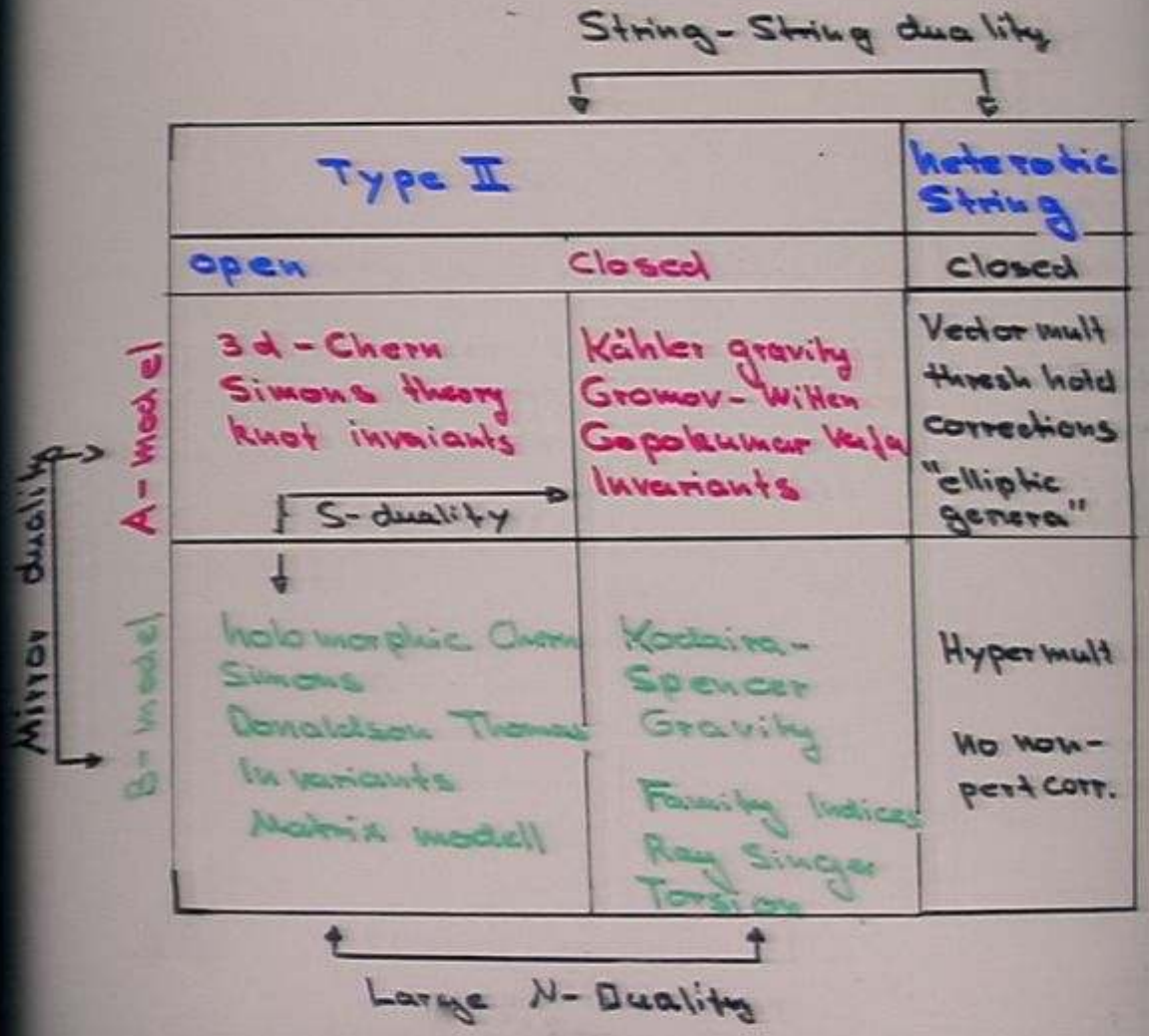
Title: Higher Genus Amplitudes on Compact Calabi-Yau and Threshold Corrections

Date: Nov 20, 2004 05:00 PM

URL: http://pirsa.org/04110023

Abstract:

Topological String Dualities:



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Symplectic Invariants

Gromov-Witten Inv. $\sim$ Gopakumar-Vafa Inv.

$\wr$ $\wr$

Donaldson-Thomas Inv.

Gromov-Witten: M symplectic manifold

$X: \Sigma_g \longrightarrow M$ (or)

$[X(\Sigma_g)] = \beta \in H_2(M, \mathbb{Z})$; $\overline{M}_{g,\beta}$ stable compactification of moduli space of map (or)

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Define a generating function "the free energy" of the topological A-model

$$F^{top}(\lambda, t) = \sum_{g=0}^{\infty}{}' \lambda^{2g-2} F^{(g)}(t)$$

$$F^{(g)} = \sum_{\beta \in H_2(M,\mathbb{Z})} r^g_\beta \; q^\beta$$

r^g_β Gromow-Witten invariant

$$r^g_\beta := \int_{\overline{M}_{g\beta}} c_{vir}(g,\beta) \in \mathbb{Q}$$

If M is a X vacuum manifold of a gauged linear σ-model without superpotential

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or equivalenty a non compact

toric Calabi-Yau

Γ_p^1 can calculated

by localisation w.r.t $(\mathbb{C}^*)^r$

action (Kontsevich, ...)

For GLd-model with superpotential

or equivalently compact CY X

defined as Hypersurface

or Complete intersections in compact

toric ambient spaces A localisation

fails for $g>1$, because there

is no easy restriction from

$\overline{\mu}_{g,\beta}(A)$ to $\overline{\mu}_{g,\beta}(X)$

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Large N transition techniques in particular the topological Vertex calculation rely likewise on toric localisation in A and are hard to generalise to compact CY case.

possible techniques:

- B-model holomorphic anomaly
- $\mathbb{I} \leftrightarrow$ het String duality
- direct calculation of GP × DT invariants

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Gopakumar & Vafa reorganisation of $F(\lambda, t)$

Type II A-model

Free energy

$$F(\lambda,t) = \sum'_{g,\beta} \lambda^{2g-2} n^1_\beta q^\beta$$

↑

Data of individual contribution g, β of $H_2(X)$

heterotic string

BPS saturated amplitudes $F(\lambda, t)$

Data of individual Contribution

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p: Q BPS charge $Q \in H_2(\mathcal{M}, \mathbb{Z})$

$\mathcal{J}$: j_L, j_R Quantum # of

$SU(2)_L \times SU(2)_R$ of little

4d Lorentz rep.

naively: $N^Q_{j_L, j_R} \sim r^p_q$

too detailed

$$\sum_{j_L j_R} (-1)^{2j_R} (2j_R+1) N^Q_{j_L j_R} = \sum r^p_q I^q$$

$$I^q = \left[(\tfrac{1}{2})_L + 2(0)_L \right]^{\otimes q}$$

combination which contributes to BPS saturated amplitude $\mathcal{F}(\lambda, t)$

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β: Q BPS charge $Q \in H_2(M,\mathbb{Z})$

g: j_L, j_R Quantum # of

$SU(2)_L \times SU(2)_R$ offshell

4d Lorentz rep.

naively: $\underbrace{N^Q_{j_L,j_R}}_{\text{too detailed}} \sim \Gamma^\beta_g$

$$\sum_{j_L j_R} (-1)^{2j_R}(2j_R+1)\, N^Q_{j_L j_R}[\tfrac{1}{2}] = \sum n^Q_g I^g \qquad \Gamma^\beta_g \sim$$

$$I^g = \left[\left(\tfrac{1}{2}\right)_L + 2(0)_L\right]^{\otimes g}$$

combination which contributes to BPS saturated amplitude $\mathcal{F}(\lambda,t)$

$F(\lambda, t)$ depends on spectrum of heterotic string only as an index like quantity:

$$I(\kappa, \beta) = \mathrm{Tr}_{\mathcal{H}} (-1)^{2j_L} e^{-\kappa j_L - \beta H}$$

Similar to elliptic genus.

Precise contribution of h^2_Q to $F(\lambda, t)$ given by Schwinger loop computation

$$F(\lambda, t) = \frac{c(t)}{\chi(t)} + \lambda(t) + \sum_{g=0}^{\infty} \sum_{Q \in H_2(M,\mathbb{Z})} \sum_{m=1}^{\infty} n_Q^g \frac{1}{m} \left(2 \sin \frac{m\lambda}{2}\right)^{2g-2} \cdot q^{Qm}$$

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remarkably: This leads to a product form for

$$Z = e^{F}$$

$$\boxed{Z = \prod_{Q\neq 0}\Big[\Big(\prod_{r=1}^{\infty}(1-q_\lambda^{r}\, q^{Q})^{r\, n_Q^{(0)}}\Big)\times \prod_{g=1}^{\infty}\prod_{\ell=0}^{2g-2}\big(1-q_\lambda^{g-\ell-1}\, q^{Q}\big)^{(-1)^{g+r}\binom{2g-2}{\ell} n_Q^{(g)}}\Big]}$$

(*)

$$q^{Q} = e^{i\sum_{i=1}^{h_{11}} t_i \int_{[Q]} J_i} \qquad q_\lambda = e^{i\lambda}$$

$$Q=0:\ \frac{1}{m}\frac{1}{(2\sin\frac{m\lambda}{2})^2} = \sum_{g=0}^{1} \lambda^{2g-2}(-1)^{g+1}\frac{B_{2g}}{2g(2g-2)!}\, m^{2g-2}$$

$\zeta(-n) = -\frac{B_{n+1}}{n+1}$ $\Rightarrow$ constant map contributions

$n_0^{(0)} = -\frac{\chi}{2}$ $\qquad \langle 1\rangle^{\mu}_{g,0} = (-1)^{g}\frac{\chi}{2}\int_{\mathcal{M}_g} c^{g}_{g-1} = (-1)\frac{\chi}{2}\frac{|B_{2g}\, B_{2g-2}|}{(2g-2)(2g-2)!}$

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(*) resembles indices in Hilbert spaces of free bosons $\prod_{n\geq 0} \frac{1}{(1-q^n)}$ fermion $\prod_{n\geq 0} (1-q^n)$ with $U(1)$ charge measured by q^λ

Form of (*) also remarkably close to the expansion expected for Donaldson & Thomas invariants:

$$0 \to I \to \mathcal{O}_M \to \mathcal{O}_Z \to 0$$

I ideal sheave

Z subscheme of M consisting

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$$q^Q = e^{i \int_{[\Sigma]} \omega_i} \qquad q_\lambda = e^{i\lambda}$$

$$\frac{1}{\lambda} \frac{1}{(2 \sin \frac{\lambda}{2})^2} = \sum_{g=0}^{\infty} \lambda^{2g-2} (-1)^{g+1} \frac{B_{2g}}{2g(2g-2)!}$$

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$$\zeta(-1) = -\frac{B_2}{2} \qquad \langle 1 \rangle_{g,0}^{X} = (-1)^g \chi \int_{\overline{\mathcal{M}}_g} c_{3g-3} = (-1)^g \chi \frac{|B_{2g} B_{2g-2}|}{2 (2g-2)(2g-2)!}$$

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of Curve C & k points

↑ Support of D2 brane

↑ Support of D0 branes

DT find a moduli space of

ideal sheave $I_m(\mathcal{M}, Q)$

↑ $[\Sigma] \in H_2$ Charge of D2 brane

↗ $\chi(\mathcal{O}_\Sigma)$ # of D0 branes

with a perfect obstruction theory.

$$N_Q^{(m)} := \int_{I_m(\mathcal{M},Q)} c_{vir}$$

DT invariants

General conjecture Maulik

Nekrasov, Okounkov, Pandharipande

arXiv: math.AG / 0312059

introduce generating function

$$Z_{DT}(M, q_\lambda, q) = \sum'_{Q,m\in\mathbb{Z}} \tilde{n}^{(m)}_Q \, q_\lambda^m \, q^Q$$

$$Z_{GV}(M, q_\lambda, q) \cdot M(q_\lambda)^{\frac{\chi(M)}{2}}$$

$$= Z_{DT}(M, -q_\lambda, q)$$

Because of $q_\lambda = e^{i\lambda}$

$$F_{GW} = \sum' \lambda^{2g-2} \sum'_Q r^g_Q \, q^Q$$

this conjecture cannot be checked genus by genus

Inspiration & proof (local case) comes from

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topological vertex Aganagic, AK, Mariño

Vafa arXiv: hep-th / 0305132

$$M(q,\lambda) = \prod_{n\geq 0} \frac{1}{(1-q^{n+\lambda})^n}$$ Mac-Mahon

function $\sim Z = \sum'_{\text{Sol random partion}} q_\lambda^{\#\text{boxes}}$

"no brane" vertex.

↑ non-compact case ↑

Compact case

- B-model
- Type II hat string duality

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Simplest example Yau & Zaslow

arXiv:hep-th 9512121

	Type II_A	Heterotic
16 Super symmetry charges	K3	T^4

BPS count on heterotic side captured by index:

$$\mathrm{Tr}_R F\, (-1)^F\, q^{L_0 - \frac{c}{24}}\, \bar{q}^{\bar{L}_0 - \frac{c}{24}} =$$

projection right groundstate

$\rightsquigarrow$ representation of $(\text{Heisenberg Alg})^{24}$

$$= q^{-1} \prod_{n \geq 1} \frac{1}{(1-q^n)^{24}}$$

restriction of partion function of het string

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$\sim$ representation of $(\text{Heisenberg } A_{1_2})^{24}$

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restriction of partition function of het string

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On K3 side:

BPS states calculated by

Cohomology of moduli space M_g

of D2 brane : (i) deformation

space of curve in class

$Q \in H_2(M,\mathbb{Z})$ (ii) deformation of

flat $U(1)$ bundle

$$Q^2 = 2g-2$$

without restriction of generality $Q = [B] + g[F]$ F elliptic fibre

(1) choice of g points in B

(2) choice of g vertical position in P^*

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$\Rightarrow$ Orbifold model of $\mu_q = (K3^n)^q / Sym_q$

$\parallel$

$Sym^q(K3^n)$

resolution: $\hat{\mu}_q = Hilb^q(K3^n)$

(1) Euler number $\chi(q)$ individual Betinumbers of the resolution space can be obtained by orbifold resolution formulas

Göttsche & Soergel: Math Ann. 286 (1990) 235

$$S_qV = \bigoplus_{n\geq 0} q^n \, Sym^n(V)$$

(1) $$\chi_{\substack{orb \\ res}}(S_qV) = \prod_k \frac{1}{(1-q^k)^{\chi(Y)}}$$

(2) $$P(q,y) = \sum'_{\substack{n\geq 0 \\ 0\leq k\leq d}} \left(1 - y^{k-dim_{\mathbb{C}}(Y)} q^n\right)^{-(-1)^k b_k}$$

$$= \sum'_{0\leq k\leq d} (-1)^k y^k \, b_k(Y)$$

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$$= \sum_{0\leq k\leq 2d} (-1)^k y^k\, b_k(Y)$$

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Eigenvalues of J_R^3, j_L^3 labeling $N^Q_{j_L^3, j_R^3}$ more precisely of the relevant sum over j_R^3

$$\sum_{g=0}^{\infty} n_Q^{(g)} I_g = \sum_{j_L^3 J_R^3} (-1)^{2j_R^3} (2j_R^3 - 1) \cdot N^Q_{j_L^3 J_R^3} [J_L]$$

can be maped $b_2(Y)$

Hosono, Saito, Takashi arXiv:hep-th/9901151

Katz, A.K, Vafa arXiv:hep-th/9910181

$$\boxed{\tilde{F}(\lambda, t) = \left(\frac{1}{2\sin\frac{\lambda}{2}}\right)^2 \prod_{n\geq 1} \frac{1}{(1-e^{i\lambda}q^n)^2 (1-q^n)^{20} \overline{(1-e^{-i\lambda}q^n)^2}}}$$

(Q, q) label $\mathcal{N}=2$ BPS states

8 supercharges $\longleftarrow$ 16 supercharges

SU(3) holonomy $\leftarrow$ However we just control Hilbert scheme of surfaces

SU(2) holonomy

2) Applications :

1) local CY $\mathcal{O}(K) \to B$

rational elliptic surface

good invariants

$N^{Q}_{j_L, j_S} \Leftarrow$ "rigid case"

HST 99

2) CY fibred by surface $\cong$ K3

"non rigid case" $N^{Q}_{j_L, j_S}$ not invariant under complex structure deformations but n^{Q}_{g} is!

K3 fibration case one finds

$$F_{K3}(\lambda,t) = \frac{\Theta(q)}{q}\left(\frac{1}{2\sin\frac{\lambda}{2}}\right)^2 \cdot \prod_{n\geq 1}\frac{1}{(1-e^{i\lambda}q^n)^2(1-q^n)^{20}(1-e^{-i\lambda}q^n)^2}$$

Interesting global information here is in $\Theta(q)$

1) How does $\Theta(q)$ depends on intersection form

2) How does $\Theta(q)$ depends on degenerate fibers of K3.

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Restrict to case with no reducible

Fibres : K3 fibrations

Pic (K3) = 1 $E^2 = 2$

	χ	E^2	$c_2 H$	H^3
$\mathbb{P}\begin{pmatrix}3&3&2&2&1&1&0\\3&1&0&0&1&1&2\end{pmatrix}[\frac{84}{6}]$	-84	2	22	1
$\mathbb{P}\begin{pmatrix}3&1&1&1&1&1&0\\1&0&0&0&1&1&1\end{pmatrix}[\frac{84}{6}]$	-140	2	24	2
$\mathbb{P}\begin{pmatrix}3&2&1&1&1&1&0\\3&2&0&0&1&1&1\end{pmatrix}[\frac{84}{6}]$	-196	2	$\frac{84}{42}$	3
$\mathbb{P}^5_{11226}[12]$	-252	2	52	4

10^8 new (2,2) vacua

constructed most

of them as hypersurfaces

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$$\chi = -28(1+2k)$$

$$\Theta(q) = \Theta^{(k)}(q) = 2^{-8} W E_4 E_6^{(k)}$$

$$E_6^{(k)}(q) = 4W^{12} - (76+7k)W^8X^4 - (180-6k)W^4X^8 - (4-k)X^{12}$$

$$W = \Theta_3\left(\frac{\tau}{2}\right)$$

$$X = \Theta_4\left(\frac{\tau}{2}\right)$$

generate modular forms of $\Gamma^0(4)$

Note that $\underline{F} = K3 \to M \to \mathbb{P}^1$ does not admit elliptic fibration for smooth M.

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physically the following is very interesting:

S-S duality

Vector moduli space of Type II

" " " heterotic

Kähler def space of K3 fibred C

finite R' non perturbative region $e^{-S} \sim 0$

non perturbative heterotic string

perturbative region

$S = \int_{R'} \omega + i \int_{R'} \Phi$

region of large R'

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F-S duality

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Kähler def space of K3 fibred CY

finite Σ' non perturbative region $e^{-S} \sim 0$

non perturbative heterotic string

↑ ↑ ↑

perturbative region $S = \int_{\Sigma'} \omega + i \int_{\Sigma'} B$

region of large Σ'

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We find series

		χ	H^3	E^2	Hc_2
A	$\mathbb{P}^4_{11226}[12]$	-252	4	2	
B	$\mathbb{P}\left(\begin{smallmatrix}4&2&1&1&1&0\\3&1&0&0&1&1\end{smallmatrix}\right)\left[\begin{smallmatrix}8&2\\6&1\end{smallmatrix}\right]$	-252	2	2	
C	$\mathbb{P}\left(\begin{smallmatrix}3&3&1&1&1&0&0\\0&0&0&2&2&1&1\end{smallmatrix}\right)\left[\begin{smallmatrix}6&3\\6&2\end{smallmatrix}\right]$	-252	0	2	

$\downarrow$ $-28(1+2W) = -252$

$\Theta(q) = \Theta^{(4)}(q)$ $\forall$ examples

$F_{K3}(\lambda,t) = \frac{\Theta(q)}{q}\,\tilde{F}(\lambda,t)$

fixes the complete perturbative BPS spectrum.

Yet the moduli space of A,B,C is globally different.

Topological subsector of non pert het. String does not fix compl.!!

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Relation to threshold corrections of heterotic string:

Because of the cancellation mechanism of gauge & gravitational anomaly in the heterotic string gauge & gravitational threshold corrections are related.

$$\Delta^{(1)}_{\text{Gauge}} = -i \int \frac{d^2\tau}{\mathrm{Im}\,\tau\,\eta^2} \mathrm{Tr}_{\mathcal{H}} F_L (-1)^{F_L} q^{H_L} \bar{q}^{-H_R}$$

level of ~~Kac~~ ~~moody~~ affine alg.

$$\left[Q^2 - \frac{k}{8\pi\tau_2} \right]$$

generator of gauge group

$$\Delta^{(1)}_{grav} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left[\frac{-i}{2^2(\tau)} Tr_R \left\{ F_L (-1)^{F_L} q^{H_L} q^{H_R} \cdot \underbrace{\left(E_2(\tau) - \frac{3}{\pi \tau_2} \right)}_{\hat{E}_2 \text{ modular but not holomorphic}} - b_{grav} \right] \right.$$

$$\Delta^{(1)}_{grav} R_+^2 , \quad \Delta^{(g)}_{grav} F_+^{2g-2} R_+^2 \; ?$$

Antoniadis, Gava, Narain, Taylor
arXiv:hep-th/9507115

found generalization of $\Delta^{(0)} \to \Delta^{(g)}$

$$\hat{E}_2 \to - \sum_{k=0}^{\infty}{}' \tilde{\lambda}^{2k} S_k \left(\hat{G}_2, \tfrac{1}{2} G_4, \ldots \right)$$

$$G_k = 2\zeta(k) E_k(\tau)$$

$$\frac{\vartheta_1'(z|\tau)}{\vartheta_1(z,\tau)} = -e^{\sum_{k=0}^{\infty} G_{2k} z^{2k}} = \left(\frac{2\pi i \tilde{\lambda} z^3}{\vartheta_1(\lambda,\tau)} \right) e^{-\frac{\pi \tilde{\lambda}^2}{\tau_2}}$$

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Eg. StU model Mariño & Moore arXiv hep-th/9808131

$$F(\lambda,T,u) = \frac{1}{2\pi^2}\int_{\mathcal{F}} \frac{d^2\tau}{y}\left(\frac{E_4E_6}{\eta^{24}}\right)$$

$$\sum_{\Gamma^{2,2}}{}' q^{\frac{1}{2}|p_L|^2}\,\bar{q}^{\frac{1}{2}|p_R|^2}$$

$$\cdot\left[\left(\frac{2\pi i\lambda\eta^3}{\theta_1(\tilde{\lambda},\tau)}\right)^2 e^{-\pi\frac{\tilde{\lambda}^2}{y}}\right]$$

Borcherds integral with a very easy holomorphic limit namely

(*) with $\Theta = \frac{E_4E_6}{\eta^{24}}$

$$\theta_1(z,\tau) = i\left(y^{-\frac{1}{2}} - y^{\frac{1}{2}}\right) q^{\frac{1}{8}} \prod_{n=1} (1-q^n)$$

$$(1-q^n y)(1-q^n y^{-1})$$

$\Theta^{(4)}(q) \to$ higher genus GP BPS numbers

g	d=1	2	3	4	5	6
0	2496	223752	×	×	×	×
1		−492	−1465964	×	×	×
2		−6	7488	×	×	×
3				−902328		
4				1164		
5				12	−4527744	
6					17472	
7						205118
8						−2124
9						−22
10						0

$$\sum \tilde{p}(n) q^n := \frac{1}{(1-q)(1-q^2)\cdots} \qquad \frac{}{(1-q)^6}$$

smooth curves:

$$n^Q_g = n^{2k,0}_{k^2+1} = (-1)^{\tilde{p}(2k)}\, 2\tilde{p}(2k)$$

1 nodal curves:

$$n^Q_g = n^{(2k,0)}_{k^2} = (-1)^{\tilde{p}(2k)} \left(4k^2 \tilde{p}(2k) + 252(1-\cdots\right.$$

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using model of moduli space of
BPS states of Katz, Klemm, Vafa
hep-th/9910181

using all genus result we
can calculate the

DT invariants: $\tilde{n}^{(w)}_{(r,1,0)}$ $X_{112812}[24]$

r=0	1	2
w		
-3		23
-2		2120
-1	-6	48552
0	-3840	*
1	480	-861918	*
2	114000	-861918	*
3	17857600	2384018304	✓

empirical
Some evidence for MNOP

Conjecture for compact case

$\tilde{n}^{(w)}_{(r,1,0)} \in \mathbb{Z}$

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using model of moduli space of

BPS states of Katz, A.K., Vafa

hep-th / 9910181

using all genus result we can calculate the

DT invariants: $\tilde{n}^{(m)}_{(r,1,0)}$ $X_{112812}[24]$

	T=0	1	2
m			
-3			23
-2		-6	2120
-1		-3240	4859252
0	1	-861918	×
1	480	-861918	×
2	114000	2384018304	✓
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B-model Calculation:

Holomorphic anomaly diff eq:

Bershadsky Cecotti, Ooguri, Vafa

arXiv: hep-th/9302103

$$\bar{\partial}_{\bar{k}} F^{(g)} = \frac{1}{2} \bar{C}^{ij}_{\bar{k}} \left(D_i D_j F^{(g-1)} + \sum_{r=1}^{g-1} D_i F^{(r)} D_j F^{(g-r)} \right)$$

$g > 1$

$$\partial_{\bar{k}} \partial_m F^{(1)} = \frac{1}{2} \bar{C}^{ij}_{\bar{k}} C_{mij} + \left(\frac{\chi}{24} - 1 \right) G_{\bar{k}m}$$

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Solution is constructed as follows

$$\bar{\partial}_{\bar{k}} \mathcal{F}^{(g)} = \bar{\partial}_{\bar{k}} \left[S, S^i, S^{ij}, C_{ijk} \right]$$

$$C_{ijk} = \int \Omega \, \partial_i \partial_j \partial_k \Omega$$

Construction of the propagator

S x- - - - - - x

S^i x- - - - —x

S^{ij} x———x

in multimoduli case involves

compatibility relations

$$\text{(A)} \quad (C_k^{-1})^{ie} \left(Q^i_{kl} + A^i_{kl} \right) = (C_\mu^{-1})^{ie} \left(Q^i_{\mu l} + A^i_{\mu l} \right) \qquad k \neq \mu$$

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(B) $(C^{-1}_{\kappa})^{ij}(Q_{\kappa i} + A_{\kappa j})$

$= (C^{-1}_{\mu})^{ij}(Q_{\mu j} + A_{\mu i})$ $\forall\ \kappa, \mu$

$Q^i_{\kappa\epsilon} = (\delta^i_\kappa \partial_\epsilon + \delta^i_\epsilon \partial_\kappa) K + \Gamma^i_{\kappa\epsilon}$

Weil Peterson Kähler potential — Connection

$Q_{\kappa\epsilon} = (\partial_\kappa K \partial_\epsilon K + \partial_\kappa \partial_\epsilon K + A^i_{\kappa\epsilon} \partial_i K)$

We could always solve (A) (B) with A^i_{jk} A_{ij} simple rational functions

Since $G_{\bar{i}j} = \partial_{\bar{i}} \partial_j t_p$

(C) $\Gamma^i_{\kappa j} = -\frac{\partial z^i}{\partial t_p} \frac{\partial}{\partial z_\kappa} \frac{\partial}{\partial z_j} t_p$

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$z_1(\underline{t})$ mirror map

$j(\tau)$ integral expasion

$z_i(\underline{t})$ empiral have integral expansion

A, B, C, are differential relation among $z_i(\underline{t}_0)$

Solution of B-model gives n_β^g for all classes β ⇝ see pap

Question for future work:

Can we solve one compact case completly?

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Best candidate from this works point of view.

$T2 \times K3 / \mathbb{Z}_2$

$\mathbb{Z}_2$ $z \to -z$ on T^2

$\mathbb{Z}_2$ Enriques involution on K3

Hol. group: $SU(2) \times \mathbb{Z}_2 \subset SU(3)$

heterotic dual is known

FHSV model orbifold

$\mathcal{F}^{(1)}$ stricingly beautiful

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$$\sum c(n) q^n = \frac{\vartheta(2\tau)^8}{\vartheta(\tau)^8 \vartheta(4\tau)^8}$$

$$= \frac{1}{q} + 8 + 36q + 128q^2 + \ldots$$

Since FHSV is an orbifold of T^6 heterotic string $\Rightarrow$ gravitation treshhold correction are computable.

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