Title: Certain non-rigid Calabi--Yau threefolds over ${f Q}$ and their modularity. (partial audio)

Date: Nov 19, 2004 03:30 PM

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Abstract: Please Note: There is no audio for most presentations from November 18th due to technical issues. We appologize to presenters and interested parties for this inconvenience.

Calabi–Yau Varieties

Definition: A smooth projective variety X of dimension d is called a *Calabi–Yau* variety if

(i) $H^i(X, \mathcal{O}_X) = 0$ for every $i, 0 < i < d$,

(ii) the canonical bundle $\mathcal{K}_X$ of X is trivial.

Introduce the Hodge numbers

$$h^{i,j}(X) := \dim H^j(X, \Omega^i_X).$$

Then X is a Calabi–Yau if

$$h^{i,0}(X) = 0 \quad \text{for every } i, 0 < i < d \text{ and } \quad h^{0,d}(X) = p_g(X) = 1.$$

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Examples

• If $d = 1$, a Calabi–Yau variety is nothing but an elliptic curve (if it is equipped with a rational point).

• If $d = 2$, the conditions $h^{1,0}(X) = 0$ and $p_g(X) = 1$ imply that dimension 2 Calabi–Yau varieties are $K3$ surfaces.

• If $d = 3$, the conditions $h^{1,0}(X) = h^{2,0}(X) = 0$ and $p_g(X) = 1$ characterize Calabi–Yau threefolds. Since Calabi–Yau threefolds are Kähler manifolds, $h^{1,1}(X) > 0$.

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- The k-th Betti number of X is

$$B_k(X) = \dim H^k(X, \mathbf{C}) = \dim H^k_{et}(\bar{X}, \mathbf{Q}_\ell)$$

Then $B_k(X) = 0$ for $k > 2d$, and the Poincaré duality implies that

$$B_k(X) = B_{2d-k}(X)$$

- There are symmetries of Hodge diamond:

$$h^{i,j}(X) = h^{j,i}(X); h^{p,q}(X) = h^{d-p,d-q}(X)$$

where the first identity follows from the operation of complex conjugation, and the latter from the Serre duality.

There is the Hodge decomposition

$$B_k(X) = \sum_{i+j=k} h^{i,j}(X)$$

Hodge diamond for elliptic curves

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$$

Hodge diamond for elliptic curves

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$$

$B_0 = 1$

$(-)B_1 = 2$

$B_2 = 1$

$E(X) = 0$

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Hodge diamond for $K3$ surfaces

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$

Hodge diamond for $K3$ surfaces

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$

$B_0 = 1$

$(-) B_1 = 0$

$B_2 = 22$

$(-) B_3 = 0$

$B_4 = 1$

$E(X) = 24$

Hodge diamond for $K3$ surfaces

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$

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Hodge diamond for Calabi–Yau 3-folds

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{1,2} & & 1 \\
 & 0 & & h^{2,2} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

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\begin{array}{ccccccc}
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 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

$h^{11} > 0$

$B_0 = 1$
$(-)\ B_1 = 0$
$B_2 = h^{1,1}$
$(-)\ B_3 = 2(1 + h^{2,1})$
$B_4 = h^{2,2}$
$(-)\ B_5 = 0$
$B_6 = 1$

$E(X) = 2(h^{1,1} - h^{2,1})$

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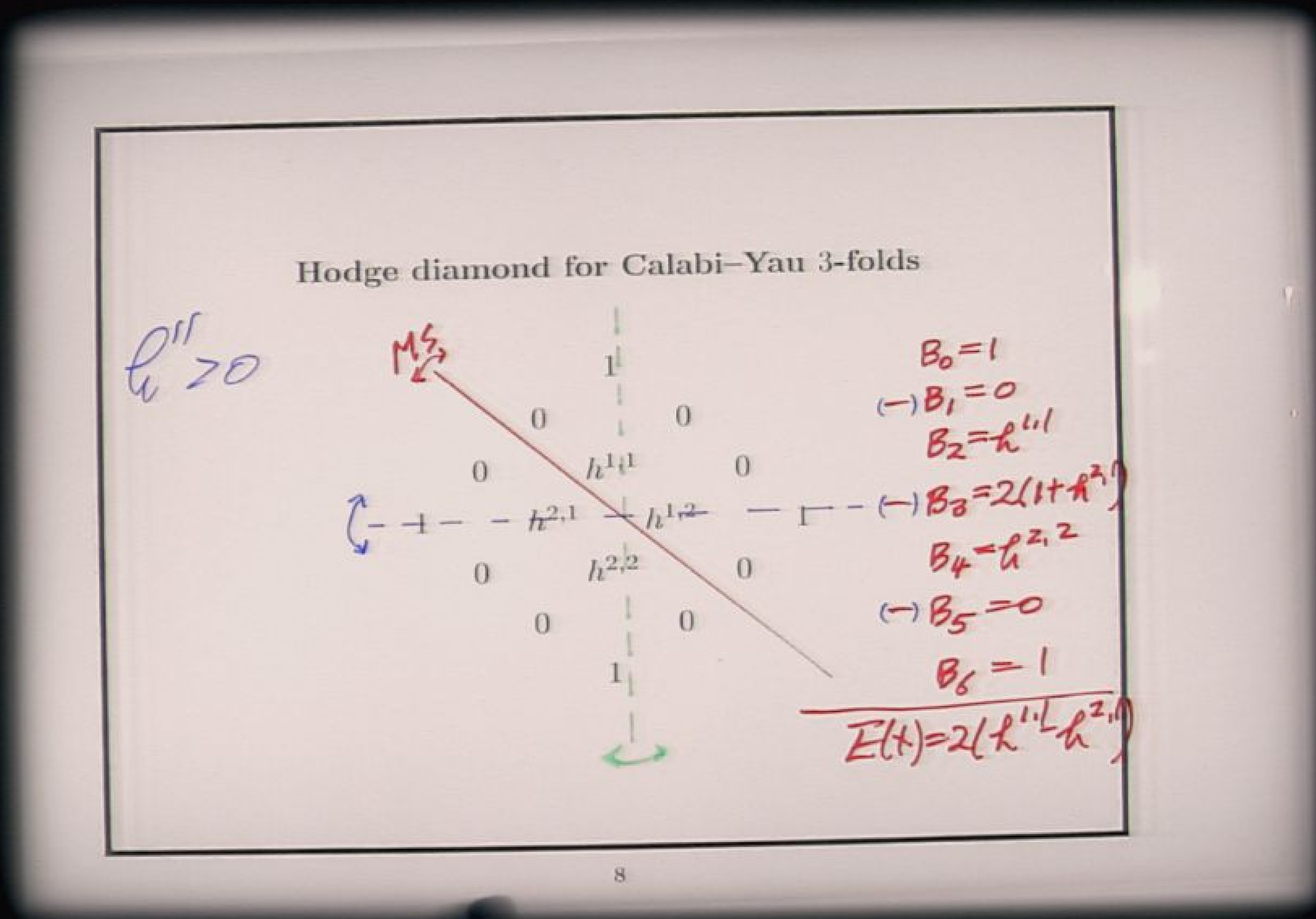
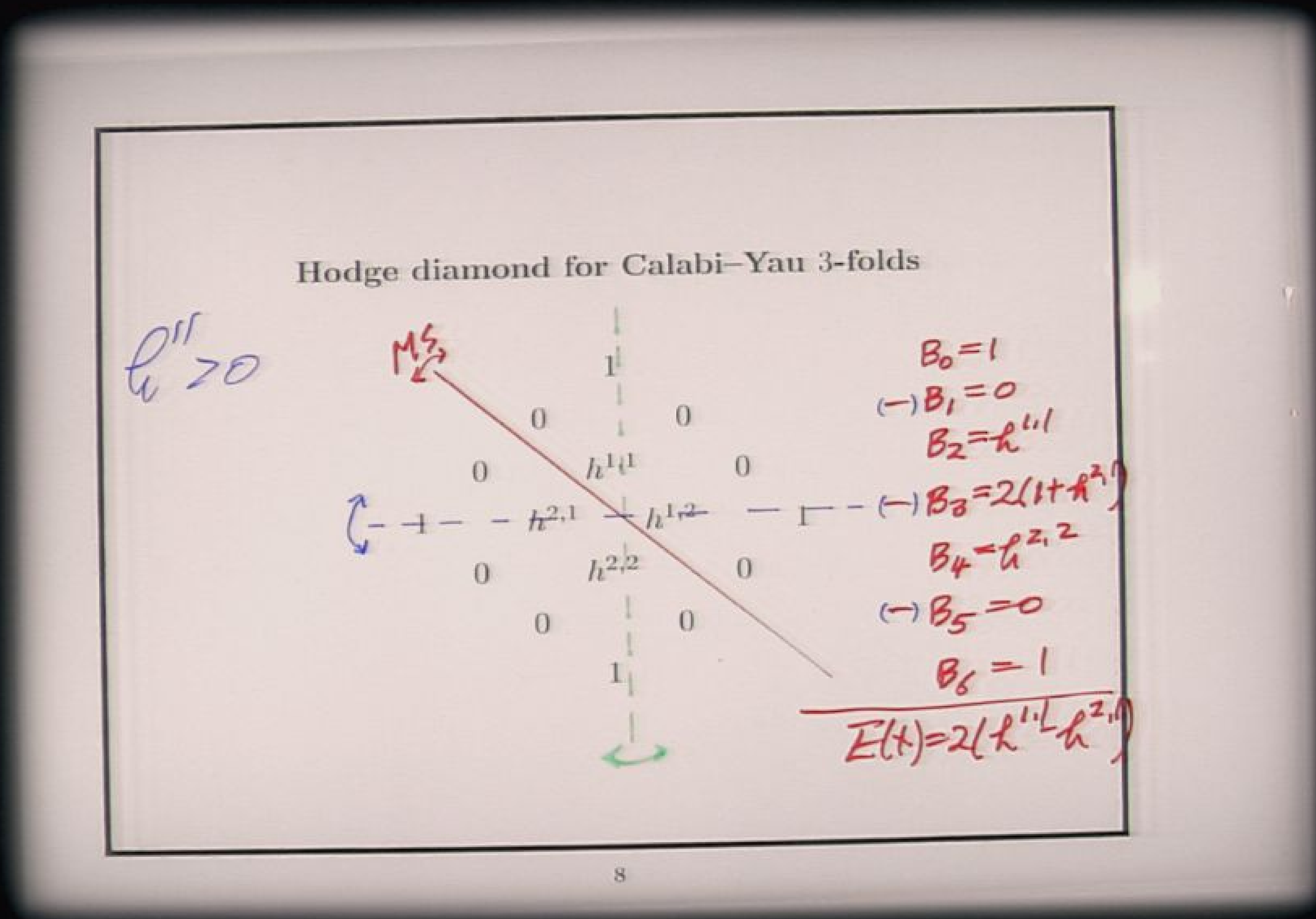

Hodge diamond for Calabi–Yau 3-folds
1
0 0
0 $h^{1,1}$ 0
1 $h^{2,1}$ $h^{1,2}$ 1
0 $h^{2,2}$ 0
0 0
1
8


Hodge diamond for Calabi–Yau 3-folds
1
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$E(t)=2(h^{1,1}-h^{2,1})$
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The L-series and zeta-function

Let X be a Calabi–Yau variety of dimension d defined over $\mathbf{Q}$. Let p be a prime, and assume that the reduction of $X_p := X \bmod p$ is smooth over $\bar{\mathbf{F}}_p$. Such a prime p is called *good*. For a good prime p, let Frob_p denote the Frobenius morphism on X induced from the map $x \to x^p$. Let ℓ be a prime $\neq p$. Then Frob_p acts on the ℓ-adic étale cohomology groups $H^i_{et}(\bar{X}_p, \mathbf{Q}_\ell) \simeq H^i_{et}(\bar{X}, \mathbf{Q}_\ell)$ for $i = 0, 1, \cdots, 2d$. Let

$$P^i_p(T) := \det(1 - \mathrm{Frob}^*_p T \,|\, H^i_{et}(\bar{X}, \mathbf{Q}_\ell))$$

be the characteristic polynomial of the endomorphism Frob^*_p on $H^i_{et}(\bar{X}, \mathbf{Q}_\ell)$. Then $P^i_p(T) \in 1 + T\mathbf{Z}[T]$ with $\deg P^i_p(T) = B_i$, and its reciprocal roots $\alpha_{i,j}$, $j = 1, \cdots, B_i$ are algebraic integers with absolute value $p^{i/2}$ (Deligne).

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Let $\mathcal{G} = \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ denote the absolute Galois group. Then there is an ℓ-adic Galois representation

$$\rho^i_{X,\ell} : \mathcal{G} \to GL(H^i_{et}(\bar{X}, \mathbf{Q}_\ell)).$$

In particular, using $\rho^i_{X,\ell}(\mathrm{Frob}_p)$ we define the i-th L-series of X.

Definition: The i-th cohomological L-series of X is defined by the Euler product

$$L_i(X,s) := L(H^i_{et}(\bar{X}, \mathbf{Q}_\ell), s) := \prod_{p \neq \ell} \det(1 - \rho^i_{X,\ell}(\mathrm{Frob}_p)p^{-s})^{-1}$$

$$\times (\text{similar factor at } \ell)$$

where the product runs over good primes.

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Digressing, we have

$$L_i(X,s) = \prod_{p \neq \ell} P_p^i(p^{-s})^{-1} \times (\text{similar factor at } \ell)$$

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In particular, when i =dimension of X, the d-th L-series $L_d(X,s) = L(H^d_{et}(\bar{X}, \mathbf{Q}_\ell), s)$ is simply called the L-series of X, and denoted by $L(X,s)$, and the local p-factor is denoted by $P_p(T)$.

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Definition: The full zeta-function of X is defined by taking the alternative product of the partial L-series:

$$\zeta(X,s) = \prod_{i=0}^{2d} L_i(X,s)^{(-1)^i}$$

In general we expect $\zeta(X,s)$ to

- have meromorphic continuation to $\mathbf{C}$,
- satisfy a functional equation relating the value at s to the value at $1+d-s$, and
- encode important arithmetic information about X.

The modularity of elliptic curves over $\mathbf{Q}$

Theorem ($d = 1$)(Wiles, et al.): *Every elliptic curve E defined over $\mathbf{Q}$ is modular. That is, there exists a cusp newform f of weight $2 = 1 + 1$ on $\Gamma_0(N)$ (N=the conductor of E) such that*

$$L(E, s) = L(f, s).$$

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$$L(E, s) = L(f, s).$$

- The L-series $L(E,s)$ of E is:

$$L(E,s) = \prod_p \frac{1}{1 - a(p)p^{-s} + \varepsilon(p)p^{1-2s}} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

where p runs over all rational primes, and

$$a(p) = \begin{cases} p+1-\#E(\mathbf{F}_p),\ \varepsilon(p)=1 & \text{if } p \not| \Delta \\ 0, \pm 1,\ \varepsilon(p)=0 & \text{if } p|\Delta \end{cases}$$

Here Δ is the discriminant of E.

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• A cusp form f of weight 2 on $\Gamma_0(N)$ is a modular form vanishing at all cusps of $\Gamma_0(N)$. Write $f(q) = \sum_{n=1}^{\infty} a_f(n) q^n$ with $q = e^{2\pi i z}$. The L-series of $f(q)$ is:

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Theorem of Wiles et al. says that

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Singular (extremal) K3 surfaces over Q

Let X be an algebraic K3 surface. Let $NS(X)$ be the Néron-Severi group of X generated by algebraic cycles on X. Then $NS(X)$ is a free finitely generated abelian group ($\subseteq H^2(X,\mathbf{Z}) \cap H^{1,1}(X,\mathbf{R})$) of $\mathbf{Z}$-rank at most 20. Let $T(X) := NS(X)^{\perp}_{H^2(X,\mathbf{Z})}$ be the group of transcendental cycles on X. The $\mathbf{Z}$-rank of $NS(X)$ is called the Picard number of X, and denoted by $\rho(X)$.

Definition: A K3 surface X over $\mathbf{Q}$is called *singular* (or *extremal*) if $\rho(X_{\bar{\mathbf{Q}}}) = \text{rank}_{\mathbf{Z}} NS(X_{\bar{\mathbf{Q}}}) = 20$.

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The modularity of singular (extremal) K3 surfaces

$H^2(X, \mathbf{Z}) \otimes \mathbf{Q} = NS(X)_{\mathbf{Q}} \oplus T(X)_{\mathbf{Q}}$, we have the decomposition the L-series of X as

$$L(X, s) = L(NS(X) \otimes \mathbf{Q}_\ell, s)\, L(T(X) \otimes \mathbf{Q}_\ell, s).$$

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Theorem $(d = 2)$(Livné) : *Every singular (extremal) K3 surface X defined over $\mathbf{Q}$ is modular. That is, $L(T(X) \otimes \mathbf{Q}_\ell, s)$ comes [illegible]m a modular form.*

More precisely, assuming that all 20 algebraic cycles generating $NS(X)$ are defined over $\mathbf{Q}$. Then

$$L(NS(X) \otimes \mathbf{Q}_\ell, s) = \zeta(s-1)^{20}$$

where $\zeta(s)$ is the Riemann zeta-function, and

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where g is a cusp form of weight $3 = 2 + 1$ on some $\Gamma_1(N)$ or $\Gamma_0(N)$ with a character.

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A crude classification of Calabi-Yau threefolds

Definition: A Calabi-Yau threefold X is called *rigid* if $H^{2,1}(X) = H^{1,2}(X) = 0$. Otherwise, X is *non-rigid.*

A rigid Calabi–Yau threefold X has $B_3 = 2$, and $E(X) = 2h^{1,1}(X)$.

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Rigid CY 3-folds

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\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & 0 & & 0 & & 1 \\
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 & & & 1 & & &
\end{array}
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$h^{1,1} > 0$

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The modularity conjecture for rigid Calabi–Yau threefolds over $\mathbb{Q}$

Let X be a rigid Calabi-Yau threefold defined over $\mathbb{Q}$. Then X is modular. That is, there exists a cusp form f of weight $4 = \underset{d}{\underset{\|}{3}} + 1$ on some $\Gamma_0(N)$ such that

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Theorem: *Up to date, there are "about" 50 rigid Calabi-Yau threefolds over* $\mathbf{Q}$ *for which the modularity is established.*

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Method 2: Wiles' method

In Wiles' proof of the Shimura–Taniyama conjecture for elliptic curves, prime 3 played very crucial role (backed up by prime 5), i.e, in the proof of the modularity of the 2-dimensional Galois representation $GL_2(\mathbf{Z}_3)$.

For rigid Calabi–Yau threefolds over $\mathbf{Q}$, a similar criterion has been established.

Theorem (Dieulefait and Manoharmayum, You-Chiang Yi):

Let X be a rigid Calabi–Yau threefold over $\mathbf{Q}$. Suppose that X satisfies one of the following two conditions:

(1) X has good reduction at 3 and 7, or

(2) X has good reduction at 5 and some prime $p \equiv \pm 2 \pmod 5$ with $t_3(p)$ not divisible by 5.

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Furthermore, there is an algorithm to determine the level of weight 4 cusp newform associated to a rigid Calabi–Yau threefold. (Dieulefait)

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The modularity of non-rigid Calabi–Yau threefolds over $\mathbb{Q}$

Definition: A Calabi–Yau threefold X is said to be *non-rigid* if $h^{2,1} \neq 0$ (so that $B_3(X) \geq 4$).

For a non-rigid Calabi–Yau threefold defined over $\mathbb{Q}$, there associates the Galois representation of dimension $B_3 \geq 4$.

Goal: Establish the modularity of the L-series of non-rigid Calabi–Yau threefolds over $\mathbf{Q}$.

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Methods for establishing the modularity

There are several methods to establish the modularity of non-rigid Calabi–Yau threefolds over $\mathbb{Q}$.

- When the B_3-dimensional Galois representation splits into a sum of several 2-dimensional pieces, apply the Serre–Faltings criterion, or Wiles' method (used for the modularity of rigid Calabi–Yau threefolds) to these rank 2 motives.
- Construct modular non-rigid Calabi–Yau threefolds (e.g., Hilbert modular varieties, Siegel modular varieties, fiber-products of modular varieties). Construct a birational transformation defined over $\mathbb{Q}$ to one of these modular non-rigid Calabi–Yau threefolds.
- Algebraic correspondence (Tate's conjecture).

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Examples of Livné and Yui

Idea: Let E be an elliptic curve, and let Y be a singular (extremal) K3 surface, both defined over $\mathbb{Q}$. Then we know that both E and Y are modular, that is, there exist cusp forms of weight 2 and 3 associated to E and Y, respectively, on some congruence subgroups of $PSL_2(\mathbb{Z})$.

Consider the product $Y \times E$. Then the Hodge numbers of $Y \times E$ are computed by Künneth formula:

$$h^{0,3}(Y \times E) = 1,\ h^{1,0}(Y \times E) = 1 = h^{2,0}(Y \times E)$$

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Consequently $Y \times E$ is not a Calabi–Yau threefold.

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However, with **an appropriate choice** of Y, there is the action of ± 1 on $Y \times E$. Blowing up the locus of points of order 2, let X be a smooth resolution of $Y \times E / \pm 1$. Let $N_{\pm}$ be the motive of algebraic cycles on Y invariant (resp. anti-invariant) under the action of ± 1 and let $n_{\pm} = \mathrm{rank} N_{\pm}$ (so $n_{+} + n_{-} = 20$). Then X is a non-rigid Calabi–Yau threefold with

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$$h^{3,0}(X) = 1,\ h^{1,0}(X) = h^{2,0}(X) = 0,$$

$$h^{1,1}(X) = 1 + n_+,\ h^{2,1}(X) = 1 + n_-$$

and

$$B_3(X) = 4 + 2n_-.$$

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Construction of Modular Singular (Extremal) $K3$ Surfaces
Group Theoretic Approach

Let $\bar{\Gamma} \subset PSL_2(\mathbb{Z})$ be a genus zero torsion-free congruence subgroup of index $\mu = [PSL_2(\mathbb{Z}) : \bar{\Gamma}] < \infty$. The natural morphism from the compact Riemann surface $(\mathfrak{H}/\bar{\Gamma})^* \to (\mathfrak{H}/PSL_2(\mathbb{Z}))^*$ is ramified only at cusps, and we call the ramification index the *cusp width*. Let n_i $(1 \leq i \leq t)$ be the cusp widths of t inequivalent cusps. Then $\mu = \sum_{i=1}^{t} n_i$. The Riemann–Hurwitz formula gives

$$g = 1 + \frac{1}{2}(\frac{\mu}{6} - t)$$

In our case, $g = 0$, so this formula is read as

$$\mu = 6(t - 2)$$

Table lists all genus zero torsion-free congruence subgroups (Sebbar)

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A. Sebbar

33 groups

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Shioda

Elliptic Modular Surfaces

Let $\bar{\Gamma} \subset PSL_2(\mathbb{Z})$ be a genus zero, torsion-free congruence subgroup of index $\mu < \infty$. Then $\bar{\Gamma}$ can be lifted to a congruence subgroup $\Gamma \subset SL_2(\mathbb{Z})$ of finite index with the property that Γ has no elliptic elements (in particular $-Id \notin \Gamma$) and Γ contains no elements of trace -2.

Let $X_\Gamma = (\mathfrak{H}/\Gamma)^*$ be the corresponding modular curve. Consider the automorphism of $\mathfrak{H} \times \mathbb{C}$ defined by

$$(\tau, z) \mapsto (\frac{a\tau + b}{c\tau + d}, \frac{z + m\tau + n}{c\tau + d})$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $\tau \in \mathfrak{H}$, $z \in \mathbb{C}$ and $(m, n) \in \mathbb{Z}^2$.

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Elliptic $K3$ Surfaces

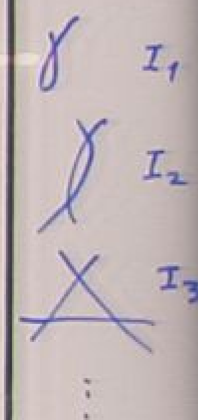
Now assume that X is an elliptic $K3$ surface with the elliptic fibration $\pi : X \to C$.

Hypothesis: The elliptic fibration $\pi : X \to \mathbb{P}^1$ has only $I_n, n > 0$ type fibers (*semi-stable*).

Under this hypothesis, $C \simeq \mathbb{P}^1$ and the J-map $C \to \mathbb{P}^1$ has degree at most 24, indeed, equal to 24. This is equivalent to $\sum_{i=1}^{t} n_i = 24$.

Miranda–Persson Classification: (1989)

Miranda and Persson studied all possible configurations of I_n, $n > 0$ fibers on elliptic K3 surfaces. In the case of exactly 6 singular fibers, they obtained 112 possible configurations. All these K3 surfaces have the maximal possible Picard number 20, and hence these are singular (extremal). Here is the list.



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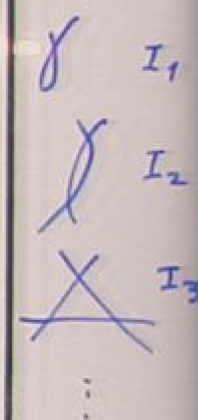

Miranda and Persson studied all possible configurations of I_n, $n > 0$ fibers on elliptic K3 surfaces. In the case of exactly 6 singular fibers, they obtained 112 possible configurations. All these K3 surfaces have the maximal possible Picard number 20, and hence these are singular (extremal). Here is the list.

Elliptic $K3$ Surfaces

Now assume that X is an elliptic $K3$ surface with the elliptic fibration $\pi : X \to C$.

Hypothesis: The elliptic fibration $\pi : X \to \mathbb{P}^1$ has only $I_n, n > 0$ type fibers (*semi-stable*).

Under this hypothesis, $C \simeq \mathbb{P}^1$ and the J-map $C \to \mathbb{P}^1$ has degree at most 24, indeed, equal to 24. This is equivalent to $\sum_{i=1}^{t} n_i = 24$.

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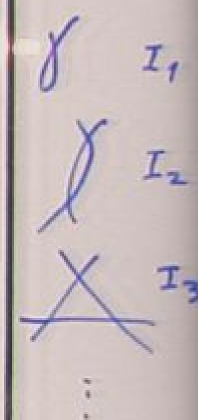

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Miranda-Persson

[1,1,1,1,1,19]	[1,1,1,1,2,18]	[1,1,1,1,3,17]	[1,1,1,1,4,16]
[1,1,1,1,5,15]	[1,1,1,1,6,14]	[1,1,1,1,7,13]	[1,1,1,1,9,11]
[1,1,1,1,10,10]	[1,1,1,2,2,17]	[1,1,1,2,3,16]	[1,1,1,2,4,15]
[1,1,1,2,5,14]	[1,1,1,2,6,13]	[1,1,1,2,7,12]	[1,1,1,2,8,11]
[1,1,1,2,9,10]	[1,1,1,3,3,15]	[1,1,1,3,4,14]	[1,1,1,3,5,13]
[1,1,1,3,6,12]	[1,1,1,3,7,11]	[1,1,1,3,8,10]	[1,1,1,4,6,11]
[1,1,1,4,7,10]	[1,1,1,5,5,11]	[1,1,1,5,6,10]	[1,1,1,5,7,9]
[1,1,1,6,7,8]	[1,1,1,7,7,7]	[1,1,2,2,2,16]	[1,1,2,2,3,15]
[1,1,2,2,4,14]	[1,1,2,2,5,13]	[1,1,2,2,6,12]	[1,1,2,2,7,11]
[1,1,2,2,9,9]	[1,1,2,3,3,14]	[1,1,2,3,4,13]	[1,1,2,3,5,12]
[1,1,2,3,6,11]	[1,1,2,3,7,10]	[1,1,2,3,8,9]	[1,1,2,4,4,12]
[1,1,2,4,5,11]	[1,1,2,4,6,10]	[1,1,2,4,7,9]	[1,1,2,4,8,8]
[1,1,2,5,5,10]	[1,1,2,5,6,9]	[1,1,2,5,7,8]	[1,1,2,6,6,8]
[1,1,3,3,4,12]	[1,1,3,3,5,11]	[1,1,3,3,8,8]	[1,1,3,4,4,11]
[1,1,3,4,6,9]	[1,1,3,4,7,8]	[1,1,3,5,6,8]	[1,1,3,5,7,7]
[1,1,4,4,7,7]	[1,1,4,5,6,7]	[1,1,4,6,6,6]	[1,1,5,5,6,6]
[1,2,2,2,3,14]	[1,2,2,2,5,12]	[1,2,2,2,7,10]	[1,2,2,3,3,13]
[1,2,2,3,4,12]	[1,2,2,3,5,11]	[1,2,2,3,6,10]	[1,2,2,3,7,9]
[1,2,2,4,5,10]	[1,2,2,4,7,8]	[1,2,2,5,5,9]	[1,2,2,5,6,8]
[1,2,2,6,6,7]	[1,2,3,3,3,12]	[1,2,3,3,4,11]	[1,2,3,3,6,9]
[1,2,3,3,7,8]	[1,2,3,4,4,10]	[1,2,3,4,5,9]	[1,2,3,4,6,8]
[1,2,3,5,6,7]	[1,2,4,4,6,7]	[1,2,4,5,5,7]	[1,2,4,5,6,6]
[1,3,3,3,5,9]	[1,3,3,4,5,8]	[1,3,3,5,6,6]	[1,3,4,4,4,8]
[1,3,4,4,5,7]	[2,2,2,2,8,8]	[2,2,2,3,3,12]	[2,2,2,3,5,10]
[2,2,2,4,6,8]	[2,2,2,6,6,6]	[2,2,3,3,4,10]	[2,2,3,3,7,7]
[2,2,3,4,5,8]	[2,2,3,5,5,7]	[2,2,4,4,4,8]	[2,2,4,4,6,6]
[2,2,5,5,5,5]	[2,3,3,3,4,9]	[2,3,3,4,5,7]	[2,3,3,4,6,6]
[2,3,4,4,5,6]	[3,3,3,3,6,6]	[3,3,4,4,5,5]	[4,4,4,4,4,4]

If $MW \supseteq (\mathbb{Z}/2\mathbb{Z})^2$, then the corresponding elliptic K3 surface is unique!

Miranda-Persson

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[1,1,1,1,5,15]	[1,1,1,1,6,14]	[1,1,1,1,7,13]	[1,1,1,1,9,11]
[1,1,1,1,10,10]	[1,1,1,2,2,17]	[1,1,1,2,3,16]	[1,1,1,2,4,15]
[1,1,1,2,5,14]	[1,1,1,2,6,13]	[1,1,1,2,7,12]	[1,1,1,2,8,11]
[1,1,1,2,9,10]	[1,1,1,3,3,15]	[1,1,1,3,4,14]	[1,1,1,3,5,13]
[1,1,1,3,6,12]	[1,1,1,3,7,11]	[1,1,1,3,8,10]	[1,1,1,4,6,11]
[1,1,1,4,7,10]	[1,1,1,5,5,11]	[1,1,1,5,6,10]	[1,1,1,5,7,9]
[1,1,1,6,7,8]	[1,1,1,7,7,7]	[1,1,2,2,2,16]	[1,1,2,2,3,15]
[1,1,2,2,4,14]	[1,1,2,2,5,13]	[1,1,2,2,6,12]	[1,1,2,2,7,11]
[1,1,2,2,9,9]	[1,1,2,3,3,14]	[1,1,2,3,4,13]	[1,1,2,3,5,12]
[1,1,2,3,6,11]	[1,1,2,3,7,10]	[1,1,2,3,8,9]	[1,1,2,4,4,12]
[1,1,2,4,5,11]	[1,1,2,4,6,10]	[1,1,2,4,7,9]	[1,1,2,4,8,8]
[1,1,2,5,5,10]	[1,1,2,5,6,9]	[1,1,2,5,7,8]	[1,1,2,6,6,8]
[1,1,3,3,4,12]	[1,1,3,3,5,11]	[1,1,3,3,8,8]	[1,1,3,4,4,11]
[1,1,3,4,6,9]	[1,1,3,4,7,8]	[1,1,3,5,6,8]	[1,1,3,5,7,7]
[1,1,4,4,7,7]	[1,1,4,5,6,7]	[1,1,4,6,6,6]	[1,1,5,5,6,6]
[1,2,2,2,3,14]	[1,2,2,2,5,12]	[1,2,2,2,7,10]	[1,2,2,3,3,13]
[1,2,2,3,4,12]	[1,2,2,3,5,11]	[1,2,2,3,6,10]	[1,2,2,3,7,9]
[1,2,2,4,5,10]	[1,2,2,4,7,8]	[1,2,2,5,5,9]	[1,2,2,5,6,8]
[1,2,2,6,6,7]	[1,2,3,3,3,12]	[1,2,3,3,4,11]	[1,2,3,3,6,9]
[1,2,3,3,7,8]	[1,2,3,4,4,10]	[1,2,3,4,5,9]	[1,2,3,4,6,8]
[1,2,3,5,6,7]	[1,2,4,4,6,7]	[1,2,4,5,5,7]	[1,2,4,5,6,6]
[1,3,3,3,5,9]	[1,3,3,4,5,8]	[1,3,3,5,6,6]	[1,3,4,4,4,8]
[1,3,4,4,5,7]	[2,2,2,2,8,8]	[2,2,2,3,3,12]	[2,2,2,3,5,10]
[2,2,2,4,6,8]	[2,2,2,6,6,6]	[2,2,3,3,4,10]	[2,2,3,3,7,7]
[2,2,3,4,5,8]	[2,2,3,5,5,7]	[2,2,4,4,4,8]	[2,2,4,4,6,6]
[2,2,5,5,5,5]	[2,3,3,3,4,9]	[2,3,3,4,5,7]	[2,3,3,4,6,6]
[2,3,4,4,5,6]	[3,3,3,3,6,6]	[3,3,4,4,5,5]	[4,4,4,4,4,4]

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Miranda-Persson

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[1,1,1,1,5,15]	[1,1,1,1,6,14]	[1,1,1,1,7,13]	[1,1,1,1,9,11]
[1,1,1,1,10,10]	[1,1,1,2,2,17]	[1,1,1,2,3,16]	[1,1,1,2,4,15]
[1,1,1,2,5,14]	[1,1,1,2,6,13]	[1,1,1,2,7,12]	[1,1,1,2,8,11]
[1,1,1,2,9,10]	[1,1,1,3,3,15]	[1,1,1,3,4,14]	[1,1,1,3,5,13]
[1,1,1,3,6,12]	[1,1,1,3,7,11]	[1,1,1,3,8,10]	[1,1,1,4,6,11]
[1,1,1,4,7,10]	[1,1,1,5,5,11]	[1,1,1,5,6,10]	[1,1,1,5,7,9]
[1,1,1,6,7,8]	[1,1,1,7,7,7]	[1,1,2,2,2,16]	[1,1,2,2,3,15]
[1,1,2,2,4,14]	[1,1,2,2,5,13]	[1,1,2,2,6,12]	[1,1,2,2,7,11]
[1,1,2,2,9,9]	[1,1,2,3,3,14]	[1,1,2,3,4,13]	[1,1,2,3,5,12]
[1,1,2,3,6,11]	[1,1,2,3,7,10]	[1,1,2,3,8,9]	[1,1,2,4,4,12]
[1,1,2,4,5,11]	[1,1,2,4,6,10]	[1,1,2,4,7,9]	[1,1,2,4,8,8]
[1,1,2,5,5,10]	[1,1,2,5,6,9]	[1,1,2,5,7,8]	[1,1,2,6,6,8]
[1,1,3,3,4,12]	[1,1,3,3,5,11]	[1,1,3,3,8,8]	[1,1,3,4,4,11]
[1,1,3,4,6,9]	[1,1,3,4,7,8]	[1,1,3,5,6,8]	[1,1,3,5,7,7]
[1,1,4,4,7,7]	[1,1,4,5,6,7]	[1,1,4,6,6,6]	[1,1,5,5,6,6]
[1,2,2,2,3,14]	[1,2,2,2,5,12]	[1,2,2,2,7,10]	[1,2,2,3,3,13]
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[1,2,2,4,5,10]	[1,2,2,4,7,8]	[1,2,2,5,5,9]	[1,2,2,5,6,8]
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[1,2,3,3,7,8]	[1,2,3,4,4,10]	[1,2,3,4,5,9]	[1,2,3,4,6,8]
[1,2,3,5,6,7]	[1,2,4,4,6,7]	[1,2,4,5,5,7]	[1,2,4,5,6,6]
[1,3,3,3,5,9]	[1,3,3,4,5,8]	[1,3,3,5,6,6]	[1,3,4,4,4,8]
[1,3,4,4,5,7]	[2,2,2,2,8,8]	[2,2,2,3,3,12]	[2,2,2,3,5,10]
[2,2,2,4,6,8]	[2,2,2,6,6,6]	[2,2,3,3,4,10]	[2,2,3,3,7,7]
[2,2,3,4,5,8]	[2,2,3,5,5,7]	[2,2,4,4,4,8]	[2,2,4,4,6,6]
[2,2,5,5,5,5]	[2,3,3,3,4,9]	[2,3,3,4,5,7]	[2,3,3,4,6,6]
[2,3,4,4,5,6]	[3,3,3,3,6,6]	[3,3,4,4,5,5]	[4,4,4,4,4,4]

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[1,1,1,1,5,15]	[1,1,1,1,6,14]	[1,1,1,1,7,13]	[1,1,1,1,9,11]
[1,1,1,1,10,10]	[1,1,1,2,2,17]	[1,1,1,2,3,16]	[1,1,1,2,4,15]
[1,1,1,2,5,14]	[1,1,1,2,6,13]	[1,1,1,2,7,12]	[1,1,1,2,8,11]
[1,1,1,2,9,10]	[1,1,1,3,3,15]	[1,1,1,3,4,14]	[1,1,1,3,5,13]
[1,1,1,3,6,12]	[1,1,1,3,7,11]	[1,1,1,3,8,10]	[1,1,1,4,6,11]
[1,1,1,4,7,10]	[1,1,1,5,5,11]	[1,1,1,5,6,10]	[1,1,1,5,7,9]
[1,1,1,6,7,8]	[1,1,1,7,7,7]	[1,1,2,2,2,16]	[1,1,2,2,3,15]
[1,1,2,2,4,14]	[1,1,2,2,5,13]	[1,1,2,2,6,12]	[1,1,2,2,7,11]
[1,1,2,2,9,9]	[1,1,2,3,3,14]	[1,1,2,3,4,13]	[1,1,2,3,5,12]
[1,1,2,3,6,11]	[1,1,2,3,7,10]	[1,1,2,3,8,9]	[1,1,2,4,4,12]
[1,1,2,4,5,11]	[1,1,2,4,6,10]	[1,1,2,4,7,9]	[1,1,2,4,8,8]
[1,1,2,5,5,10]	[1,1,2,5,6,9]	[1,1,2,5,7,8]	[1,1,2,6,6,8]
[1,1,3,3,4,12]	[1,1,3,3,5,11]	[1,1,3,3,8,8]	[1,1,3,4,4,11]
[1,1,3,4,6,9]	[1,1,3,4,7,8]	[1,1,3,5,6,8]	[1,1,3,5,7,7]
[1,1,4,4,7,7]	[1,1,4,5,6,7]	[1,1,4,6,6,6]	[1,1,5,5,6,6]
[1,2,2,2,3,14]	[1,2,2,2,5,12]	[1,2,2,2,7,10]	[1,2,2,3,3,13]
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[2,2,5,5,5,5]	[2,3,3,3,4,9]	[2,3,3,4,5,7]	[2,3,3,4,6,6]
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$PSL_2(\mathbb{R})$-conj. classes

Extremal congruence K3 surfaces

[1, 5, 8]

[2, 7]

[3]

[4, 6, 9]

#	The group Γ	Widths of the cusps	Cusp form of weight 3
1	$\Gamma(4)$	4, 4, 4, 4, 4, 4	$\eta(q)^6$
2	$\Gamma_0(3) \cap \Gamma(2)$	6, 6, 6, 2, 2, 2	$\eta(q)^3\eta(q^3)^3$
3	$\Gamma_1(7)$	7, 7, 7, 1, 1, 1	$\eta(q)^3\eta(q^7)^3$
4	$\Gamma_1(8)$	8, 8, 4, 2, 1, 1	$\eta(q)^2\eta(q^2)\eta(q^4)\eta(q^8)^2$
5	$\Gamma_0(8) \cap \Gamma(2)$	8, 8, 2, 2, 2, 2	$\eta(q^2)^6$
6	$\Gamma_1(8; 4, 1, 2)$	8, 4, 4, 4, 2, 2	?
7	$\Gamma_0(12)$	12, 4, 3, 3, 1, 1	$\eta(q^2)\eta(q^6)^3$
8	$\Gamma_0(16)$	16, 4, 1, 1, 1, 1	$\eta(q^4)^6$
9	$\Gamma_1(16; 16, 2, 2)$	16, 2, 2, 2, 1, 1	?

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Extremal congruence K3 surfaces

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1	$\Gamma(4)$	$4, 4, 4, 4, 4, 4$	$\eta(q)^6$
2	$\Gamma_0(3) \cap \Gamma(2)$	$6, 6, 6, 2, 2, 2$	$\eta(q)^3\eta(q^3)^3$
3	$\Gamma_1(7)$	$7, 7, 7, 1, 1, 1$	$\eta(q)^3\eta(q^7)^3$
4	$\Gamma_1(8)$	$8, 8, 4, 2, 1, 1$	$\eta(q)^2\eta(q^2)\eta(q^4)\eta(q^8)^2$
5	$\Gamma_0(8) \cap \Gamma(2)$	$8, 8, 2, 2, 2, 2$	$\eta(q^2)^6$
6	$\Gamma_1(8; 4, 1, 2)$	$8, 4, 4, 4, 2, 2$	?
7	$\Gamma_0(12)$	$12, 4, 3, 3, 1, 1$	$\eta(q^2)\eta(q^6)^3$
8	$\Gamma_0(16)$	$16, 4, 1, 1, 1, 1$	$\eta(q^4)^6$
9	$\Gamma_1(16; 16, 2, 2)$	$16, 2, 2, 2, 1, 1$	?

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Extremal congruence K3 surfaces

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1	$\Gamma(4)$	$4, 4, 4, 4, 4, 4$	$\eta(q)^6$
2	$\Gamma_0(3) \cap \Gamma(2)$	$6, 6, 6, 2, 2, 2$	$\eta(q)^3\eta(q^3)^3$
3	$\Gamma_1(7)$	$7, 7, 7, 1, 1, 1$	$\eta(q)^3\eta(q^7)^3$
4	$\Gamma_1(8)$	$8, 8, 4, 2, 1, 1$	$\eta(q)^2\eta(q^2)\eta(q^4)\eta(q^8)^2$
5	$\Gamma_0(8) \cap \Gamma(2)$	$8, 8, 2, 2, 2, 2$	$\eta(q^2)^6$
6	$\Gamma_1(8; 4, 1, 2)$	$8, 4, 4, 4, 2, 2$	?
7	$\Gamma_0(12)$	$12, 4, 3, 3, 1, 1$	$\eta(q^2)\eta(q^6)^3$
8	$\Gamma_0(16)$	$16, 4, 1, 1, 1, 1$	$\eta(q^4)^6$
9	$\Gamma_1(16; 16, 2, 2)$	$16, 2, 2, 2, 1, 1$	?

$PSL_2(\mathbb{R})$-conj. classes

Extremal congruence K3 surfaces

[1, 5, 8]
[2, 7]
[3]
[4, 6, 9]

#	The group Γ	Widths of the cusps	Cusp form of weight 3
1	$\Gamma(4)$	$4,4,4,4,4,4$	$\eta(q)^6$
2	$\Gamma_0(3)\cap\Gamma(2)$	$6,6,6,2,2,2$	$\eta(q)^3\eta(q^3)^3$
3	$\Gamma_1(7)$	$7,7,7,1,1,1$	$\eta(q)^3\eta(q^7)^3$
4	$\Gamma_1(8)$	$8,8,4,2,1,1$	$\eta(q)^2\eta(q^2)\eta(q^4)\eta(q^8)^2$
5	$\Gamma_0(8)\cap\Gamma(2)$	$8,8,2,2,2,2$	$\eta(q^2)^6$
6	$\Gamma_1(8;4,1,2)$	$8,4,4,4,2,2$	?
7	$\Gamma_0(12)$	$12,4,3,3,1,1$	$\eta(q^2)\eta(q^6)^3$
8	$\Gamma_0(16)$	$16,4,1,1,1,1$	$\eta(q^4)^6$
9	$\Gamma_1(16;16,2,2)$	$16,2,2,2,1,1$	?

The modularity of the eight non-rigid Calabi-Yau threefolds

Theorem: *Let Y be one of the nine singular $K3$ surfaces in Table, and let E be an elliptic curve. Then the following holds true:*

(a) The product $Y \times E$ has the Hodge numbers $h^{0,3}(Y \times E) = 1$, $h^{1,0}(Y \times E) = 1$ and $B_3(Y \times E) = 44$ (so that $Y \times E$ is not a Calabi–Yau threefold).

(b) The motive $T(Y \times E) = T(Y) \times H^1(E)$ is a submotive of $H^3(Y \times E)$ of rank 4. If E and Y are both defined over $\mathbb{Q}$, this submotive is modular, in the sense that its L-series is associated to a cusp form g_Y of weight 3 on $T(Y)$ and a cusp form g_E of weight 2 on E.

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(c) Assume that both Y and E are defined over $\mathbb{Q}$. Let

$$a(p) = \text{the } p\text{-th Fourier coefficient of } g_E$$

$$b(p) = \text{the } p\text{-th Fourier coefficient of } g_Y$$

Then fo[illegible]y good prime p, the local Euler p-factor of the L-series

$$L(T(Y \times E) \otimes \mathbb{Q}_\ell, s) = L(g_E \otimes g_Y, s)$$

is [illegible]icitly given by

$$1 - a(p)b(p)p^{-s} + (b(p)^2 + pa(p)^2 - 2p^2\varepsilon_Y(p))p^{1-2s}$$
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$$h^{3,0} = 1,\ h^{1,0} = h^{2,0} = 0, h^{2,1} = 1 + n_-.$$

It has a model defined over $\mathbb{Q}$ if both Y and E are defined over $\mathbb{Q}$, and X is modular, that is,

$$L(X, s) = L(g_E \otimes g_Y, s)L(E \otimes \chi_\delta, s-1)^{n_-}$$

where χ_δ is the quadratic character cut by $\mathbb{Q}(\sqrt{\delta})$.

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Motivations for Examples of Livné and Yui

We consider a Calabi-Yau threefold $f : X \to \mathbb{P}^1$ fibered by semi-stable K3 surfaces. Let $S \subset \mathbb{P}^1$ be a finite set of points, and let $\Delta \subset X$ be the pull-back of S. Let $\omega_{X/\mathbb{P}^1}$ be the canonical sheaf. The Kodaira-Spencer maps $\theta^{2,0}$ and $\theta^{1,1}$ are defined, respectively, by:

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If f has exactly four singular fibers, then X is rigid, and birational to the Nikulin-Kummer construction of a symmetric square of a family of elliptic curves $f : E \to \mathbb{P}^1$. After passing to a double cover $E' \to E$ (if necessary), the family $g' : E' \to \mathbb{P}^1$ is a modular family of elliptic curves from the Beauville six curves.

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index	*Number of cusps*	*Group*	*Cusp widths*	#
12	4	$\Gamma(3)$	3, 3, 3, 3	
		$\Gamma_0(4) \cap \Gamma(2)$	4, 4, 2, 2	
		$\Gamma_1(5)$	5, 5, 1, 1	
		$\Gamma_0(6)$	6, 3, 2, 1	
		$\Gamma_0(8)$	8.2.1.1	
		$\Gamma_0(9)$	9, 1, 1, 1	
24	6	$\Gamma(4)$	4, 4, 4, 4, 4, 4	#1
		$\Gamma_0(3) \cap \Gamma(2)$	6, 6, 6, 2, 2, 2	#2
		$\Gamma_1(7)$	7, 7, 7, 1, 1, 1	#3 *
		$\Gamma_1(8)$	8, 8, 4, 2, 1, 1	#4
		$\Gamma_0(8) \cap \Gamma(2)$	8, 8, 2, 2, 2, 2	#5
		$\Gamma(8; 4, 1, 2)$	8, 4, 4, 4, 2, 2	#6
		$\Gamma_0(12)$	12, 4, 3, 3, 1, 1	#7
		$\Gamma_0(16)$	16, 4, 1, 1, 1, 1	#8
		$\Gamma(16; 16, 2, 2)$	16, 2, 2, 2, 1, 1	#9

rational elliptic surfaces with section

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Extremal elliptic K3 surfaces with section

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		$\Gamma(8; 4, 1, 2)$	8, 4, 4, 4, 2, 2	#6
		$\Gamma_0(12)$	12, 4, 3, 3, 1, 1	#7
		$\Gamma_0(16)$	16, 4, 1, 1, 1, 1	#8
		$\Gamma(16; 16, 2, 2)$	16, 2, 2, 2, 1, 1	#9

rational elliptic surfaces with section

$\theta^{1,1}\theta^{2,0} \neq 0$

Extremal elliptic K3 surfaces with section

$\theta^{1,1}\theta^{2,0} = 0$