Title: Higher Genus Amplitudes on Compact Calabi-Yau and Threshold Corrections

Date: Nov 20, 2004 05:00 PM

URL: http://pirsa.org/04110023

Abstract:

1)

Topological String Dualities:

String-String duality

	Type II		heterotic String
	open	Closed	closed
A-model	3d-Chern Simons theory knot invariants	Kähler gravity Gromov-Witten Gopakumar Vafa Invariants	Vector mult threshold corrections "elliptic genera"
B-model	holomorphic Chern Simons Donaldson Thomas Invariants Matrix modell	Kodaira-Spencer Gravity Family Indices Ray Singer Torsion	Hypermult no non-pert corr.

S-duality

Mirror duality

Large N-Duality

2

Symplectic Invariants

Gromov-Witten Inv. $\sim$ Gopakumar-Vafa Inv.

$\wr$ $\quad$ $\wr$

Donaldson-Thomas Inv.

Gromov-Witten: M symplectic manifold

$$X : \Sigma_g \longrightarrow M \qquad (*)$$

$[X(\Sigma_g)] = \beta \in H_2(M, \mathbb{Z})$; $\overline{M}_{g,\beta}$ stable compactification of moduli space of map $(*)$

Define a generating function "the free energy" of the topological A-model

$$\mathcal{F}^{top}(\lambda,t) = \sum_{g=0}^{\infty} \lambda^{2g-2}\, \mathcal{F}^{(g)}(t)$$

$$\mathcal{F}^{(g)} = \sum_{\beta\in H_2(M,\mathbb{Z})} r^{g}_{\beta}\; q^{\beta}$$

r^{g}_{β} Gromow-Witten Invariant

$$r^{g}_{\beta} := \int_{\overline{\mathcal{M}}_{g\beta}} c_{vir}(g,\beta) \in \mathbb{Q}$$

If M is a X vacuum manifold of a gauged linear σ-model without superpotential or equivalenty a non compact

toric Calabi-Yau,

r_β^g can calculated

by localisation w.r.t $(\mathbb{C}^*)^r$

action (Kontsevich, ...)

For GLσ-model with superpotential

or equivalently compact CY X

defined as Hypersurface

or Complete intersections in compact

toric ambient spaces A localisation

fails for $g>1$, because there

is no easy restriction from

$\overline{\mathcal{M}}_{g,\beta}(A)$ to $\overline{\mathcal{M}}_{g,\beta}(X)$

Large N transition techniques
in particular the topological
Vertex calculation rely likewise
on toric localisation in A
and are hard to generalise to
compact CY case.

possible techniques:

- B-model holomorphic anomaly
- II ↔ het String duality
- direct calculation of
  GP × DT invariants

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6

Gopakumar & Vafa reorganisation of $F(\lambda, t)$

Type II A-model

Free energy

$$F(\lambda,t) = \sum_{g,\beta}' \lambda^{2g-2}\, r^g_\beta\, q^\beta$$

Data of individual contribution $g, \beta \in H_2(X)$

heterotic string

BPS saturated amplitudes $\mathcal{F}(\lambda,t)$

Data of individual contribution

β: Q BPS charge $Q \in H_2(M, \mathbb{Z})$

g: j_L, j_R Quantum # of

$SU(2)_L \times SU(2)_R$ offshell

4d Lorentz rep.

naively: $\underbrace{N^{Q}_{j_L, j_R}}_{\text{too detailed}} \sim \Gamma^{\beta}_{g}$

$$\sum_{j_L\, j_R} (-1)^{2j_R} (2j_R+1)\, N^{Q}_{j_L\, j_R} = \sum n^{Q}_{g} I^{g} \quad \Gamma^{\beta}_{g}$$

$$I^{g} = \left[\left(\tfrac{1}{2}\right)_L + 2(0)_L \right]^{\otimes g}$$

Combination which contributes to BPS saturated amplitude $F(\lambda, t)$

7

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$r^{\beta}_{g} \sim$

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combination which contributes to BPS saturated amplitude $F(\lambda, t)$

8

$\mathcal{F}(\lambda,t)$ depends on spectrum of heterotic string only as an index like quantity:

$$I(\alpha,\beta) = \mathrm{Tr}_{\mathcal{H}} (-1)^{2j_L} e^{-\alpha j_L - \beta H}$$

Similar to elliptic genus.

Precise contribution of n^g_Q to $\mathcal{F}(\lambda,t)$ given by Schwinger loop computation

$$\mathcal{F}(\lambda,t) = \frac{c(t)}{\lambda^2} + \ell(t)$$

$$+ \sum_{g=0}^{\infty}{}' \sum_{Q \in H_2(M,\mathbb{Z})}{}' \sum_{m=1}^{\infty} n^g_Q \frac{1}{m} \left(\sin \frac{m\lambda}{2}\right)^{2g-2} \cdot q^{Qm}$$

9

remarkably: This leads to a product form for

$$Z = e^{F}$$

$$Z = \prod_{Q \neq 0} \Big[\Big(\prod_{r=1}^{\infty} (1 - q_\lambda^{r} q^{Q})^{r n_Q^{(0)}} \Big) \times \prod_{g=1}^{\infty} \prod_{\ell=0}^{2g-2} (1 - q_\lambda^{g-\ell-1} q^{Q})^{(-1)^{g+r} \binom{2g-2}{\ell} n_Q^{(g)}} \Big] \tag{3A}$$

$$q^{Q} = e^{i \sum_{i=1}^{b_{11}} t_i \int_{[Q]} J_i} \qquad q_\lambda = e^{i\lambda}$$

$Q=0$: $\frac{1}{n} \frac{1}{(2 \sin \frac{n\lambda}{2})^2} = \sum_{g=0}' \lambda^{2g-2} (-1)^{g+1} \frac{B_{2g}}{2g(2g-2)!} n^{2g-3}$

$\zeta(-n) = -\frac{B_{n+1}}{n+1} \Rightarrow$ constant map contribution

$n_0^{(0)} = -\frac{\chi}{2} \qquad \langle 1 \rangle_{g,0}^{M} = (-1)^{g} \frac{\chi}{2} \int_{M_g} c_{g-1}^{3} = (-1)^{g} \frac{\chi}{2} \frac{|B_{2g} B_{2g-2}|}{(2g-2)(2g-2)!}$

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10

(*) resembles indices in Hilbert spaces of free bosons $\prod_{n>0} \frac{1}{(1-q^n)}$

fermion $\prod_{n>0} (1-q^n)$ with

U(1) charge measured by $q\lambda$

Form of (*) also remarkably close to the expansion expected for Donaldson & Thomas Invariants:

$$0 \to I \to O_X \to O_Z \to 0$$

I ideal sheave

Z subscheme of X consisting

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of curve C & points

support of D2 brane

support of D0 branes

DT find a moduli space of

ideal sheave $I_n(X, Q)$

$\chi(\mathcal{O}_Z)$

# of D0 branes

$[Z] \in H_2$

Charge of D2 brane

with a perfect obstruction theory.

$$\tilde{N}^{(n)}_Q := \int_{I_n(X,Q)} c_{vir} \quad \text{DT invariants}$$

General conjecture Maulik

Nekrasov, Okounkov, Pandharipande

arXiv: math.AG/0312058

12

introduce generating function

$$Z_{DT}(M, q_\lambda, q) = \sum_{Q, m \in \mathbb{Z}}' \tilde{n}^{(m)}_Q \, q_\lambda^m \, q^Q$$

$$\boxed{Z_{GV}(M, q_\lambda, q) \cdot M(q_\lambda)^{\frac{\chi(M)}{2}} \doteq Z_{DT}(M, -q_\lambda, q)}$$

Because of $q_\lambda = e^{i\lambda}$

$$F_{GW} = \sum' \lambda^{2g-2} \sum_Q' r^g_Q \, q^Q$$

this conjecture cannot be checked genus by genus

inspiration & proof (local case) comes from

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inspiration & proof (local case) comes from topological vertex Aganagic, AK, Mariño

Vafa arXiv: hep-th/0305132

$$M(q_\lambda) = \prod_{n>0} \frac{1}{(1-q^n)^n} \quad \text{Mac-Mahon}$$

$$\text{function} \quad \sim Z = \sum_{\substack{\text{3d random} \\ \text{partition}}} q_\lambda^{\#\text{ boxes}}$$

"no brane" vertex.

↑ non-compact case ↑

Compact case

- B-model
- Type II het string duality

14

Simplest example   Yau & Zaslow
arXiv:hep-th 9512121

	Type II_A	Heterotic
16 Super symmetry charges	K3	T^4

BPS count on heterotic side captured by index:

$$\mathrm{Tr}_R\, F\, (-1)^F\, q^{L_0-\frac{c}{24}}\, \bar{q}^{\bar{L}_0-\frac{c}{24}} =$$

↑ projection right groundstate

~ representation of (Heisenberg Alg)^{24}

$$= q^{-1}\prod_{n\geq 1}\frac{1}{(1-q^n)^{24}}$$

restriction of partition function of het string

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$\sim$ representation of $(\text{Heisenberg Alg})^{24}$

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restriction of partition function of het string

On K3 side:

BPS states calculated by Cohomology of moduli space $\mathcal{M}_g$ of D2 brane: (1) deformation space of Curve in class $Q \in H_2(M, \mathbb{Z})$ x (2) deformation of flat $U(1)$ bundle

$$Q^2 = 2g-2$$

without restriction of generality $Q = [B] + g[F]$ F elliptic fibre

(1) choice of g points in B

(2) choice of g vertical position in F^*

16

$\Rightarrow$ Orbifold model of $\mathcal{M}_g = (K3^*)^g / \mathrm{Sym}_g$

$$\mathrm{Sym}^g(K3^*)$$

resolution: $\hat{\mathcal{M}}_g = \mathrm{Hilb}^g(K3^*)$

(1) Euler number & (2) individual Betti numbers of the resolution space can be obtained by orbifold resolution formulas

Göttsche & Soergel: Math. Ann. 296 (1993) 235

$$S_q V = \bigoplus_{n \geq 0} q^n \, \mathrm{Sym}^n(V)$$

(1) $$\chi_{\substack{orb\\res}}(S_q V) = \prod_k \frac{1}{(1-q^k)^{\chi(Y)}}$$

(2) $$P(S_q Y, y) = \sum_{\substack{n>0\\0 \leq k \leq d}} \left(1 - y^{k - \dim_{\mathbb{C}}(Y)} q^n\right)^{-(-1)^k b_k}$$

$$= \sum_{0 \leq k \leq d} (-1)^k y^k \, b_k(S_q Y)$$

16

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17

Eigenvalues of J_R^3, j_L^3 labeling $N^Q_{j_L^3, j_R^3}$ more precisely of the relevant sum over j_R^3

$$\sum_{g=0}^{\infty} n_Q^{(g)} I_g = \sum_{j_L^3, j_R^3}' (-1)^{2j_R^3} (2j_R^3 - 1) \cdot N^Q_{j_L^3 j_R^3} [J_L]$$

can be maped $b_Q(x)$

Hosono, Saito, Takachi arXiv:hep-th/9901151

Katz, A.K., Vafa arXiv:hep-th/9910181

$$\boxed{\tilde{F}(\lambda, t) = \left(\frac{1}{2 \sin\frac{\lambda}{2}}\right)^2 \prod_{n \geq 1} \frac{1}{(1-e^{i\lambda}q^n)^2 (1-q^n)^{20} (1-e^{-i\lambda}q^n)^2}}$$

(Q, q) label $\mathcal{N}=2$ BPS states

8 super charges $\longleftarrow$ 16 supercharges

SU(3) holonomy

SU(2) holonomy

However we just control Hilbert scheme of surfaces

2) Applications :

1) local CY $\mathcal{O}(K) \to B$

rational elliptic surface

HST 88

good invariants

$\mathcal{N}^{Q}_{j_L, j_R} \Leftarrow$ "rigid case"

2) CY fibred by Surface $\equiv$ K3

"non rigid case" $\mathcal{N}^{Q}_{j_L, j_R}$ not invariant under complex structure deformations but n^{Q}_{g} is!

K3 fibration case one finds

$$\boxed{\begin{aligned} F_{K3}(\lambda, t) &= \frac{\Theta(q)}{q}\left(\frac{1}{2\sin\frac{\lambda}{2}}\right)^2 \\ &\cdot \prod_{n\geq 1} \frac{1}{(1-e^{i\lambda}q^n)^2(1-q^n)^{20}(1-e^{-i\lambda}q^n)^2} \end{aligned}}$$

Interesting global information here is in $\Theta(q)$

1) How does $\Theta(q)$ depends on intersection form

2) How does $\Theta(q)$ depends on degenerate fibres of K3.

Restrict to case with no reducible

Fibres : K3 fibrations

$\mathrm{Pic}(K3) = 1$ $\qquad E^2 = 2$

	χ	E^2	$c_2 H$	H^3
$\mathbb{P}\begin{pmatrix} 3&3&2&2&1&1&0 \\ 3&1&0&0&1&1&2 \end{pmatrix}\left[\begin{array}{c\|c} 6&6 \\ 6&2 \end{array}\right]$	-84	2	22	1
$\mathbb{P}\begin{pmatrix} 3&1&1&1&1&1&0 \\ 1&0&0&0&1&1&1 \end{pmatrix}\left[\begin{array}{c\|c} 6&2 \\ 6&0 \end{array}\right]$	-140	2	24	2
$\mathbb{P}\begin{pmatrix} 3&2&1&1&1&1&0 \\ 3&2&0&0&1&1&1 \end{pmatrix}\left[\begin{array}{c\|c} 6&3 \\ 6&2 \end{array}\right]$	-196	2	~~36~~ 42	3
$\mathbb{P}^4_{11226}[12]$	-252	2	52	4

10^8 new $(2,2)$ vacua

constructed most

of them as hypersurfaces

$$\chi = -28(1+2k)$$

$$\Theta(q) = \Theta^{(k)}(q) = 2^{-8}\, W\, E_4\, E_6^{(k)}$$

$$E_6^{(k)}(q) = 4W^{12} - (76+7k)\, W^8 X^4 - (180-6k)\, W^4 X^8 - (4-k)\, X^{12}$$

$$W = \Theta_3\left(\tfrac{\tau}{2}\right)$$
$$X = \Theta_4\left(\tfrac{\tau}{2}\right)$$

generate modular forms of $\Gamma^0(4)$

Note that $\underline{F} = K3 \to \mathcal{M} \to \mathbb{P}^1$ does not admit elliptic fibration for smooth $\mathcal{M}$.

21

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physically the following is very interesting:

S-S duality

Vector moduli space of Type II

" " " heterotic

Kähler dof space of K3 fibred C

finite $\mathbb{P}^1$ non perturbative region

$e^{-S} \sim 0$

non perturbative heterotic string

↑ ↑ ↑

perturbative region

$S = \int_{\mathbb{P}^1} \omega + i \int_{\mathbb{P}^1} B$

region of large $\mathbb{P}^1$

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We find series

		χ	H^3	E^2	$H c_2$
A	$\mathbb{P}^4_{11226}[12]$	-252	4	2	
B	$\mathbb{P}\begin{pmatrix} 4\,2\,1\,1\,1 & 0 \\ 3\,1\,0\,0\,1 & 1 \end{pmatrix}\left[\begin{smallmatrix} 8 & 2 \\ 6 & 1 \end{smallmatrix}\right]$	-252	2	2	
C	$\mathbb{P}\begin{pmatrix} 3\,3 & 1\,1\,1\,0\,0 \\ 0\,0 & 0\,2\,2\,1\,1 \end{pmatrix}\left[\begin{smallmatrix} 6 & 3 \\ 6 & 3 \end{smallmatrix}\right]$	-252	0	2	

$\downarrow$ $-28(1+2k) = -252$

$$\Theta(q) = \Theta^{(4)}(q) \qquad \forall \text{ examples}$$

$$\mathcal{F}_{K3}(\lambda,t) = \frac{\Theta(q)}{q}\,\tilde{\mathcal{F}}(\lambda,t)$$

fixes the complete perturbative BPS spectrum.

Yet the moduli space of A,B,C is globally different.

Topological sub sector of het. String does not fix non pol compl.!!

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Relation to threshold corrections of heterotic string:

Because of the cancellation mechanism of gauge & gravitational anomaly in the heterotic string gauge & gravitational threshold corrections are related.

$$\Delta^{(1)}_{\text{gauge}} = -i \int \frac{d^2\tau}{\mathrm{Im}\,\tau\, \eta^2} \mathrm{Tr}_{\mathcal{R}}\, F_L (-1)^{F_L} q^{H_L} \bar{q}^{H_R} \left[Q^2 - \frac{k}{8\pi \tau_2} \right]$$

k: level of ~~Kac moody~~ affine alg.

Q: generator of gauge group

$$\Delta^{(1)}_{grav} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left[\frac{-i}{\eta^2(\tau)} \mathrm{Tr}_R \left\{ F_L (-1)^{F_L} q^{H_L} \bar{q}^{H_R} \cdot \right.\right.$$

$$\left.\left. \underbrace{\left(E_2(\tau) - \frac{3}{\pi \tau_2} \right)}_{\hat{E}_2 \text{ modular but not holomorphic}} - \mathrm{brav} \right]$$

$\Delta^{(1)}_{grav} R_+^2 \,, \quad \Delta^{(g)}_{grav} F_+^{2g-2} R_+^2$?

Antoniadis, Gava, Narain, Taylor
arXiv/hep-th/9507115

found generalisation of $\Delta^{(1)} \to \Delta^{(g)}$

$$\hat{E}_2 \to -\sum_{k=0}^{\infty} \tilde{\lambda}^{2k} S_k \left(\hat{G}_2, \tfrac{1}{2} G_4, \ldots \right)$$

$G_k = 2\zeta(k) E_k(\tau)$

$$\frac{\eta^3(\tau)\,\tilde{\lambda}}{\theta_1(\tilde{\lambda},\tau)} = -e^{\sum_{k=1}^{\infty} \frac{G_{2k}}{2k} \tilde{\lambda}^{2k}} = \left(\frac{2\pi i \tilde{\lambda} \eta^3}{\theta_1(\lambda,\tau)} \right) e^{-\frac{\pi \tilde{\lambda}^2}{\tau_2}}$$

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26

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27

Eg. STU model   Mariño & Moore arXiv hep-th/9808131

$$F(\lambda,T,U) = \frac{1}{2\pi^2}\int_{\mathcal{F}} \frac{d^2\tau}{y}\left(\frac{E_4 E_6}{\eta^{24}}\right) \sum'_{\Gamma^{2,2}} q^{\frac{1}{2}|p_L|^2}\bar{q}^{\frac{1}{2}|p_R|^2} \cdot \left[\left(\frac{2\pi i\lambda\eta^3}{\Theta_1(\tilde{\lambda},\tau)}\right)^2 e^{-\pi\frac{\tilde{\lambda}^2}{y}}\right]$$

Borcherds integral with a very easy holomorphic limit namely

(*)   with $\Theta = \frac{E_4 E_6}{\eta^{24}}$

$$\Theta_1(z,\tau) = i\,(y^{-\frac{1}{2}} - y^{\frac{1}{2}})\, q^{\frac{1}{8}} \prod_{n=1} (1-q^n)(1-q^n y)(1-q^n y^{-1})$$

$\Theta^{(4)}(q) \to$ higher genus GP BPS numbers

g	$d=1$	2	3	4	5	6
0	2496	223752	x	x	x	x
1		-492	-1465364	x	x	x
2		-6	7488	x	x	x
3				-902328		
4				1164		
5				12	-4527744	
6					17472	
7						20511
8						-2124
9						-22
10						0

$$\sum' \tilde{p}(n) q^n := \frac{1}{(1-q)(1-\ldots}$$

$$\overline{(1-q)^6}$$

smooth curves:

$$n^Q_g = n^{2k,0}_{k^2+1} = (-1)^{\tilde{p}(2k)} \, 2\tilde{p}(2k)$$

1 nodal curves:

$$n^Q_g = n^{(2k,0)}_{k^2} = (-1)^{\tilde{p}(2k)} \left(4k^2 \tilde{p}(2k) + 252(1-\ldots\right.$$

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$$\sum{}' \tilde{\beta}(n) q^n := \frac{1}{(1-q)(1-q^2)^2} \qquad \overline{(1-q)^6}$$

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using model of moduli space of
BPS states of Katz, Ak, Vafa
hep-th/9910181

using all genus result we can calculate the

DT invariants: $\tilde{n}^{(m)}_{(r,1,0)}$ $X_{112812}[24]$

m	$r=0$	1	2
-3			23
-2		-6	21120
-1		-3840	4853252
0	1	-861918	x
1	480	-861918	x
2	114000	2384018304	✓
3	17857600		

~~Some~~ empirical evidence for MNOP conjecture for compact case

$$\tilde{n}^{(m)}_{(r,1,0)} \in \mathbb{Z}$$

28

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$$\tilde{n}^{(m)}_{(r,1,0)} \in \mathbb{Z}$$

B-model calculation:

Holomorphic anomaly diff eq:

Bershadsky Cecotti, Ooguri, Vafa

arXiv: hep-th/9302123

$$\bar{\partial}_{\bar{k}} F^{(g)} = \frac{1}{2}\bar{C}_{\bar{k}}^{ij}\Big(D_i D_j F^{(g-1)} +$$

$$g \geq 1 \qquad \sum_{r=1}^{g-1} D_i F^{(r)} D_j F^{(g-r)}\Big)$$

$$\partial_{\bar{k}} \partial_m F^{(1)} = \frac{1}{2}\bar{C}_{\bar{k}}^{ij} C_{mij} + \left(\frac{\chi}{24} - 1\right) G_{\bar{k}m}$$

Solution is constructed as follows

$$\bar{\partial}_{\bar{k}} F^{(g)} = \bar{\partial}_{\bar{k}} \left[S, S^i, S^{ij}, C_{ijk} \right]$$

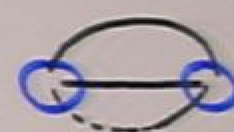
$$C_{ijk} = \int \Omega \, \partial_i \partial_j \partial_k \, \Omega$$

Construction of the propagator

S

S^i

S^{ij}



in multimoduli case involves

compatibility relations

$$\text{(A)} \quad (C_k^{-1})^{ie} \left(Q^i_{ke} + A^i_{ke} \right) = (C_\mu^{-1})^{ie} \left(Q^i_{\mu e} + A^i_{\mu e} \right) \qquad k \neq \mu$$

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(B) $(G_k^{-1})^{ij}(Q_{ki} + A_{kij})$

$$= (G_\mu^{-1})^{ij}(Q_{\mu j} + A_{\mu ij}) \quad \forall\, k \neq \mu$$

$$Q^i_{k\ell} = (\delta^i_k \partial_\ell + \delta^i_\ell \partial_k) K + \Gamma^i_{k\ell}$$

Weil Peterson Kähler potential

Connection

$$Q_{k\ell} = (\partial_k K \partial_\ell K + \partial_k \partial_\ell K + A^i_{k\ell} \partial_i K)$$

We could always solve (A) (B) with A^i_{jk} A_{ji} simple rational functions

Since $G_{\bar{p}j} = \delta_{\bar{p}p} \frac{\partial t_j}{\partial z_p}$

(C) $\Gamma^i_{kj} = -\frac{\partial z^i}{\partial t_p} \frac{\partial}{\partial z_k} \frac{\partial}{\partial z_j} t_p$

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$z_i(\underline{t})$ mirror map

$j(\tau)$ integral expasion

$z_i(\underline{t})$ empiral have integral expansion

A, B, C, are differential relation among $z_i(\underline{t}_0)$

Solution of B-model gives n^g_β for all classes β → see pap

Question for future work:

Can we solve one compact case completly?

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Best candidate from this worlds

point of view.

$T2 \times K3 / \mathbb{Z}_2$

$\mathbb{Z}_2 \quad z \to -z \quad$ on T^2

$\mathbb{Z}_2$ Enriques involution on K3

Hol. group: $SU(2) \times \mathbb{Z}_2 \subset SU(3)$

heterotic dual is known

FHSV model orbifold

$\mathcal{F}^{(1)}$ striking ly beautiful

$$\mathcal{F}^{(1)} = \log \left\| \frac{1}{\eta(2t^8)\,\Phi(y)} \right\|^2 \quad \frac{\cdot\, c\,(r,t)t_2}{(r,t,p^2-p)} \; c_{-1}$$

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$c(n)$

$$\sum' c(n) q^n = \frac{\eta(2\tau)^8}{\eta(\tau)^8 \, \eta(4\tau)^8}$$

$$= \frac{1}{q} + 8 + 36q + 128q^2 + \ldots$$

Since FHSV is an orbifold of T^6 heterotic string $\rightsquigarrow$ gravitation treshhold correction are computable.

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- Indication of decoupling of $S \leftrightarrow T_i$

$\Rightarrow$ work in progress.

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