Title: Arithmetic Varieties from String Theory and D-branes

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Abstract:

Arithmetic Varieties from String Theory & D-Branes

(Work in progress)

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References: RS, JGP **44** (2003)
RS & S. Underwood, JGP **48** (2003) 169
M. Lynker & RS, hep-th/0410189
RS, J. Tyson, to appear

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The basic picture of string theory

Naive picture:

The 1-dimensional string sweeps out a Riemann surface Σ_g of genus g – the string world sheet - which is embedded in 10-dimensional spacetime $M^{1,9}$

$$\begin{aligned} x : \Sigma_g &\longrightarrow M^{1,d} \\ (\tau,\sigma) &\longmapsto (x^0(\tau,\sigma), \ldots, x^d(\tau,\sigma)). \end{aligned}$$

d is a priori an undetermined number. Naive considerations would expect

$$d = 3. \qquad (!?)$$

(x has (super) symmetry partners carrying half intregral representations.)

Spacetime $M^{1,d}$ comes equipped with different fields:

- Metric g
- Connection A in a principal bundle with structure group $E_8 \times E_8$
- 2-Form B

Consistency constraints arise via

- supersymmetry (world sheet and spacetime)
- conformal invariance (world sheet)

Different solutions of consistency constraints:

- $d = 9$
- Calabi-Yau varieties (geometries)

$$M^{1,9} = M^{1,3} \times X,$$

with X a compact complex Kähler threefold with vanishing first Chern class.

Set

$$\mathcal{A} = \text{set of CYs} \quad \cup \quad \text{Special Fanofolds} \cup \cdots$$

- Conformal field theories/vertex algebras

$$M^{1,9} = M^{1,3} \times \text{CFT},$$

where CFT is a 2−dimensional field theory on the world sheet Riemann surface which has to satisfy certain constraints.

Set

$$\mathcal{B} = \text{set of CFTs}.$$

Obvious Questions:

Difficult:
Is it possible to find an underlying string CFT for all CYs, i.e. find a map

$$\tau : \mathcal{A} \longrightarrow \mathcal{B}.$$

(Has been achieved at cohomological level in many examples.)

Even more difficult:
Is it possible to find a geometric interpretation for all string theoretic CFTs:

$$\mathcal{C} : \mathcal{B} \longrightarrow \mathcal{A}.$$

(Has not been achieved in a single example in dimensions larger than one).

String theory induced recurring fantasy:

- Let $\mathcal{CY}$ be a hypothetical subcategory in the category of varieties (CYs + Special Fano).
- Let $\mathcal{CFT}$ be a subcategory in some hypothetical category of CFTs/VOAs
- Then there exists a string theoretic functor $\mathcal{S}$

$$\mathcal{S} : \mathcal{CY} \longrightarrow \mathcal{CFT}$$

String theory theme: modularity

Idea for contact between string world sheet and spacetime: **modularity**, one of the important themes in arithmetic geometry.

Hecke (28), Eichler(54), Taniyama(55,57)
Shimura(50s,60s,70s), Weil(67)
Langlands(60s,70s), Deligne(60s,70s), Serre(70s,80s)
Wiles(95), Breuil-Conrad-Diamond-Taylor(01)
Consani, Dieulefait, Hulek, Saito, Scholten, van Geemen, van Straten, Verrill, Yui and others

Observation/Idea:
Geometry leads to modular forms and vice versa.

Not good enough –
string theory has particular needs:
Calabi-Yau varieties must lead to particular types of modular forms coming from the string theoretic conformal field theory/affine Lie algebra.

Task: Find them.

Problem: It is not clear what the best framework is to make this question precise.

Here: Consider conformal field theories based on affine Lie algebras.

String theory and affine Lie algebras

– Want exactly solvable string theory models
(simplicity, rich structure, obvious symmetries, computability)
(no path integral, no action, no Lagrangian)

– Ingredients: affine Lie algebras/Virasoro algebra

– Physical version: OPEs, correlation functions

– Mathematical versions: VOAs, Operads

Needed in the present context: affine Lie algebra structures

Ingredients:

- Connected compact Lie group G
- Lie algebra $\mathfrak{g}$
- Loop algebra

$$\mathrm{L}\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$$

- Central extension: exact sequence

$$0 \longrightarrow \mathfrak{a} \longrightarrow \hat{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

with central $\mathfrak{a}$, i.e.

$$[a, x] = 0 \quad \forall a \in \mathfrak{a},\ x \in \hat{\mathfrak{g}}.$$

• Special case: 1-dimensional extension

$$\mathfrak{a} \cong \mathbb{C} \cdot \mathbf{1}$$

• Affine Lie algebra $\hat{\mathfrak{g}}$: central extension of the loop algebra

$$0 \longrightarrow \mathbb{C}K \longrightarrow \hat{\mathfrak{g}} \longrightarrow L\mathfrak{g} \longrightarrow 0$$

with central element K.

• Virasoro algebra Vir: central extension of the Lie algebra Der $\mathcal{K} = \mathbb{C}((t))\partial_t$ of derivatives

$$0 \longrightarrow \mathbb{C}C \longrightarrow \mathrm{Vir} \longrightarrow \mathrm{Der}\,\mathcal{K} \longrightarrow 0$$

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Example: $\hat{A}_1^{(1)}$

Lie algebra: A_1 (su(2), sl(2))

Cartan matrix: $A = (2)$

Generalized Cartan matrix: $\hat{A} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

Character maps on the upper half-plane

$$\chi_\ell : \mathfrak{H} \times C \longrightarrow \mathbf{C}$$

defined as

$$\begin{aligned} \chi_\ell^k(\tau, u) : &= \mathrm{tr}_{\mathcal{H}_\ell} q^{L_0 - \frac{c}{24}} e^{2\pi i u J_0} \\ &= \sum_{\substack{m=-k+1 \\ m=\ell \bmod 2}}^{k} c_{\ell,m}^k \theta_{m,k}(\tau, u), \end{aligned}$$

with string functions

$$c_{\ell,m}^k(\tau) = \frac{1}{\eta^3(\tau)} \sum_{\substack{-|x|<y\le|x| \\ (x,y) \text{ or } (\frac{1}{2}-x, \frac{1}{2}+y) \\ \in \mathbb{Z}^2 + \left(\frac{\ell+1}{2(k+2)}, \frac{m}{2k}\right)}} \mathrm{sign}(x) e^{2\pi i \tau((k+2)x^2 - ky^2)}$$

theta functions

$$\theta_{m,n}(\tau, z, u) = e^{-2\pi i u} \sum_{\ell \in \mathbb{Z} + \frac{m}{2n}} e^{2\pi i \tau n \ell^2 + 2\pi i z \ell}$$

and

$$c = \frac{3k}{k+2}.$$

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$$c = \frac{3k}{k+2}.$$

Closed under modular transformations:

$\tau \mapsto \tau + 1$:

$$\chi_\ell(\tau + 1, u) = e^{2\pi i(\Delta_\ell^k - \frac{c}{24})} \chi_\ell(\tau, u)$$

$\tau \mapsto -1/\tau$:

$$\chi_\ell\left(-\frac{1}{\tau}, \frac{u}{\tau}\right) = e^{\pi i k u^2/2} \sum_m S_{\ell m} \chi_m(\tau, u),$$

with modular S-matrix

$$S_{\ell m} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{(\ell+1)(m+1)\pi}{k+2}\right)$$

and

$$\Delta_\ell^k = \frac{\ell(\ell+2)}{4(k+2)}.$$

Remark.
The rational numbers given by the level k, the central charge c, and the anomalous dimensions Δ_ℓ, characterize the representations of the model (field content), and determine the structure of the correlation functions (Yukawa couplings).

Hence physically important, and arithmetically boring.

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More interesting parametrization:

(Motivated by 2-dimensional critical systems in 1980s and early 1990s)

Define the quantum dimensions:

$$Q_{\ell m} = \frac{S_{\ell m}}{S_{0m}}.$$

These are more interesting numbers number theoretically.

Even better, they know about the anomalous dimensions and the central charge.

Define the Rogers dilogarithm function as

$$L(z) = Li_2(z) + \frac{1}{2}\log(z)\log(1-z),$$

where

$$Li_2(z) = \sum_{n\in\mathbb{N}} \frac{z^n}{n^2}$$

is Euler's dilogarithm.

This function leads to the rather surprising

Theorem. (Nahm-Recknagel-Terhoeven, Kuniba-Nakanishi)

$$\frac{1}{L(1)}\sum_{i=1}^{k} L\left(\frac{1}{Q_{ij}^2}\right) = \frac{3k}{k+2} - 24\Delta_j^{(k)} + 6j.$$

Thus we have two equivalent ways of parametrizing the behavior of the fields at fixed central charge.

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Questions beyond the spectrum:

There are a great many detailed questions one can ask in the context of the CY/CFT connection and D-branes.

Perhaps the first step would be to understand the characters.

Problem:
Understand the geometric origin of CFT characters.

Observation:
Characters are modular forms.

Question:
Where do modular forms come from?

3. Modular forms:

Denote by $\Gamma_0(N) \subset \mathrm{SL}(2,\mathbb{Z})$ Hecke's congruence group

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c = 0(\mathrm{mod}\ N) \right\}.$$

Definition. *A function on the upper half plane $\mathfrak{H}$ which transforms under*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \subset \mathrm{SL}(2,\mathbb{Z})$$

as

$$f(g\tau) = (c\tau + d)^k f(\tau)$$

is called a modular form of weight k with respect to Γ.

Hecke construction of modular forms:

Consider L-series

$$L(s) = \sum_n \frac{a_n}{n^s}$$

which satisfy a functional equation

$$L(s) = wL(k-s)$$

for some k. Then their Mellin transform f defined via

$$L(f,s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^{i\infty} (-iz)^s f(z) \frac{dz}{z}$$

defines a modular form of weight k.

More precisely:

Converse Theorems:

In practice: one computes L-series via some interesting structure and wants to know whether they are modular.

Hecke's Converse Theorem.
Suppose $N = 1, 2, 3, 4$ *and* $L(s) = \sum_n a_n n^{-s}$ *converges in some right half plane and continues to an entire function such that* $\Gamma(s)L(s)$ *is entire and bounded in vertical strips. If*

$$\Lambda(s) := \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(s)$$

satisfies the functional equation

$$\Lambda(s) = \pm(-1)^{k/2}\Lambda(k-s)$$

then $f(q) = \sum_n a_n q^n$, $q = e^{2\pi i \tau}$, *is a modular form of weight* k *for* $\Gamma_0(N)$.

Generalizations:

- Weil 1967 for any N via twistings (unpleasant).
- Conrey/Farmer: $N \leq 15$ (via Euler product).

Q: Where do L-series come from?

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Q: Where do L-series come from?

L-functions appear in a great many different contexts:

Themes:

- Number theory (Euler, Riemann, Dirichlet, Dedekind, Hecke)
- Galois representations (Artin, Tate, Langlands)
- Algebraic varieties (Hasse-Weil L-functions)
- Modular forms

— Sometimes all these themes are connected:

(*) Abelian varieties with complex multiplication.

(*) Étale cohomology of varieties (schemes).

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Geometric Zeta-functions

Hasse (1930), Weil (1950s): Consider

$$Z(X/\mathbb{F}_p, t) \equiv exp\left(\sum_{r\in\mathbb{N}} \#(X/\mathbb{F}_{p^r})\frac{t^r}{r}\right)$$

with

• For p prime $\mathbb{F}_{p^r}$ = extension of

$$\mathbb{F}_p = \{0, 1, 2, ..., p-1\}$$

of degree r.

• t is a formal variable, and F.K.Schmidt (1930s) showed rationality:

$$Z(X/\mathbb{F}_p, t) = \frac{\mathcal{P}^{(p)}(t)}{(1-t)(1-pt^2)}$$

with

$$\mathcal{P}^{(p)}(t) = 1 - \beta(p)t + pt^2$$

and

$$\beta(p) = p + 1 - \#(X/\mathbb{F}_p).$$

Cohomological L-functions:

There still is an arbitrary choice of the prime p in $Z(X/\mathbb{F}_p, t)$. Physically this is unjustified and should be avoided (no 'scale').

Define the Hasse-Weil L-function of a Calabi-Yau d-fold as the cohomological L-function associated to $H^d(X)$.

$$\begin{aligned} L_{\rm HW}(X,s) &= \prod_{\substack{\mathbb{Z}\ni p \\ p \text{ good prime}}} \frac{1}{\mathcal{P}_d^{(p)}(p^{-s})} \\ &= \prod_{\substack{\mathbb{Z}\ni p \\ p \text{ good prime}}} \frac{1}{\prod_{j=1}^{b_d}\left(1-\beta_j^{(d)}(p)p^{-s}\right)} \\ &=: \sum_n \frac{a_X^d(n)}{n^s}, \end{aligned}$$

where the coefficients contain arithmetic/cohomological information about the variety.

Problem:
Find a string theoretic interpretation of the coefficients $a_X^d(n)$.

Idea: Look for stringy modular forms.
But: string theory only provides a selection of rather specific modular forms.
Not any old modular form will be useful.

Geometric Modularity

Is it possible to understand the modular symmetry of the string geometrically?

Simplest framework:
Consider the simplest possible string compactification from 10 to 8 dimensions (unphysical toy models)

$$M^{1,9} \longrightarrow M^{1,7} \times E$$

with elliptic curves E.

Examples:

The simplest class of models is provided by the class of elliptic Brieskorn-Pham curves for which physicists have conjectured a relation with exactly solvable conformal field theories.

They are described as polynomials in weighted projective space

$$\begin{aligned} E^3 &= \{(x:y:z) \in \mathbb{P}_2 \mid x^3+y^3+z^3=0\}, \\ E^4 &= \{(x:y:z) \in \mathbb{P}_{1,1,2} \mid x^4+y^4+z^2=0\}, \\ E^6 &= \{(x:y:z) \in \mathbb{P}_{1,2,3} \mid x^6+y^3+z^2=0\} \end{aligned}$$

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These lead to the Hasse-Weil series

$$L_{\rm HW}(E^3,s) = 1 - \frac{2}{4^s} - \frac{1}{7^s} + \frac{5}{13^s} + \frac{4}{16^s} - \frac{7}{19^s} + \cdots,$$
$$L_{\rm HW}(E^4,s) = 1 + \frac{2}{5^s} - \frac{3}{9^s} - \frac{6}{13^s} + \frac{2}{17^s} - \frac{1}{25^s} + \cdots,$$
$$L_{\rm HW}(E^6,s) = 1 + \frac{4}{7^s} + \frac{2}{13^s} - \frac{8}{19^s} - \frac{5}{25^s} + \frac{4}{31^s} + \cdots,$$

Associated to these are Hasse-Weil forms defined by the Mellin transforms

$$f_{\rm HW}(E^3,q) = q - 2q^4 - q^7 + 5q^{13} + 4q^{16} - 7q^{19} + \cdots$$
$$f_{\rm HW}(E^4,q) = q + 2q^5 - 3q^9 - 6q^{13} + 2q^{17} - q^{25} + \cdots$$
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These forms are normalized cusp Hecke eigenforms of weight 2 and levels 27, 64, 144, respectively.

Goal: Derive these modular forms from string theory.

Specifically:
We expect these elliptic curves to correspond to a affine Lie algebras constructed from two or three copies of the algebra $A_{1,k}^{(1)}$ at different levels k. Models:

$$C_n = \left(\bigotimes_{i=1}^{n_i} A_{1,k_i}^{(1)}\right)_{\rm GSO}$$

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These forms are normalized cusp Hecke eigenforms of weight 2 and levels 27, 64, 144, respectively.

Goal: Derive these modular forms from string theory.

Specifically:
We expect these elliptic curves to correspond to a affine Lie algebras constructed from two or three copies of the algebra $A_{1,k}^{(1)}$ at different levels k. Models:

$$C_n = \left(\bigotimes_{i=1}^{n_i} A_{1,k_i}^{(1)}\right)_{\rm GSO}.$$

Conformal field theoretic models (at $c = 3$):
(rough indication)

$$\begin{aligned} C_3 &= (A_{1,1}^{(1)})^{\otimes 3}_{\text{GSO}} \\ C_4 &= (A_{1,2}^{(1)})^{\otimes 2}_{\text{GSO}} \\ C_6 &= (A_{1,1}^{(1)} \otimes A_{1,4}^{(1)})_{\text{GSO}}, \end{aligned}$$

where GSO stands for a supersymmetric projection.

These models have a great many q–expansions in terms of various quantities:

Objects:	Level k	Number (rough)
String functions	1	1
	2	3
	4	7
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Remarks.

- It is not obvious from a physical perspective whether this is possible.

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• At $k=1$ none of the affine characters χ_ℓ of $A_1^{(1)}$ are useful.

• Neither are the $N=2$ characters $\chi^{\ell}_{q,s}$ or theta functions.

• Nor are the string functions c^{ℓ}_{m}, which (essentially) describe the characters of parafermionic theories.

• The weight of c^{ℓ}_{m} is $-\frac{1}{2}$, hence the weight of

$$\Theta^{\ell}_{m} = \eta^{3}(\tau) c^{\ell}_{m}$$

is unity.

• Can make weight 2 with products

$$\Theta^{\ell}_{m}(a\tau)\Theta^{\ell'}_{m'}(b\tau)$$

with integers a, b.

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$\underline{E_3}$:

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Compute and compare:

$$\begin{aligned} \Theta^1_{1,1}(q^3)\Theta^1_{1,1}(q^9) &= q - 2q^4 - q^7 + 5q^{13} + 4q^{16} - 7q^{19} + \cdots \\ f_{\rm HW}(E_3, q) &= q - 2q^4 - q^7 + 5q^{13} + 4q^{16} - 7q^{19} + \cdots \end{aligned}$$

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We had the Hasse-Weil modular form

$$f_{\rm HW}(E_4, q) = q + 2q^5 - 3q^9 - 6q^{13} + 2q^{17} - q^{25} + 10q^{29} + \cdots$$

The conductor of this curve is 64, hence we would expect the level to be 64, or perhaps a divisor of it.

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Needed: new rules.

Need some new gadget that changes sign in just the right way (and some rationale why it didn't appear for E_3!).

We want to twist the form with a character

$$\chi : \mathbb{Z} \longrightarrow \mathbb{Z}^\times$$

via

$$f_\chi(q) := \sum_n \chi(n) a_n q^n.$$

Class of candidates:

$$\chi_n(p) = \left(\frac{n}{p}\right) = \left\{ \begin{array}{ll} 1 & n \text{ is a square in } \mathbb{F}_p \\ -1 & n \text{ is not a square in } \mathbb{F}_p \end{array} \right\}.$$

What works is $n = 2$:

The Hasse-Weil L-function $L_{\rm HW}(E_4, s)$ of the quartic elliptic curve E_4 factors into the twisted product

$$f(E_4, q) = \Theta_1^1(4\tau)^2 \otimes \chi_2.$$

The twist character is $\chi_2(\cdot) = \left(\frac{2}{\cdot}\right)$ and $\Theta_1^1(\tau)$ is the affine $A_1^{(1)}$ theta function at conformal level $k = 2$.

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This result can be turned around: it provides an arithmetic interpretation of a string theoretic object (the parafermionic string functions).

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$\underline{E_6}$:

This case is more complicated for two reasons:
1) Different levels: $k = 1, 4$

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$$k = 1 \;:\; \Theta^1_{1,1}$$
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It is not obvious which ones to use, but.......

Simplifying trick:
Go into homogeneous coordinates. Then the only relevant level is $k = 1$.

Hence, consider forms of weight two of the form

$$\Theta^1_{1,1}(q^a)\Theta^1_{1,1}(q^b).$$

The conductor of the Hasse-Weil form is 144, hence the simplest case is

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Theorem. *The Mellin transforms of the Hasse-Weil L-functions of the elliptic curves E_i are modular forms that are determined by Hecke indefinite modular forms of weight one, given as*

$$\begin{aligned}
k=1: \quad f_{\rm HW}(E^3, q) &= \Theta^1_{1,1}(q^3)\Theta^1_{1,1}(q^9) &&\in S_2(\Gamma_0(27))\\
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Corollary. *The $SU(2)$ string function $c^1_{1,1}(\tau)$ at level $k=2$ is determined by the twisted Hasse-Weil modular form*

$$c_{1,1}^1(\tau) = \frac{1}{\eta^3(\tau)}\sqrt{f_{\rm HW}(E, q^{1/8})}.$$

Result: It is possible to provide a string theoretic interpretation of the Hasse-Weil modular forms for the class of Brieskorn-Pham elliptic curves in terms of modular forms derived from Kac-Moody algebras.

The link:

Arithmetic geometry $\longrightarrow$ Affine Lie algebras

provides a string ... between ... considerations ...

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Result: It is possible to provide a string theoretic interpretation of the Hasse-Weil modular forms for the class of Brieskorn-Pham elliptic curves in terms of modular forms derived from Kac-Moody characters.

The link

Arithmetic geometry $\longrightarrow$ Affine Lie algebras

provides a stronger link between the string and spacetime than the considerations of the past.

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Arithmetic geometry —— Affine Lie algebras

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Theorem. *The Mellin transforms of the Hasse-Weil L-functions of the elliptic curves E_i are modular forms that are determined by Hecke indefinite modular forms of weight one, given as*

Corollary. *The ... string functions ... are determined by the twisted Hasse-Weil modular form*

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Remaining questions:

1) What is the physical meaning meaning of the characters χ_i (if any)?

2) Why no field at $k = 1$?

Theorem. *The twist characters χ_n are the quadratic characters associated to the field of quantum dimensions of the underlying affine Lie algebra $A_1^{(1)}$.*

Proof. The field $K = \mathbb{Q}(\{Q_{\ell,m}\})$ of quantum dimensions

$$Q_{\ell,m} := \frac{S_{\ell m}}{S_{0m}}$$

arises from the

Here the $S_{\ell m}$ are elements of the matrix

$$S_{\ell m} = \sqrt{\frac{2}{k+2}}\ \sin\left(\frac{(\ell+1)(m+1)\pi}{k+2}\right)$$

introduced earlier which describes the modular transformation

$$\tau \mapsto -1/\tau$$

on the $A_{1,k}^{(1)}$ characters

$$\chi_\ell\left(-\frac{1}{\tau},\frac{u}{\tau}\right) = e^{\pi i k u^2/2}\sum_m S_{\ell m}\chi_m(\tau,u).$$

Computing these fields explicitly gives the result. □

Remark.

For $k = 1$ the field is $K = \mathbb{Q}$, which is why there is no twist character.

Interesting number fields abound:

1) The field of quantum dimensions $K = \mathbb{Q}(\{Q_{\ell m}\})$.

2) A priori different are the fields associated to the twist characters χ_n.

3) The elliptic curves of Brieskorn-Pham type admit complex multiplication, hence there is a CM field.

4) The Hecke indefinite modular forms $\Theta^k_{\ell m}$ are associated to number fields, hence there are fields K_Θ.

Explicitly:

Field	$k=1$	$k=2$	$k=4$
Quantum dimensions	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(\sqrt{3})$
Twist field	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(\sqrt{3})$
CM field	$\mathbb{Q}(\sqrt{-3})$	$\mathbb{Q}(\sqrt{-1})$	$\mathbb{Q}(\sqrt{-3})$
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String applications:

1) Brieskorn-Pham elliptic curves can be viewed as the simplest compactifications from a 10 dimensional spacetime to 8 dimensions. That's a little boring.

2) They also appear as important components of higher dimensional Calabi-Yau varieties, e.g. in elliptic K3 surfaces, such as

$$S_6 = \{(z_0 : z_1 : z_2 : z_3) \in \mathbb{P}_{(1,1,2,2)} \mid z_0^6 + z_1^6 + z_2^3 + z_3^3 = 0\}$$

as well as higher dimensional elliptic fibrations, e.g.

$$X_{12} = \{(z_0 : z_1 : \cdots : z_4) \in \mathbb{P}_{(1,1,2,4,4)} \mid z_0^{12} + z_1^{12} + z_2^6 + z_3^3 + z_3^4 = 0\}$$

3) They generate factors in the cohomology of higher dimensional varieties via the factorization of Jacobians of higher genus curves a la Faddeev, Gross, Rohrlich and others.

The simplest examples are the Fermat curves of degree 4 and 6.

This establishes the following direction (in this special class)

$$\text{Spacetime} \implies \text{String Theory}$$

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From strings to spacetime: elliptic case

Two steps:

1) Identify 'good' string theoretic modular forms.

2) Use ideas of Eichler-Shimura-Deligne-Serre..... to construct varieties.

Question:

What are 'good' string theoretic forms?
What are the criteria that single out the three string theoretic forms of theorem 3?

Input:

Level k	Hecke modular form Θ_m^ℓ
1	$\Theta_{1,1}^1(\tau) = q^{1/12}(1 - 2q - q^2 + 2q^3 + q^4 + 2q^5 + \cdots)$
2	$\Theta_{0,0}^2(\tau) = q^{1/16}(1 - 2q + q^3 + q^5 - 2q^6 + 2q^7 + \cdots)$
	$\Theta_{1,1}^2(\tau) = q^{1/8}(1 - q - 2q^2 + q^3 + 2q^5 + 2q^6 + \cdots)$
	$\Theta_{2,0}^2(\tau) = q^{9/16}(1 - 2q^2 + 2q^4 + 2q^5 + \cdots)$
4	$\Theta_{0,0}^4(\tau) = q^{1/24}(1 - 2q + 2q^3 - q^5 + q^7 + q^{11} + \cdots)$
	$\Theta_{0,2}^4(\tau) = q^{19/24}(1 - q - q^2 - q^4 + q^5 + q^6 + q^9 + \cdots)$
	$\Theta_{0,4}^4(\tau) = q^{25/24}(1 - q - q + q^5 + q^7 + q^8 + \cdots)$
	$\Theta_{1,1}^4(\tau) = q^{5/48}\,(1 - q - q^2 + q^3 - q^4 + q^5 + q^8 + \cdots)$
	$\Theta_{1,3}^4(\tau) = q^{29/48}(1 - 2q - q^3 + q^5 + q^6 + 2q^7 + \cdots$
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Hecke indefinite modular forms of weight 1 at conformal levels $k = 1, 2, 4$.

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Hecke indefinite modular forms of weight 1 at conformal levels

Constraint 1.

Constraint 2.
Consider only forms of weight 2 of the form:

Constraint 3.

Constraint 4.
Consider only normalized cusp forms.

Constraint 5.
Consider only Hecke [illegible] of the above type.

Constraint 6.
The level of the [illegible] form must be divisible by the prime factors of [illegible] conformal level.

Constraint 7.
Consider [illegible] characters associated to the fields of quantum dimension [illegible] by the affine factors $A^{(1)}_{1,k}$.

Constraint 1.
Assume building blocks that are cusp forms.

Choice: Hecke indefinite modular forms $\Theta^k_{\ell,m}$

Constraint 2.
Consider only forms of weight 2 of the form

$$\Theta^{k_1}_{\ell_1,m_1}(q^a)\Theta^{k_2}_{\ell_2,m_2}(q^b).$$

Constraint 3.
Consider only 'integral' forms.

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Consider only Hecke eigenforms of the above type.

Constraint 6.
The level of the modular form must be divisible by the prime factors of $k+2$, where k is the conformal level.

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Result: For all three exactly solvable string models there is a **unique** form satisfies all constraints.

Possibly relevant for higher dimensions:

Constraint 8.
Consider forms with complex multiplication.

Construction of spacetime from conformal field theory:

Eichler-Shimura Theory

• Suppose we have a cusp form $f \in S_2(\Gamma_0(N))$ with an expansion

$$f = \sum_n a_n q^n.$$

• Associated to this cusp form is a field

$$K = \mathbb{Q}(\{a_n\}).$$

• The ES construction leads to an abelian variety A_f of dimension

$$\dim_{\mathbb{C}} A_f = [K : \mathbb{Q}].$$

• f is the Mellin transform of the Hasse-Weil L-function $L_{\mathrm{HW}}(A_f, s)$.

• Arithmetically, and therefore string theoretically, the variety A_f carries the important information.

• More generally the idea is to think of the motives $H^1(A_f)$ as building blocks of string theoretically induced motives.

Affine Lie algebras and generalized Weierstrass curves

- The interpretation of the Hasse-Weil L-function as the Mellin transform of Θ−products is not restricted to the three elliptic Brieskorn-Pham curves.

- The starting point can be more general: consider Θ−quotients of

$$\Theta^1 = \Theta^1_{1,1}, \qquad \Theta^2 = \Theta^2_{1,1},$$

which are of weight 2 and Hecke eigenforms.

- Result:

Θ − Quotient	Conductor	Elliptic curve
$\Theta^1(\tau)\Theta^1(11\tau)$	11	$y^2 + y = x^3 - x^2$
$\Theta^2(\tau)\Theta^2(7\tau)$	14	$y^2 + xy + y = x^3 - x$
$\Theta^1(2\tau)\Theta^1(10\tau)$	20	$y^2 = x^3 + x^2 - x$
$\Theta^2(2\tau)\Theta^2(6\tau)$	24	$y^2 = x^3 + x^2 - x$
$\Theta^1(3\tau)\Theta^1(9\tau)$	27	$y^2 + y = x^3$
$\Theta^2(4\tau)^2$	32	$y^2 = x^3 - x$
$\Theta^1(6\tau)^2$	36	$y^2 = x^3 + 1$
$\frac{\Theta^1(8\tau)^4}{\Theta^1(4\tau)\Theta^1(16\tau)}$	64	$y^2 = x^3 + x$
$\frac{\Theta^1(4\tau)^2\Theta^1(20\tau)^2}{\Theta^1(2\tau)\Theta^1(8\tau)\Theta^1(10\tau)\Theta^1(40\tau)}$	80	$y^2 = x^3 - x^2 - x$
$\frac{\Theta^1(12\tau)^6}{\Theta^1(6\tau)^2\Theta^1(24\tau)^2}$	144	$y^2 = x^3 - 1$

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• The curves of conductor $N = 27, 64, 144$ in the list are the ones in the theorem:

$$\frac{\Theta^1(8\tau)}{\Theta^1(4\tau)\Theta^1(16\tau)} = \Theta^1(4\tau)^2 \otimes \chi_2$$
$$\frac{\Theta^1(12\tau)}{\Theta^1(6\tau)^2\Theta^1(24\tau)} = \Theta^1(6\tau)^2 \otimes \chi_3.$$

• The curves with $N = 27, 32, 36, 64, 144$ have complex multiplication and

$$\begin{aligned} f(E_{64}, q) &= f(E_{32}, q) \otimes \chi_2 \\ f(E_{144}, q) &= f(E_{36}, q) \otimes \chi_3 \end{aligned}$$

• Hence, modulo twists, the only elliptic curves in this list with a known string interpretation are the ones with complex multiplication, confirming our conjecture that exactly solvable Calabi-Yau varieties admit complex multiplication (in an appropriate sense).

Arithmetic aspects of D-branes

1) **Exactly solvable D-branes and number fields**

General boundary state is expanded in terms of Ishibashi states $|j\rangle\rangle_\Omega$ as

$$|\alpha\rangle_\Omega = \sum_{j\in E} \frac{\psi^j_\alpha}{\sqrt{S_{0j}}} |j\rangle\rangle_\Omega,$$

where

$$E = \{j \in \mathcal{T} \mid j = w(\bar{j}), N_{w(j)\bar{j}} \neq 0\}.$$

The coefficients ψ^j_α are determined by the Cardy condition

$$n^i_{\alpha\beta} = \sum_{j\in E} \frac{S_{ij}}{S_{0j}} \psi^j_\alpha (\psi^j_\beta)^*,$$

where

$$\mathcal{H}_{\alpha\beta} = \bigoplus n^i_{\alpha\beta} \mathcal{H}_i$$

with bulk decomposition

$$\mathcal{H} = \bigoplus_i \mathcal{H}_i.$$

Here the S_{ij} are elements of the matrix

$$S_{ij} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{(i+1)(j+1)\pi}{k+2}\right)$$

which describes the modular transformation $\tau \mapsto -1/\tau$ on the $A^{(1)}_{1,k}$ characters

$$\chi_i\left(-\frac{1}{\tau}, \frac{u}{\tau}\right) = e^{\pi i k u^2/2} \sum_j S_{ij} \chi_j(\tau, u).$$

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