = \{\lambda \in \mathbb{C},\ \lambda^{a+b+c} = \frac{(a+b+c)^{a+b+c}}{a^a b^b c^c}\} = \{\lambda_j,\ 0 \le j < N\}$$

Reference fiber: $\Sigma_0 = W^{-1}(0)$: arcs γ_j = straight lines $\Rightarrow$ vanishing cycles $L_j \subset \Sigma_0$.

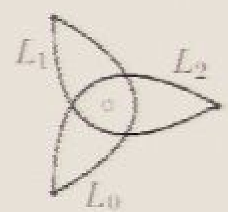
If ω is $\mathbb{Z}_N$-invariant, then $L_j = \exp(\frac{-2\pi i j}{a+b+c}) \cdot L_0$.

Visualize L_j and intersections via projection $\pi_x : \Sigma_0 \to \mathbb{C}^*$.

($b + c$-fold branched covering, with $a + b + c$ branch points)



$(a, b, c) = (4, 2, 1)$



$(a, b, c) = (1, 1, 1)$

8

Outline of argument

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Then pass to dual exceptional collection by "full mutation" (change $\{\gamma_j\}$ to $\{\gamma'_j\}$ with base point at infinity) $\Rightarrow$ exterior algebra becomes truncated symmetric algebra.

Example 2: Del Pezzo surfaces

(Auroux-Katzarkov-Orlov, in progress)

$X = \mathbb{CP}^2$ blown up at $k \leq 9$ points, $-K_X$ ample (or more generally, nef).

$D^bCoh(X)$ has an exceptional collection $\mathcal{O},\ \pi^*T_{\mathbb{P}^2}(-1),\ \pi^*\mathcal{O}_{\mathbb{P}^2}(1),\ \mathcal{O}_{E_1}, \ldots, \mathcal{O}_{E_k}$

Compositions encode coordinates of blown up points. For generic blowups, $\mathrm{Hom}(\mathcal{O}_{E_i}, \mathcal{O}_{E_j}) = 0$.

Infinitely close blowups give pairs of morphisms in deg. 0 and 1 (recover $\mathcal{O}_C$ (-2-curve) as a cone).

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Mirror: mirror to $\mathbb{CP}^2$ compactifies to $\overline{M}$ = resolution of $\{XYZ = T^3\} \subset \mathbb{CP}^3$,
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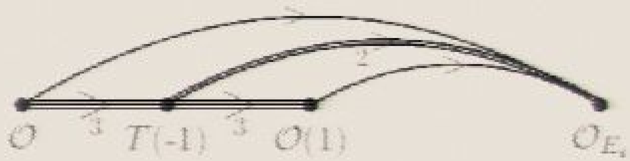

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W is Morse, with 3 crit. pts. in $\{|W| < \infty\}$; fiber at infinity has 9 components.

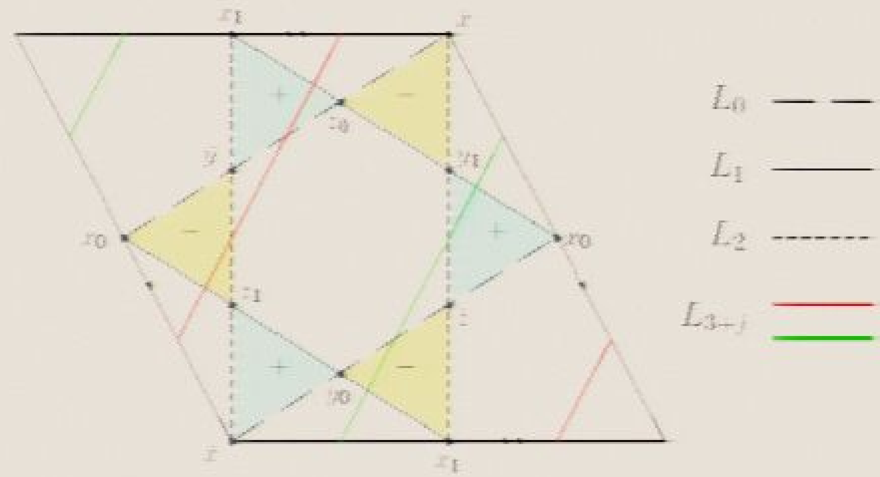
Mirror to X = deform $(\overline{M}, W)$ to bring k of the crit. pts. over ∞ into finite part.

Get an elliptic fibration over $\{|W_k| < \infty\}$: $W_k : M_k \to \mathbb{C}$, with $3+k$ sing. fibers.

(symplectic form to be specified later)

Theorem. *For suitable choice of* $[B + i\omega]$, $D\,Lag(W_k) \simeq D^bCoh(X_k)$.

The vanishing cycles of W_k



Symplectic deformation parameters: $[B+i\omega] \in H^2(M_k,\mathbb{C})$:

- Area of fiber: $\tau = \frac{1}{2\pi}\int_\Sigma (B+i\omega)$ $\longleftrightarrow$ cubic curve $\mathbb{CP}^2 \supset E \simeq \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$

  (all blowups are at points of E: think of E as zero set of $\beta \in H^0(\Lambda^2 T)$.)
- Area of C ($\partial C = L_0 + L_1 + L_2$): $t = \frac{1}{2\pi}\int_C (B+i\omega)$ $\longleftrightarrow$ $\sigma \in \mathrm{Pic}_0(E)$

(same parameter as in Example 1: commutative deformations correspond to $t=0$: takes values in $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$.)

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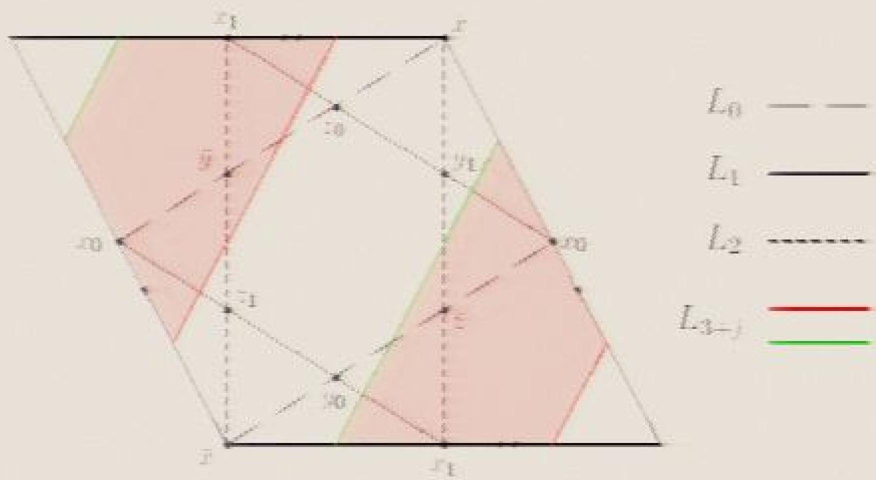

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- Areas of cycles C_j ($\partial C_j = L_{3+j} + \dots$): $t_j = \frac{1}{2\pi}\int_{C_j}(B+i\omega)$, take values in $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$.

= positions of blown up points on E.

For $t_i - t_j = 0 \mod (\mathbb{Z}+\tau\mathbb{Z})$, L_{3+i}, L_{3+j} become Ham. isotopic, acquire $HF^*(L_{3+i}, L_{3+j}) \simeq H^*(S^1)$.

This corresponds to infinitely close blowups, where -2-curves appear.