Title: The Arithmetic of Calabi-Yau Manifolds

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Abstract:

Arithmetic of Calabi-Yau Manifolds

with

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AIMS:

- To explain the fact that the periods of a Calabi–Yau manifold in terms of which we compute many observables of the effective low energy limit of string theory encode important arithmetic information about the manifold.

- To speculate about the role of 'quantum corrections' and mirror symmetry.

Periods of the Quintic

Consider for definiteness, the one parameter family of quintics in $\mathbb{P}_4$

$$\mathcal{M} \; : \; P(x,\psi) \; = \; \sum_{i=1}^{5} x_i^5 - 5\psi \, x_1 x_2 x_3 x_4 x_5 \; .$$

$\mathcal{M}$ has $h^{11} = 1$ and $h^{21} = 101$.

In this simple case there is a simple relation between $\mathcal{M}$ and its mirror

$$\mathcal{W} \; = \; \mathcal{M}/\Gamma$$

$$\Gamma \; : \; (x_1,\, x_2,\, x_3,\, x_4,\, x_5) \; \longmapsto \; (\zeta^{n_1} x_1,\, \zeta^{n_2} x_2,\, \zeta^{n_3} x_3,\, \zeta^{n_4} x_4,\, \zeta^{n_5} x_5)$$

where $\zeta^5 = 1$ and $\sum_i n_i \equiv 1 \; mod \; 5$.

Parametrise the deformations of the complex structure by the periods of the holomorphic $(3,0)$-form Ω

$$\varpi_j(\psi) = \int_{\gamma_j} \Omega\,, \qquad \forall \quad \gamma_j \in H_3(\mathcal{M})$$

$\mathcal{M}$ has $h^{21} = 101$ and $204 = 2\times 100 + 4$ periods while $\mathcal{W}$ has $h^{21} = 1$ and 4 periods.

These periods are (generalised) hypergeometric functions and satisfy a differential equation of order b_3. In the case of the principal periods

$$\mathcal{L}\,\varpi(\lambda) = 0\ ; \quad \lambda = \frac{1}{(5\psi)^5}$$

where

$$\mathcal{L} = \vartheta^4 - 5\lambda\prod_{i=1}^{4}(5\vartheta + i)\ , \quad \text{with} \ \ \vartheta = \lambda\frac{d}{d\lambda}\,.$$

The operator $\mathcal{L}$ is of fourth order and $\lambda = 0$ is a regular singular point with all four indices equal to zero. Thus the solutions near the origin are asymptotic to

$$1,\ \log\lambda,\ \log^2\lambda,\ \log^3\lambda\,.$$

The solution that has no logarithm is the series

$$f_0(\lambda) = \sum_{m=0}^{\infty} \frac{(5m)!}{(m!)^5} \lambda^m .$$

more generally the solutions are of the form

$$\begin{aligned}
\varpi_0(\lambda) &= f_0(\lambda) \\
\varpi_1(\lambda) &= f_0(\lambda) \log\lambda + f_1(\lambda) \\
\varpi_2(\lambda) &= f_0(\lambda) \log^2\lambda + 2 f_1(\lambda) \log\lambda + f_2(\lambda) \\
\varpi_3(\lambda) &= f_0(\lambda) \log^3\lambda + 3 f_1(\lambda) \log^2\lambda + 3 f_2(\lambda) \log\lambda + f_3(\lambda)
\end{aligned}$$

where the $f_j(\lambda)$ are power series. These series will enter into our calculation of the number of rational points of $\mathcal{M}$. Recall that these solutions may be found by the method of Frobenius. That is by seeking solutions of the form

$$\varpi(\lambda, \varepsilon) = \sum_{m=0}^{\infty} a_m(\varepsilon)\, \lambda^{m+\varepsilon} \quad \text{to the equation} \quad \mathcal{L}\, \varpi(\lambda, \varepsilon) = \varepsilon^4 \lambda^\varepsilon .$$

Integral Series

We know what the integers mean for the q-expansion of the yukawa coupling:

$$y_{ttt} = 5\left(\frac{2\pi i}{5}\right)^3 \frac{\psi^2}{\varpi_0(\psi)^2(1-\psi^5)}\left(\frac{d\psi}{dt}\right)^3 = 5 + \sum_{k=0}^{\infty}\frac{n_k k^3 q^k}{1-q^k},$$

where in this expression

$$q = \exp(2\pi i t) \quad \text{and} \quad t = \frac{1}{2\pi i}\frac{\varpi_1(\lambda)}{\varpi_0(\lambda)}.$$

Integers however appear also in the mirror map

$$\begin{aligned}\lambda = \ & q + 154\, q^2 + 179139\, q^3 + 313195944\, q^4 \\ & + 657313805125\, q^5 + 1531113959577750\, q^6 \\ & + 3815672803541261385\, q^7 \\ & + 9970002717955633142112\, q^8 + \dots .\end{aligned}$$

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Rational Points

Now ask a very strange (for a physicist) question:

For the quintic $\mathcal{M}$

$$P(x,\psi) = \sum_{i=1}^{5} x_i^5 - 5\psi\, x_1 x_2 x_3 x_4 x_5$$

how many solutions of the equation $P(x,\psi) = 0$ are there with integer x_i and how does this number vary with ψ?

Since the x_i are coordinates in a projective space and we are free to multiply the coordinates by a common scale there is no difference between seeking an integral solution and a rational solution, $x_i \in \mathbb{Q}$. This formulation is better because $\mathbb{Q}$ is a field but it is still very hard to answer in general. An easier but still interesting question is how many solutions are there over a finite field.

Field Theory While Standing on One Leg

A field $\mathbb{F}$ is a set on which $+$ and $\times$ are defined and have the usual associative and distributive properties. $\mathbb{F}$ is an abelian group with respect to addition and $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$ is an abelian group with respect to multiplication.

Finite fields are uniquely classified by the number of elements which is p^N for some prime p and integer N.

The simplest finite field is $\mathbb{F}_p$ the set of integers mod p

$\mathbb{F}_7$							
x	0	1	2	3	4	5	6
x^{-1}	*	1	4	5	2	3	6

An old result, going back to Fermat, is $a^p \equiv a$ write this

$$a(a^{p-1} - 1) \equiv 0$$

it follows that

$$a^{p-1} \equiv \begin{cases} 1, & \text{if } a \neq 0 \\ 0, & \text{if } a = 0 . \end{cases}$$

There is another elementary fact that is also useful. Consider

$$\sum_{a \in \mathbb{F}_p} a^n = \sum_{a \in \mathbb{F}_p} (ba)^n = b^n \sum_{a \in \mathbb{F}_p} a^n .$$

It follows now that

$$\sum_{a \in \mathbb{F}_p} a^n \equiv \begin{cases} 0, & \text{if } p-1 \text{ does not divide } n \\ -1, & \text{if } p-1 \text{ divides } n . \end{cases}$$

A Zero'th Order Result

Take now $x \in \mathbb{F}_p^5$ and $5\psi \in \mathbb{F}_p$, $(p \neq 5)$ and let

$$\nu_\lambda = \#\{x \mid P(x,\psi) \equiv 0\}, \ \ \lambda = \frac{1}{(5\psi)^5}\,.$$

This number can be computed mod p with relative ease

$$\nu_\lambda \equiv \sum_{x \in \mathbb{F}_p^5} \left(1 - P(x,\psi)^{p-1}\right)$$

Expand the power and use the fact that $\sum x_i^n \equiv \begin{cases} 0, & \text{if } p-1 \text{ does not divide } n \\ -1, & \text{if } p-1 \text{ divides } n\,. \end{cases}$

The result is that

$$\nu_\lambda \equiv {}^{[p/5]}\varpi_0(\lambda) = \sum_{m=0}^{[p/5]} \frac{(5m)!}{(m!)^5}\,\lambda^m\,.$$

p-Adic Numbers

ν_λ is a definite number so we may seek to compute it exactly. We expand

$$\nu_\lambda = \nu_\lambda^{(0)} + \nu_\lambda^{(1)} p + \nu_\lambda^{(2)} p^2 + \nu_\lambda^{(3)} p^3 + \nu_\lambda^{(4)} p^4 + \ldots$$

with $0 \leq \nu_\lambda^{(j)} \leq p - 1$ and evaluate $\text{mod } p^2$, $\text{mod } p^3$, and so on.

This leads naturally into p-adic analysis. Given an $r \in \mathbb{Q}$ we write

$$r = \frac{m}{n} = \frac{m_0}{n_0} p^\alpha$$

where m_0, n_0 and p have no common factor. The p-adic norm of r is defined to be

$$\|r\|_p = p^{-\alpha}, \quad \|0\|_p = 0$$

and is a norm, that is it has the properties:

$$\|r\|_p \geq 0,$$

$$\|r_1 r_2\|_p = \|r_1\|_p \|r_2\|_p$$

$$\|r_1 + r_2\|_p \leq \|r_1\|_p + \|r_2\|_p$$

Counting the Number of Points Exactly

Denote by ν_λ the number of solutions to the equation $P(x,\psi)=0$ over $\mathbb{F}_p$.

$$\nu_\lambda = {}^p f_0(\Lambda) + \left(\frac{p}{1-p}\right)\ {}^p f_1'(\Lambda) + \frac{1}{2!}\left(\frac{p}{1-p}\right)^2\ {}^p f_2''(\Lambda) + \frac{1}{3!}\left(\frac{p}{1-p}\right)^3\ {}^p f_3'''(\Lambda) + \frac{1}{4!}\left(\frac{p}{1-p}\right)^4\ {}^p f_4''''(\Lambda) + \mathcal{O}(p^5)\ .$$

This expression holds for $5 \not| \, p-1$. In the expression

$$\Lambda \ = \ \mathrm{Teich}(\lambda) \ = \ \lim_{n\to\infty} \lambda^{p^n} \quad \text{and} \quad {}^p f_0(\Lambda) \ = \ \sum_{m=0}^{p-1} \frac{(5m)!}{(m!)^5}\,\Lambda^m\ .$$

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Now, as we have said, the number of rational points is determined by the periods and there are $b^3 = 2h^{21} + 2$ of these. The Hodge number h^{21} counts the number of parameters on which the complex structure depends and, in simple cases, this corresponds to the number of ways of deforming the defining polynomial

$$P(x,c) = \sum_{\vec{v}} c_{\vec{v}}\, x^{\vec{v}} \;\; ; \;\; x^{\vec{v}} = x_1^{v_1}\, x_2^{v_2}\, x_3^{v_3}\, x_4^{v_4}\, x_5^{v_5}\, .$$

The directions in which $P(x,c)$ can be deformed correspond to the monomials $x^{\vec{v}}$ considered subject to the ideal $(\partial P/\partial x_i)$. A special role is played by fundamental monomial

$$Q = x_1 x_2 x_3 x_4 x_5$$

which is related by mirror symmetry to the Kähler form of the mirror.

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Return now to our special one parameter family of polynomials

$$P(x,\psi) \;=\; \sum_{i=1}^{5} x_i^5 - 5\psi\, x_1x_2x_3x_4x_5 \,.$$

$\mathcal{M}$ **has** $2h^{21}(\mathcal{M}) + 2 = 204 = 2\times 100 + 4$ **periods while** $\mathcal{W}$ **has** $2h^{21}(\mathcal{W}) + 2 = 4\,.$

$$1 \;\longrightarrow\; Q \;\longrightarrow\; Q^2 \;\longrightarrow\; Q^3$$

$$x^v \;\longrightarrow\; Q\, x^v$$

This leads to 1 fourth order differential operator $\mathcal{L}_{\vec{1}}$ **and 100 second order operators** $\mathcal{L}_{\vec{v}}$**.**

There are tenth order monomials that are not included in the above scheme and which require special attention. The generators of the ideal are

$$x_1^4 \simeq \psi\, x_2x_3x_4x_5 \;\; \textbf{\& cyclic.}$$

Thus

$$x^{(4,3,2,1,0)} \;\simeq\; \psi\, x^{(0,4,3,2,1)} \;\simeq\; \cdots \;\simeq\; \psi^5\, x^{(4,3,2,1,0)} \,.$$

We can also perform the sum in our expression for the number of points to give

$$\nu_\lambda = \sum_{m=0}^{p-1} \beta_m \Lambda^m$$

with coefficients

$$\beta_m = \lim_{n\to\infty} \frac{a_{m(1+p+p^2+\ldots+p^{n+1})}}{a_{m(1+p+p^2+\ldots+p^n)}} = (-1)^m G_{5m} G^5_{-m}$$

When we include the contributions of the other periods for the case $5|p-1$ we find

$$p\nu^*_\lambda = (p-1)^5 + \sum_{\vec{v}} \sum_{m=0}^{p-2} (-1)^m \Lambda^m G_{5m} \prod_{j=1}^{5} G_{-(m+kv_j)}$$

where $k = (p-1)/5$. The contribution of $\vec{v} = (0,0,0,0,0)$ gives our previous expression. The quintic $\vec{v}$'s correspond to the other 200 periods and give the extra terms that arise when $5|p-1$. These terms have a natural interpretation as the exceptional divisors of the mirror manifold. The monomial of degree 10 contributes only for the conifold when $\psi^5 = 1$.

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Rational Points over $\mathbb{F}_p$: Dwork's Character

Let

$$\Theta : \mathbb{F}_p \longrightarrow \mathbb{C}_p^*$$

be a non-trivial additive ($\Theta(x + y) = \Theta(x)\Theta(y)$) character of order p ($\Theta(x)^p = 1$). (This is a p-adic version of a character of a commutative group $G \to \mathbb{C}$.) Thus

$$\sum_{y \in \mathbb{F}_p} \Theta(yP(x, \psi)) = p\, \delta(P(x, \psi))$$

$$p\, \nu_\lambda = \sum_{x \in \mathbb{F}_p^5} \sum_{y \in \mathbb{F}_p} \Theta(yP(x, \psi))$$

Dwork constructed such character in terms of Gauss sums

$$G_n = \sum_{x \in \mathbb{F}_p^*} \Theta(x)\, \mathrm{Teich}^n(x)$$

and in terms of these one can expand the character in the form

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Incorporating these considerations

$$\nu_\lambda = p^4 + \sum_{\vec{v}} \gamma_{\vec{v}} \sum_{m=0}^{p-2} \beta_{\vec{v},m}\, \mathrm{Teich}^m(\Lambda)\,,$$

where the $\beta_{\vec{v},m}$ are given in terms of the Gauss sums or, equivalently, in terms of p-adic Γ functions.

- **For 5 $\not| (p-1)$ we only have a contribution from $\vec{v} = (0,0,0,0,0)$**
- **The coefficients $\beta_{\vec{v},m}$ are closely related to the coefficients in the series expansions of the periods around the regular singular point $\lambda = 0$.**

Explicitly to order p:

$$\nu_\lambda = {}^{[p-1]}f_0(\lambda^p) + p\ {}^{[p-1]}f_1'(\lambda^p)$$

$$- \delta_p\, p \sum_{\vec{v}} \frac{\gamma_{\vec{v}}}{\prod_{i=1}^5 (v_i k)!}\ {}^{[p-1]}{}_2F_1(a_{\vec{v}}, b_{\vec{v}}; c_{\vec{v}}; \lambda^p) + \ldots$$

The tenth order polynomial $\vec{v} = (4, 3, 2, 1, 0)$, corresponds to a "period" that is zero everywhere, except when $\psi^5 = 1$. For these values of ψ the variety is not smooth anymore: it has 125 isolated singularities that are double points ("conifold" singularities). The calcualtion for the number of rational points makes sense even for these singular cases. A little simplification reveals the contribution to ν_λ of $\vec{v} = (4, 3, 2, 1, 0)$ as

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$$\nu_\lambda = {}^{[p-1]}f_0(\lambda^p) + p\ {}^{[p-1]}f_1'(\lambda^p)$$

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The tenth order polynomial $\vec{v} = (4, 3, 2, 1, 0)$, corresponds to a "period" that is zero everywhere, except when $\psi^5 = 1$. For these values of ψ the variety is not smooth anymore: it has 125 isolated singularities that are double points ("conifold" singularities). The calcualtion for the number of rational points makes sense even for these singular cases. A little simplification reveals the contribution to ν_λ of $\vec{v} = (4, 3, 2, 1, 0)$ as

$$24p^2(p-1)\delta(Teich(\psi)^5 - 1)\,.$$

The Zeta-Function

Consider now $N_r(\lambda) = \frac{\nu_\lambda - 1}{p-1}$ which are the numbers of projective solutions of $P = 0$ over $\mathbb{F}_{p^r}$ and form

$$\zeta(T, \lambda) = \exp\left(\sum_{r=1}^{\infty} \frac{N_r(\lambda)\, T^r}{r}\right) .$$

If $\mathcal{M}$ is a point then $N_r = 1$ for all r and

$$\sum_{r=1}^{\infty} \frac{N_r T^r}{r} = \sum_{r=1}^{\infty} \frac{T^r}{r} = -\log(1 - T) \implies \zeta_{\text{pt}}(T) = \frac{1}{1-T}$$

Thus for a point

$$\prod_p \zeta_{\text{pt}}(p^{-s}) = \prod_p \frac{1}{1 - p^{-s}} = \zeta_R(s) .$$

The Weil Conjectures

- **Rationality (Dwork):** $\zeta(T)$ **is a rational function of** T

- **Functional equation (Groethendieck):**

$$\zeta\left(\frac{1}{p^d T}\right) = \pm p^{d\chi/2}\, T^{\chi}\, \zeta(T)$$

where χ **is the Euler characteristic and** d **is the real dimension of** $\mathcal{M}$**.**

- **Riemann Hypothesis (Deligne):**

$$\zeta(T) = \frac{P_1(T)P_3(T)\ldots P_{2d-1}(T)}{P_0(T)P_2(T)\ldots P_{2d}(T)}$$

with $P_i(T)$ **a polynomial with coefficients in** $\mathbb{Z}$ **of degree** b_i**. Furthermore**

$$P_i(T) = \prod_{j=1}^{b_i}(1-\alpha_{ij}\,T)\,,\ |\alpha_{ij}| = p^{i/2} \ \text{and}\ P_0(T) = 1-T\,,\ P_{2d}(T) = 1-p^d T\,.$$

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We now work over $\mathbb{F}_{p^r}$ and let $N_r(\psi)$ denote the number of projective solutions to $P(x, \psi) = 0$. The **ζ-function** is defined by the expression

$$\zeta(T, \psi) = \exp\left(\sum_{r=1}^{\infty} \frac{N_r(\psi) T^r}{r}\right)$$

We are led to decompose N_r into a sum of contributions $N_r = N_{r,0} + \sum_v N_{r,v}$.

$$\zeta_{\mathcal{M}}(T, \psi) = \frac{R_0(T, \psi) \prod_v R_v(T, \psi)}{(1 - T)(1 - pT)(1 - p^2 T)(1 - p^3 T)}$$

$$\zeta_{\mathcal{W}}(T, \psi) = \frac{R_0(T, \psi)}{(1 - T)(1 - pT)^{101}(1 - p^2 T)^{101}(1 - p^3 T)} .$$

In all cases, apart from the conifold, R_0 is a quartic

$$R_0 = 1 + a_0 T + b_0 p T^2 + a_0 p^3 T^3 + p^6 T^4 .$$

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The Euler Curves

Classical analysis gives an expression for the hypergeometric functions in terms of Euler's integral which is of the form

$$\int dx\, x^{-\alpha/5}(1-x)^{-\beta/5}(1-x/\psi^5)^{-(1-\beta/5)}\,.$$

If we think of Euler's integral as $\int \frac{dx}{y}$ then we are led to curves

$$\mathcal{E}_{\alpha\beta}(\psi):\quad y^5 = x^\alpha(1-x)^\beta(1-x/\psi^5)^{5-\beta}\,.$$

v	α	β
$(4,1,0,0,0)$	2	3
$(3,2,0,0,0)$	1	4
$(3,1,1,0,0)$	2	4
$(2,2,1,0,0)$	4	3

$$\mathcal{E}_{\alpha\beta} = \begin{cases} \mathcal{A} & \alpha+\beta=5 \\ \mathcal{B} & \alpha+\beta\neq 5 \text{ and } \alpha\neq\beta\,. \end{cases}$$

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The Conifold

For the conifold $\psi^5 = 1$ the ζ-function seems to be especially simple

$$\zeta(T,1) \;=\; \frac{(1-\epsilon\, pT)\,(1-a_p\,T+p^3T^2)\,(1-pT)^{100}}{(1-T)(1-pT)(1-p^2T)(1-p^3T)\,(1-p^2T)^{24}} \quad ; \;\; \rho = 1$$

where $\epsilon = \left(\frac{5}{p}\right) = \pm 1$ and a_p is the p-th coefficient in the q-expansion of the eigenform, g, found by Schoen; it is the unique cusp form of weight 4 for the group $\Gamma_0(25)$.

$$g = \eta(q^5)^4\left[\eta(q)^4 + 5\eta(q)^3\eta(q^{25}) + 20\eta(q)^2\eta(q^{25})^2 + 25\eta(q)\eta(q^{25})^3 + 25\eta(q^{25})^4\right]$$

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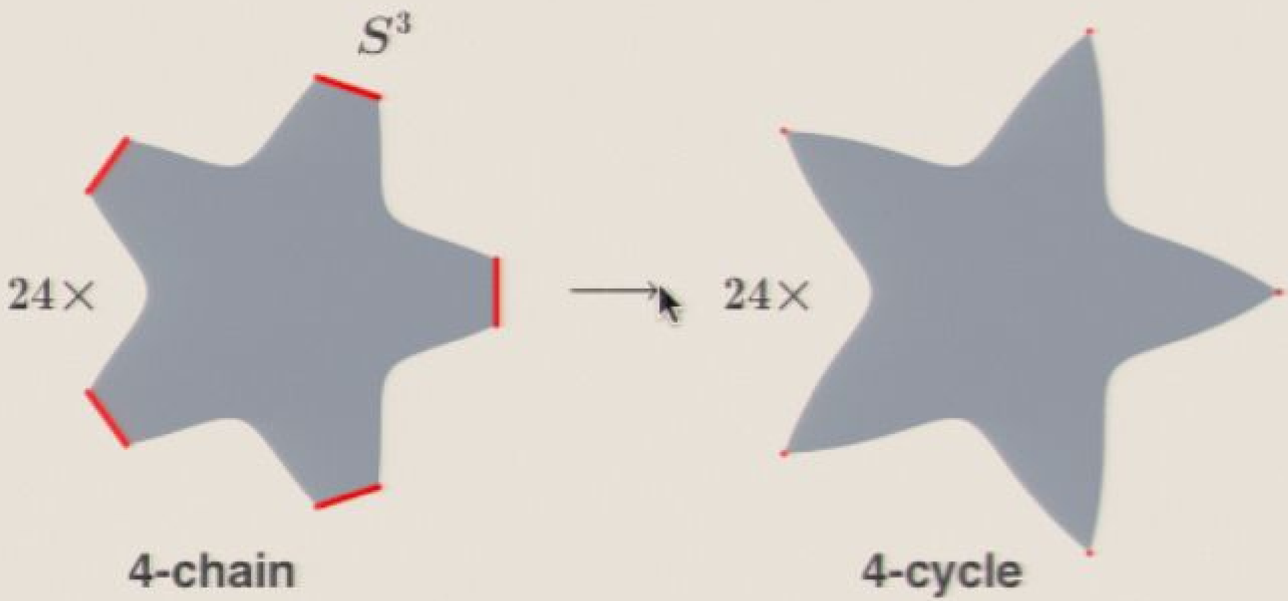
125 S^3's are blown down but only 101 are independent so 24 4-cycles are created.



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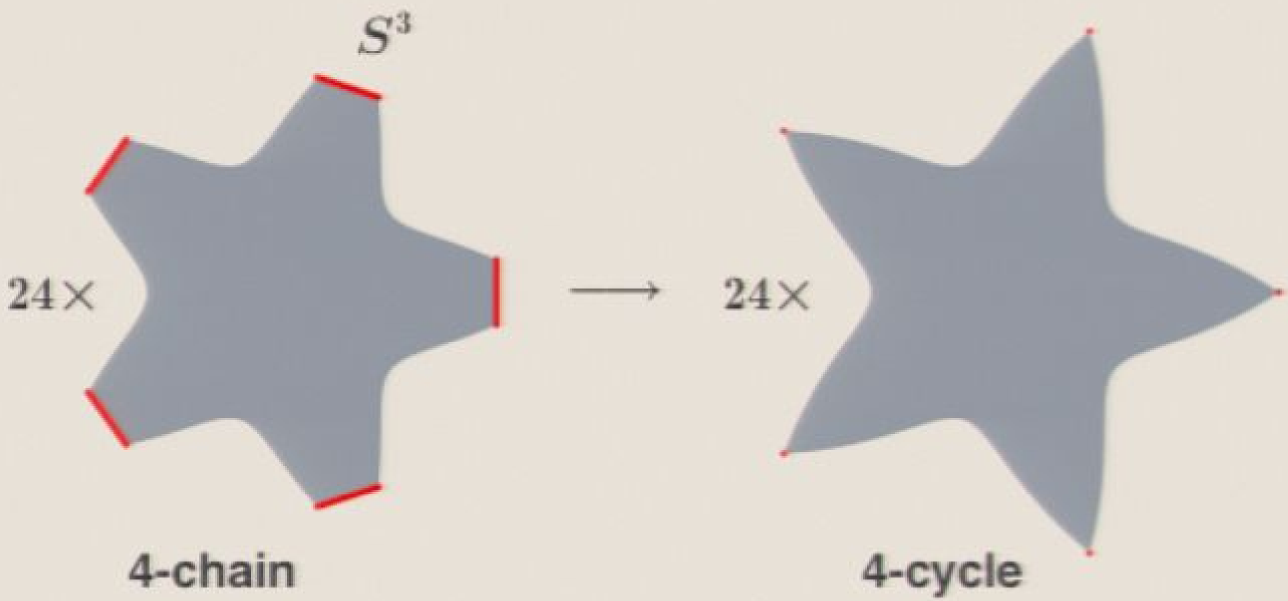

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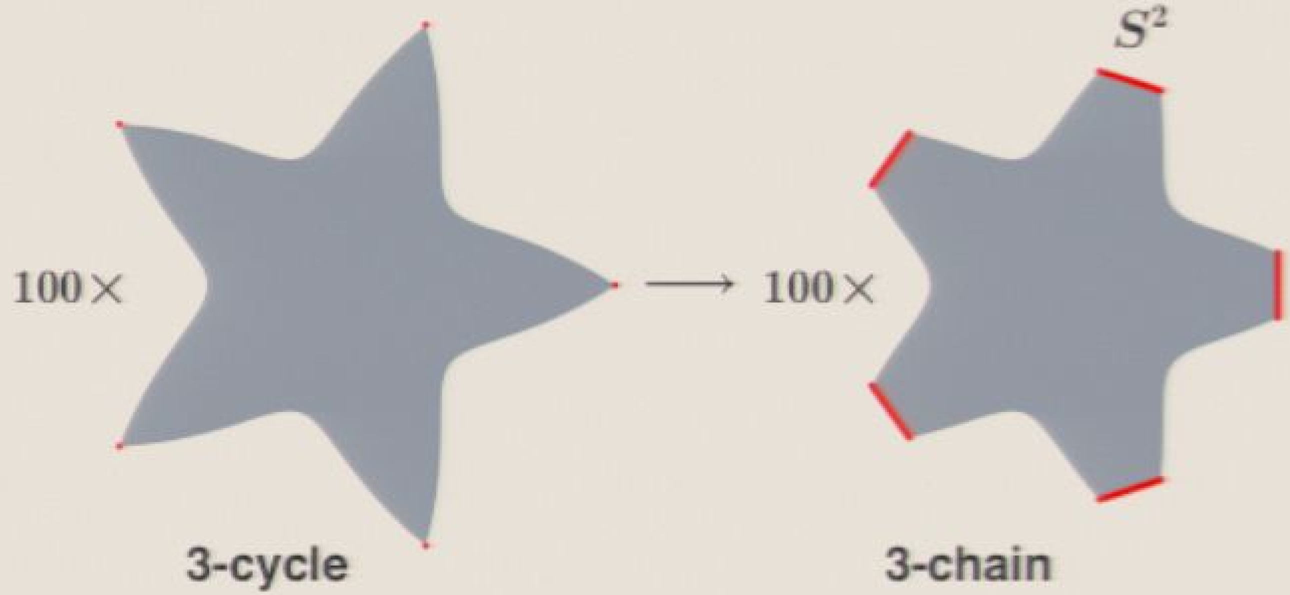
Now we resolve 125 nodes with $\mathbb{P}^1$'s, but there are 100 relations so we destroy 100 3-cycles.



$$\zeta(T,1) = \frac{(1 - a_p T + p^3 T^2)(1-pT)^{100}}{(1-T)(1-pT)^{125}(1-p^2T)^{25}(1-p^3T)}$$

$$= \frac{(1 - a_p T + p^3 T^2)}{(1-T)(1-pT)^{25}(1-p^2T)^{25}(1-p^3T)}\,.$$

The ζ-Function and Mirror Symmetry

We now work over $\mathbb{F}_{p^r}$ and let $N_r(\psi)$ denote the number of projective solutions to $P(x, \psi) = 0$.

$$\zeta(T, \psi) \;=\; \exp\left(\sum_{r=1}^{\infty} \frac{N_r(\psi) T^r}{r}\right)$$

As defined the ζ-function does not respect mirror symmetry

$$\zeta(T) \;=\; \frac{\text{Numerator of deg. } 2h^{21} + 2 \text{ depending on the cpx. structure of } \mathcal{M}}{\text{Denominator of deg. } 2h^{11} + 2}\,.$$

Explicitly for the quintic we have

$$\zeta_{\mathcal{M}}(T, \psi) \;=\; \frac{R_0(T, \psi)\, R_{\mathcal{A}}(p^\rho T^\rho, \psi)^{\frac{20}{\rho}}\, R_{\mathcal{B}}(p^\rho T^\rho, \psi)^{\frac{30}{\rho}}}{(1 - T)(1 - pT)(1 - p^2 T)(1 - p^3 T)}$$

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The 5-adic Limit

The desired relations are true in the 5-adic limit. More precisely for all p and ψ

$$R_0(T,\psi) = (1-T)(1-p\,T)(1-p^2T)(1-p^3T) + \mathcal{O}(5^2)$$

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$$\zeta_{\mathcal{W}} = \frac{1}{\zeta_{\mathcal{M}}} + \mathcal{O}(5^2)$$

Compare this with the quantum corrections to the classical Yukawa coupling which we write in the form

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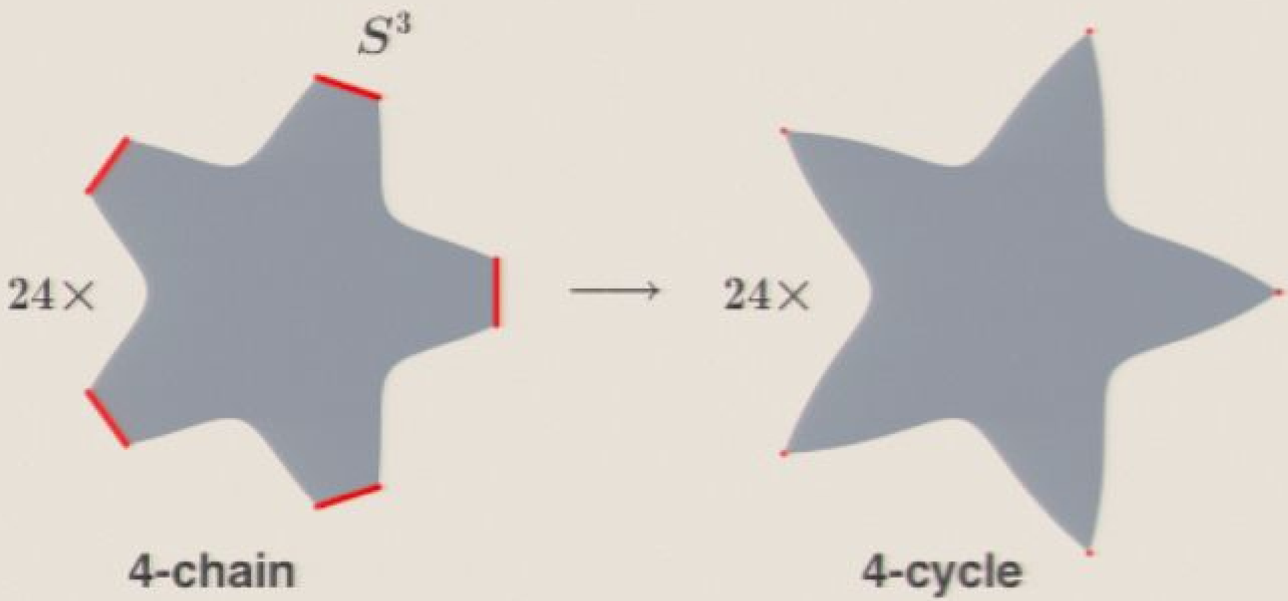

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Counting the Number of Points Exactly

Denote by ν_λ the number of solutions to the equation $P(x,\psi)=0$ over $\mathbb{F}_p$.

$$\nu_\lambda = {}^p f_0(\Lambda) + \left(\frac{p}{1-p}\right)\ {}^p f_1'(\Lambda) + \frac{1}{2!}\left(\frac{p}{1-p}\right)^2\ {}^p f_2''(\Lambda) + \frac{1}{3!}\left(\frac{p}{1-p}\right)^3\ {}^p f_3'''(\Lambda) + \frac{1}{4!}\left(\frac{p}{1-p}\right)^4\ {}^p f_4''''(\Lambda) + \mathcal{O}(p^5)\ .$$

This expression holds for $5 \not| \, p-1$. In the expression

$$\Lambda \ = \ \mathrm{Teich}(\lambda) \ = \ \lim_{n\to\infty} \lambda^{p^n} \quad \text{and} \quad {}^p f_0(\Lambda) \ = \ \sum_{m=0}^{p-1} \frac{(5m)!}{(m!)^5}\,\Lambda^m\ .$$

We can also perform the sum in our expression for the number of points to give

$$\nu_\lambda = \sum_{m=0}^{p-1} \beta_m \Lambda^m$$

with coefficients

$$\beta_m = \lim_{n\to\infty} \frac{a_{m(1+p+p^2+\ldots+p^{n+1})}}{a_{m(1+p+p^2+\ldots+p^n)}} = (-1)^m G_{5m} G^5_{-m}$$

When we include the contributions of the other periods for the case $5|p-1$ we find

$$p\nu^*_\lambda = (p-1)^5 + \sum_{\vec{v}} \sum_{m=0}^{p-2} (-1)^m \Lambda^m G_{5m} \prod_{j=1}^{5} G_{-(m+kv_j)}$$

where $k = (p-1)/5$. The contribution of $\vec{v} = (0,0,0,0,0)$ gives our previous expression. The quintic $\vec{v}$'s correspond to the other 200 periods and give the extra terms that arise when $5|p-1$. These terms have a natural interpretation as the exceptional divisors of the mirror manifold. The monomial of degree 10 contributes only for the conifold when $\psi^5 = 1$.

Counting the Number of Points Exactly

Denote by ν_λ the number of solutions to the equation $P(x,\psi)=0$ over $\mathbb{F}_p$.

$$\nu_\lambda = {}^p f_0(\Lambda) + \left(\frac{p}{1-p}\right)\ {}^p f_1'(\Lambda) + \frac{1}{2!}\left(\frac{p}{1-p}\right)^2\ {}^p f_2''(\Lambda) + \frac{1}{3!}\left(\frac{p}{1-p}\right)^3\ {}^p f_3'''(\Lambda) + \frac{1}{4!}\left(\frac{p}{1-p}\right)^4\ {}^p f_4''''(\Lambda) + \mathcal{O}(p^5)\ .$$

This expression holds for $5 \not| \, p-1$. In the expression

$$\Lambda \;=\; \mathrm{Teich}(\lambda) \;=\; \lim_{n\to\infty} \lambda^{p^n} \quad \text{and} \quad {}^p f_0(\Lambda) \;=\; \sum_{m=0}^{p-1} \frac{(5m)!}{(m!)^5}\, \Lambda^m\ .$$

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The solution that has no logarithm is the series

$$f_0(\lambda) = \sum_{m=0}^{\infty} \frac{(5m)!}{(m!)^5} \lambda^m .$$

more generally the solutions are of the form

$$\begin{aligned}
\varpi_0(\lambda) &= f_0(\lambda) \\
\varpi_1(\lambda) &= f_0(\lambda) \log\lambda + f_1(\lambda) \\
\varpi_2(\lambda) &= f_0(\lambda) \log^2\lambda + 2 f_1(\lambda) \log\lambda + f_2(\lambda) \\
\varpi_3(\lambda) &= f_0(\lambda) \log^3\lambda + 3 f_1(\lambda) \log^2\lambda + 3 f_2(\lambda) \log\lambda + f_3(\lambda)
\end{aligned}$$

where the $f_j(\lambda)$ are power series. These series will enter into our calculation of the number of rational points of $\mathcal{M}$. Recall that these solutions may be found by the method of Frobenius. That is by seeking solutions of the form

$$\varpi(\lambda, \varepsilon) = \sum_{m=0}^{\infty} a_m(\varepsilon) \lambda^{m+\varepsilon} \quad \text{to the equation} \quad \mathcal{L}\, \varpi(\lambda, \varepsilon) = \varepsilon^4 \lambda^\varepsilon .$$

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Field Theory While Standing on One Leg

A field $\mathbb{F}$ is a set on which $+$ and $\times$ are defined and have the usual associative and distributive properties. $\mathbb{F}$ is an abelian group with respect to addition and $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$ is an abelian group with respect to multiplication.

Finite fields are uniquely classified by the number of elements which is p^N for some prime p and integer N.

The simplest finite field is $\mathbb{F}_p$ the set of integers $\bmod\ p$

$\mathbb{F}_7$							
x	0	1	2	3	4	5	6
x^{-1}	*	1	4	5	2	3	6

Counting the Number of Points Exactly

Denote by ν_λ the number of solutions to the equation $P(x,\psi)=0$ over $\mathbb{F}_p$.

$$\nu_\lambda = {}^p f_0(\Lambda) + \left(\frac{p}{1-p}\right)\ {}^p f_1'(\Lambda) + \frac{1}{2!}\left(\frac{p}{1-p}\right)^2\ {}^p f_2''(\Lambda)$$

$$+\frac{1}{3!}\left(\frac{p}{1-p}\right)^3\ {}^p f_3'''(\Lambda) + \frac{1}{4!}\left(\frac{p}{1-p}\right)^4\ {}^p f_4''''(\Lambda) + \mathcal{O}(p^5)\ .$$

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