Title: Generalized Kahler geometry and T-duality (no audio)

Date: Nov 19, 2004 02:00 PM

URL: http://pirsa.org/04110016

Abstract: Please Note: There is no audio for most presentations from November 18th due to technical issues. We appologize to presenters and interested parties for this inconvenience.

Geometry of $T \oplus T^*$

- Natural split-signature metric $\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\xi(Y)+\eta(X))$ determines $O(n,n)$ structure

- Sub-bundles T and T^* are maximal isotropic

- Spinors for $T \oplus T^*$ are given by $S = \wedge^\bullet T^*$

- Exterior derivative d generates the *Courant bracket*

$$[X+\xi, Y+\eta] = [X,Y] + L_X\eta - L_Y\xi - \frac{1}{2}d(\eta(X) - \xi(Y))$$

Generalized complex structures $O(2n, 2n) - U(n, n)$

Definition: complex structure $\mathcal{J} \in O(T \oplus T^*)$ such that $[\mathcal{J}, \mathcal{J}] = 0$.

Examples: $\mathcal{J}_J = \begin{pmatrix} -J & \\ & J^* \end{pmatrix}$, $\quad \mathcal{J}_\omega = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}$

Local invariant:

- $\mathcal{J} = \begin{pmatrix} A & \pi \\ \sigma & -A^* \end{pmatrix}$, and π is a Poisson structure.

- $k := n - \frac{1}{2}\text{rk } \pi$, called type of GCS.

  complex $- n, \; n-1, \; \cdots, \; 1, \; 0 -$ symplectic

Geometry of $T \oplus T^*$

- Natural split-signature metric $\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\xi(Y)+\eta(X))$ determines $O(n,n)$ structure
- Sub-bundles T and T^* are maximal isotropic
- Spinors for $T \oplus T^*$ are given by $S = \wedge^\bullet T^*$
- Exterior derivative d generates the *Courant bracket*

$$[X+\xi, Y+\eta] = [X,Y] + L_X\eta - L_Y\xi - \frac{1}{2}d(\eta(X) - \xi(Y))$$

2

Geometry of $T \oplus T^*$

- Natural split-signature metric $\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\xi(Y)+\eta(X))$ determines $O(n,n)$ structure

- Sub-bundles T and T^* are maximal isotropic

- Spinors for $T \oplus T^*$ are given by $S = \wedge^\bullet T^*$

- Exterior derivative d generates the *Courant bracket*

$$[X+\xi, Y+\eta] = [X,Y] + L_X\eta - L_Y\xi - \frac{1}{2}d(\eta(X)-\xi(Y))$$

Generalized complex structures $O(2n, 2n) - U(n, n)$

Definition: complex structure $\mathcal{J} \in O(T \oplus T^*)$ such that $[\mathcal{J}, \mathcal{J}] = 0$.

Examples: $\mathcal{J}_J = \begin{pmatrix} -J & \\ & J^* \end{pmatrix}$, $\quad \mathcal{J}_\omega = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}$

Local invariant:

- $\mathcal{J} = \begin{pmatrix} A & \pi \\ \sigma & -A^* \end{pmatrix}$, and π is a Poisson structure.

- $k := n - \frac{1}{2}\mathrm{rk}\ \pi$, called type of GCS.

  complex $-n,\ n-1,\ \cdots,\ 1,\ 0-$ symplectic

3

Generalized complex structures $O(2n, 2n) - U(n, n)$

Definition: complex structure $\mathcal{J} \in O(T \oplus T^*)$ such that $[\mathcal{J}, \mathcal{J}] = 0$.

Examples: $\mathcal{J}_J = \begin{pmatrix} -J & \\ & J^* \end{pmatrix}$, $\quad \mathcal{J}_\omega = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}$

Local invariant:

- $\mathcal{J} = \begin{pmatrix} A & \pi \\ \sigma & -A^* \end{pmatrix}$, and π is a Poisson structure.

- $k := n - \frac{1}{2}\text{rk } \pi$, called type of GCS.

complex $-n,\ n-1,\ \cdots,\ 1,\ 0-$ symplectic

3

Generalized complex structures $O(2n, 2n) - U(n, n)$

Definition: complex structure $\mathcal{J} \in O(T \oplus T^*)$ such that $[\mathcal{J}, \mathcal{J}] = 0$.

Examples: $\mathcal{J}_J = \begin{pmatrix} -J & \\ & J^* \end{pmatrix}$, $\quad \mathcal{J}_\omega = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}$

Local invariant:

- $\mathcal{J} = \begin{pmatrix} A & \pi \\ \sigma & -A^* \end{pmatrix}$, and π is a Poisson structure.

- $k := n - \frac{1}{2}\mathrm{rk}\ \pi$, called type of GCS.

  complex $-n,\ n-1,\ \cdots,\ 1,\ 0-$ symplectic

3

Generalized complex structures $O(2n, 2n) - U(n, n)$

Definition: complex structure $\mathcal{J} \in O(T \oplus T^*)$ such that $[\mathcal{J}, \mathcal{J}] = 0$.

Examples: $\mathcal{J}_J = \begin{pmatrix} -J & \\ & J^* \end{pmatrix}$, $\quad \mathcal{J}_\omega = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}$

Local invariant:

- $\mathcal{J} = \begin{pmatrix} A & \pi \\ \sigma & -A^* \end{pmatrix}$, and π is a Poisson structure.

- $k := n - \frac{1}{2}\text{rk}\ \pi$, called type of GCS.

complex — n, $n-1, \cdots, 1, 0$ — symplectic

3

Kähler geometry

Definition: triple (g, J, ω) such that

$$\begin{tikzcd} T \arrow[rr, "g"] & & T^* \\ & T \arrow[ul, "J"] \arrow[ur, "\omega"'] & \end{tikzcd}$$

Equivalently: pair (J, ω) such that $(-\omega J)^* = -\omega J$ and $-\omega J > 0$

Equivalently: pair $(\mathcal{J}_J, \mathcal{J}_\omega)$ such that

$$\begin{pmatrix} -J & \\ & J^* \end{pmatrix} \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix} = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix} \begin{pmatrix} -J & \\ & J^* \end{pmatrix} = \begin{pmatrix} & g^{-1} \\ g & \end{pmatrix} > 0$$

Generalized complex structures $O(2n,2n) - U(n,n)$

Definition: complex structure $\mathcal{J} \in O(T \oplus T^*)$ such that $[\mathcal{J}, \mathcal{J}] = 0$.

Examples: $\mathcal{J}_J = \begin{pmatrix} -J & \\ & J^* \end{pmatrix}, \quad \mathcal{J}_\omega = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}$

Local invariant:

- $\mathcal{J} = \begin{pmatrix} A & \pi \\ \sigma & -A^* \end{pmatrix}$, and π is a Poisson structure.

- $k := n - \frac{1}{2}\mathrm{rk}\ \pi$, called type of GCS.

  complex $-n,\ n-1,\ \cdots,\ 1,\ 0-$ symplectic

Kähler geometry

Definition: triple (g, J, ω) such that

$$\begin{tikzcd} T \arrow[rr, "g"] & & T^* \\ & T \arrow[ul, "J"] \arrow[ur, "\omega"'] & \end{tikzcd}$$

Equivalently: pair (J, ω) such that $(-\omega J)^* = -\omega J$ and $-\omega J > 0$

Equivalently: pair $(\mathcal{J}_J, \mathcal{J}_\omega)$ such that

$$\begin{pmatrix} -J & \\ & J^* \end{pmatrix} \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix} = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix} \begin{pmatrix} -J & \\ & J^* \end{pmatrix} = \begin{pmatrix} & g^{-1} \\ g & \end{pmatrix} > 0$$

Kähler geometry

Definition: triple (g, J, ω) such that

$$\begin{tikzcd} T \arrow[rr, "g"] & & T^* \\ & T \arrow[ul, "J"] \arrow[ur, "\omega"'] & \end{tikzcd}$$

Equivalently: pair (J, ω) such that $(-\omega J)^* = -\omega J$ and $-\omega J > 0$

Equivalently: pair $(\mathcal{J}_J, \mathcal{J}_\omega)$ such that

$$\begin{pmatrix} -J & \\ & J^* \end{pmatrix} \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix} = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix} \begin{pmatrix} -J & \\ & J^* \end{pmatrix} = \begin{pmatrix} & g^{-1} \\ g & \end{pmatrix} > 0$$

Generalized Kähler geometry $U(n,n) \to U(n) \times U(n)$

Definition: pair $(\mathcal{J}_A, \mathcal{J}_B)$ of generalized complex structures such that

$$\mathcal{J}_A \mathcal{J}_B = \mathcal{J}_B \mathcal{J}_A = G > 0$$

Properties:

- $\text{type}(\mathcal{J}_A) + \text{type}(\mathcal{J}_B) \begin{cases} \leq n \\ \equiv n \pmod 2 \end{cases}$

- $C_\pm := \pm 1$–eigenspaces of G. These are complex for $\mathcal{J}_A, \mathcal{J}_B$ and $\pm$-definite. By projection, this induces a Riemannian metric g, a 2-form b, and two Hermitian almost complex structures $J_\pm$ on the tangent bundle T.

Theorem 1 [MG]: Algebraic equivalence $(\mathcal{J}_A, \mathcal{J}_B) \leftrightarrows (J_+, J_-, b, g)$, given by

$$\mathcal{J}_{A/B} = \frac{1}{2}\begin{pmatrix} 1 & \\ b & 1 \end{pmatrix}\begin{pmatrix} J_+ \pm J_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(J_+^* \pm J_-^*) \end{pmatrix}\begin{pmatrix} 1 & \\ -b & 1 \end{pmatrix}$$

Theorem 2 [MG]: Generalized Kähler structure $(\mathcal{J}_A, \mathcal{J}_B)$ is integrable if and only if $J_\pm$ are integrable and

$$d_-^c \omega_- = -d_+^c \omega_+ = db + H$$

Equivalently: $\nabla^\pm J_\pm = 0$, where $\nabla^\pm = \nabla \pm \frac{1}{2} g^{-1}(db + H)$, and $db + H$ is of type $(2,1) + (1,2)$.

Gates, Hull, Roček (1984): General target space geometry for $N = (2,2)$ sigma model
Kapustin: $(\mathcal{J}_A, \mathcal{J}_B)$ are topological twists.

Theorem 1 [MG]: Algebraic equivalence $(\mathcal{J}_A, \mathcal{J}_B) \leftrightarrows (J_+, J_-, b, g)$, given by

$$\mathcal{J}_{A/B} = \frac{1}{2}\begin{pmatrix} 1 & \\ b & 1 \end{pmatrix}\begin{pmatrix} J_+ \pm J_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(J_+^* \pm J_-^*) \end{pmatrix}\begin{pmatrix} 1 & \\ -b & 1 \end{pmatrix}$$

Theorem 2 [MG]: Generalized Kähler structure $(\mathcal{J}_A, \mathcal{J}_B)$ is integrable if and only if $J_\pm$ are integrable and

$$d_-^c \omega_- = -d_+^c \omega_+ = db + H$$

Equivalently: $\nabla^\pm J_\pm = 0$, where $\nabla^\pm = \nabla \pm \frac{1}{2} g^{-1}(db + H)$, and $db + H$ is of type $(2,1) + (1,2)$.

Gates, Hull, Roček (1984): General target space geometry for $N = (2,2)$ sigma model
Kapustin: $(\mathcal{J}_A, \mathcal{J}_B)$ are topological twists.

$$\begin{array}{ccl}
\{\text{Generalized Kähler}\} & \longleftarrow & \mathcal{J}_A, \mathcal{J}_B \text{ commute in } T \oplus T^* \\
\Updownarrow & & \\
\{\text{bihermitian } (2,2)\} & \longleftarrow & J_+, J_- \text{ need not commute in } T
\end{array}$$

Remark: J_+, J_- commute if and only if type($\mathcal{J}_A$)+type($\mathcal{J}_B$) $= n$.

Examples of generalized Kähler

- any Kähler manifold ($J_\pm = J$, $b = 0$), type $= (n, 0)$ or $(0, n)$
- hyperKähler, where $(J_+, J_-, b, g) = (I, J, \omega_K, g)$, type $= (0, 0)$
- primary Hopf surfaces and finite covers thereof, type $= (1, 1)$ or $(0, 0)$ with jumping to $(2, 0)$ along anticanonical divisor
- even semi-simple Lie group $(J_+, J_-, b, g) = (J_R, J_L, 0, Kill)$, $H =$ Cartan, type jumping
- (N. Hitchin) moduli space of instantons on a generalized Kähler manifold is generalized Kähler

Hodge theory for generalized Kähler

- $(\mathcal{J}_A, \mathcal{J}_B)$ commuting endomorphisms act via Spin representation on $S = \wedge^\bullet T^*$.

- decomposition of forms into (ir, is)-eigenspaces for $(\mathcal{J}_A, \mathcal{J}_B)$:

$$\wedge^\bullet T^* \otimes \mathbb{C} = \bigoplus_{\substack{|r+s|\leq n \\ r+s\equiv n (\text{mod } 2)}} U^{r,s}$$

- decomposition is orthogonal in the Born-Infeld metric $(\, ,)$ on forms depending on $g + b$.

$$
\begin{array}{ccccccc}
 & & & U_{0,n} & & & \\
 & & \cdots & & \cdots & & \\
 & U_{-n+1,1} & & & & U_{n-1,1} & \\
U_{-n,0} & & & \cdots & & & U_{n,0} \\
 & U_{-n+1,-1} & & & & U_{n-1,-1} & \\
 & & \cdots & & \cdots & & \\
 & & & U_{0,-n} & & &
\end{array}
$$

- exterior derivative d (or $d_H = d + H$) decomposes into 4 differentials:

$$d = \delta_+ + \delta_- + \overline{\delta}_+ + \overline{\delta}_-,$$

where the differential operators act as follows:

$$\begin{tikzcd}
U_{p-1,q+1} & & U_{p+1,q+1} \\
& U_{p,q} \arrow[ul, "\delta_-"'] \arrow[ur, "\overline{\delta}_+"'] \arrow[dl, "\delta_+"] \arrow[dr, "\overline{\delta}_-"] & \\
U_{p-1,q-1} & & U_{p+1,q-1}
\end{tikzcd}$$

Lemma (generalized Kähler identities):

$$\overline{\delta}_+^* = -\delta_+ \quad \text{and} \quad \overline{\delta}_-^* = \delta_-.$$

$\Rightarrow$ classical $N = (2,2)$ SUSY algebra representation on $\Omega^\bullet(M, \mathbb{C})$

$$
\begin{array}{ccccccc}
 & & & U_{0,n} & & & \\
 & & \cdots & & \cdots & & \\
 & U_{-n+1,1} & & & & U_{n-1,1} & \\
U_{-n,0} & & & \cdots & & & U_{n,0} \\
 & U_{-n+1,-1} & & & & U_{n-1,-1} & \\
 & & \cdots & & \cdots & & \\
 & & & U_{0,-n} & & &
\end{array}
$$

- exterior derivative d (or $d_H = d + H$) decomposes into 4 differentials:

$$d = \delta_+ + \delta_- + \overline{\delta}_+ + \overline{\delta}_-,$$

where the differential operators act as follows:

$$\begin{tikzcd}
U_{p-1,q+1} & & U_{p+1,q+1} \\
& U_{p,q} \arrow[ul, "\delta_-"'] \arrow[ur, "\overline{\delta}_+"] \arrow[dl, "\delta_+"] \arrow[dr, "\overline{\delta}_-"'] & \\
U_{p-1,q-1} & & U_{p+1,q-1}
\end{tikzcd}$$

Lemma (generalized Kähler identities):

$$\overline{\delta}_+^* = -\delta_+ \quad \text{and} \quad \overline{\delta}_-^* = \delta_-.$$

$\Rightarrow$ classical $N = (2,2)$ SUSY algebra representation on $\Omega^\bullet(M, \mathbb{C})$

Theorem [MG]: The H-twisted cohomology of a compact generalized Kähler manifold carries a Hodge decomposition:

$$H^\bullet_H(M,\mathbb{C}) = \bigoplus_{\substack{|p+q|\le n \\ p+q\equiv n (\mathrm{mod}\ 2)}} \mathcal{H}^{p,q}.$$

Corollary: A 4-dimensional generalized Kähler manifold with $H = 0$ must have b_1 even and b_2^+ odd, and to admit type $= (1,1)$ it must also have b_2 even.

Note: In the usual Kähler case, this decomposition is *not* the usual (p,q)-decomposition. It was first discovered by Michelsohn and is called the Clifford decomposition. if $(\mathcal{J}_A, \mathcal{J}_B) = (\mathcal{J}_J, \mathcal{J}_\omega)$, then

$$U^{n-r-s,r-s} \cong \Omega^{r,s}.$$

- exterior derivative d (or $d_H = d + H$) decomposes into 4 differentials:

$$d = \delta_+ + \delta_- + \overline{\delta}_+ + \overline{\delta}_-,$$

where the differential operators act as follows:

$$\begin{tikzcd}
U_{p-1,q+1} & & U_{p+1,q+1} \\
& U_{p,q} \arrow[ul, "\delta_-"] \arrow[ur, "\overline{\delta}_+"'] \arrow[dl, "\delta_+"'] \arrow[dr, "\overline{\delta}_-"] & \\
U_{p-1,q-1} & & U_{p+1,q-1}
\end{tikzcd}$$

Lemma (generalized Kähler identities):

$$\overline{\delta}_+^* = -\delta_+ \quad \text{and} \quad \overline{\delta}_-^* = \delta_-.$$

$\Rightarrow$ classical $N = (2, 2)$ SUSY algebra representation on $\Omega^\bullet(M, \mathbb{C})$

Theorem [MG]: The H-twisted cohomology of a compact generalized Kähler manifold carries a Hodge decomposition:

$$H^{\bullet}_{H}(M,\mathbb{C}) = \bigoplus_{\substack{|p+q|\leq n \\ p+q\equiv n (\text{mod } 2)}} \mathcal{H}^{p,q}.$$

Corollary: A 4-dimensional generalized Kähler manifold with $H = 0$ must have b_1 even and b_2^+ odd, and to admit type $= (1,1)$ it must also have b_2 even.

Note: In the usual Kähler case, this decomposition is *not* the usual (p,q)-decomposition. It was first discovered by Michelsohn and is called the Clifford decomposition. if $(\mathcal{J}_A, \mathcal{J}_B) = (\mathcal{J}_J, \mathcal{J}_\omega)$, then

$$U^{n-r-s,r-s} \cong \Omega^{r,s}.$$

Geometric T-duality (joint with G. Cavalcanti)

- our approach: T-duality of any generalized geometry, assuming *invariance*, but allowing flux. Based on the topological approach of Bouwknegt, Evslin, Mathai.

- early ideas of Minasian on the use of $Spin(n,n)$ "bispinors" to mirror symmetry.

- recent work of Ben-Bassat on T-duality of generalized complex structures where invariance is relaxed to semi-flatness.

Theorem [MG]: The H-twisted cohomology of a compact generalized Kähler manifold carries a Hodge decomposition:

$$H_H^\bullet(M,\mathbb{C}) = \bigoplus_{\substack{|p+q|\leq n \\ p+q\equiv n(\text{mod } 2)}} \mathcal{H}^{p,q}.$$

Corollary: A 4-dimensional generalized Kähler manifold with $H = 0$ must have b_1 even and b_2^+ odd, and to admit type $= (1,1)$ it must also have b_2 even.

Note: In the usual Kähler case, this decomposition is *not* the usual (p,q)-decomposition. It was first discovered by Michelsohn and is called the Clifford decomposition. if $(\mathcal{J}_A, \mathcal{J}_B) = (\mathcal{J}_J, \mathcal{J}_\omega)$, then

$$U^{n-r-s,r-s} \cong \Omega^{r,s}.$$

Geometric T-duality (joint with G. Cavalcanti)

- our approach: T-duality of any generalized geometry, assuming *invariance*, but allowing flux. Based on the topological approach of Bouwknegt, Evslin, Mathai.

- early ideas of Minasian on the use of $Spin(n, n)$ "bispinors" to mirror symmetry.

- recent work of Ben-Bassat on T-duality of generalized complex structures where invariance is relaxed to semi-flatness.

- exterior derivative d (or $d_H = d + H$) decomposes into 4 differentials:

$$d = \delta_+ + \delta_- + \overline{\delta}_+ + \overline{\delta}_-,$$

where the differential operators act as follows:

$$\begin{tikzcd}
U_{p-1,q+1} & & U_{p+1,q+1} \\
& U_{p,q} \arrow[ul, "\delta_-"'] \arrow[ur, "\overline{\delta}_+"] \arrow[dl, "\delta_+"] \arrow[dr, "\overline{\delta}_-"] & \\
U_{p-1,q-1} & & U_{p+1,q-1}
\end{tikzcd}$$

Lemma (generalized Kähler identities):

$$\overline{\delta}_+^* = -\delta_+ \quad \text{and} \quad \overline{\delta}_-^* = \delta_-.$$

$\Rightarrow$ classical $N = (2,2)$ SUSY algebra representation on $\Omega^\bullet(M, \mathbb{C})$

Geometric T-duality (joint with G. Cavalcanti)

- our approach: T-duality of any generalized geometry, assuming *invariance*, but allowing flux. Based on the topological approach of Bouwknegt, Evslin, Mathai.

- early ideas of Minasian on the use of $Spin(n,n)$ "bispinors" to mirror symmetry.

- recent work of Ben-Bassat on T-duality of generalized complex structures where invariance is relaxed to semi-flatness.

First step: T-duality between principal S^1-bundles

Data:

$$\begin{cases} \pi : P \longrightarrow B \text{ a principal } S^1\text{-bundle} \\ u \in \Omega^1(P) \text{ a connection, with curvature } du = F \\ H \in \Omega^3_{cl}(P)^{S^1} \text{ a NS 3-flux, } [H] \in H^3(M, \mathbb{Z}). \end{cases}$$

write $H = u \wedge \tilde{F} + h, \ h \in \Omega^3(B)$.

Dual data:

$$\begin{cases} \pi_*[H] \in H^2(B, \mathbb{Z}) \text{ determines a dual principal } S^1\text{-bundle } \tilde{P} \\ \text{choose a connection } \tilde{u} \text{ on } \tilde{P} \\ \tilde{H} := \tilde{u} \wedge F + h \end{cases}$$

First step: T-duality between principal S^1-bundles

Data:

$$\begin{cases} \pi : P \longrightarrow B \text{ a principal } S^1\text{-bundle} \\ u \in \Omega^1(P) \text{ a connection, with curvature } du = F \\ H \in \Omega^3_{cl}(P)^{S^1} \text{ a NS 3-flux, } [H] \in H^3(M, \mathbb{Z}). \end{cases}$$

write $H = u \wedge \tilde{F} + h, \ h \in \Omega^3(B)$.

Dual data:

$$\begin{cases} \pi_*[H] \in H^2(B, \mathbb{Z}) \text{ determines a dual principal } S^1\text{-bundle } \tilde{P} \\ \text{choose a connection } \tilde{u} \text{ on } \tilde{P} \\ \tilde{H} := \tilde{u} \wedge F + h \end{cases}$$

Exchanging flux for topology

$$
\begin{tikzcd}[column sep=small]
 & & P \times_B \tilde{P} \arrow[dl, "f"'] \arrow[dr, "g"] & & \\
 & P \arrow[dr] & & \tilde{P} \arrow[dl] & \\
H & & B & & \tilde{H}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small]
 & & S^3 \times S^1 \arrow[dl, "f"'] \arrow[dr, "g"] & & \\
 & S^3 \arrow[dr] & & S^2 \times S^1 \arrow[dl] & \\
H = 0 & & S^2 & & \tilde{H} = v_{S^2} \wedge v_{S^1}
\end{tikzcd}
$$

Theorem [BEM]: The H-twisted K-theory of P is isomorphic to the $\tilde{H}$-twisted K-theory of $\tilde{P}$.

Theorem [Cavalcanti, MG]: Let $F = u \wedge \tilde{u}$. The map

$$\mathcal{F} = g_* \circ e^F \circ f^*$$

induces an isomorphism of Courant algebroids

$$\left(\frac{T_P \oplus T_P^*}{S^1}, H\right) \longrightarrow \left(\frac{T_{\tilde{P}} \oplus T_{\tilde{P}}^*}{S^1}, \tilde{H}\right)$$

First step: T-duality between principal S^1-bundles

Data:

$$\begin{cases} \pi : P \longrightarrow B \text{ a principal } S^1\text{-bundle} \\ u \in \Omega^1(P) \text{ a connection, with curvature } du = F \\ H \in \Omega^3_{cl}(P)^{S^1} \text{ a NS 3-flux, } [H] \in H^3(M, \mathbb{Z}). \end{cases}$$

write $H = u \wedge \tilde{F} + h, \ h \in \Omega^3(B)$.

Dual data:

$$\begin{cases} \pi_*[H] \in H^2(B, \mathbb{Z}) \text{ determines a dual principal } S^1\text{-bundle } \tilde{P} \\ \text{choose a connection } \tilde{u} \text{ on } \tilde{P} \\ \tilde{H} := \tilde{u} \wedge F + h \end{cases}$$

Exchanging flux for topology

$$
\begin{tikzcd}
 & P \times_B \tilde{P} \arrow[dl, "f"'] \arrow[dr, "g"] & \\
P \arrow[dr] & & \tilde{P} \arrow[dl] \\
H & B & \tilde{H}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & S^3 \times S^1 \arrow[dl, "f"'] \arrow[dr, "g"] & \\
S^3 \arrow[dr] & & S^2 \times S^1 \arrow[dl] \\
H = 0 & S^2 & \tilde{H} = v_{S^2} \wedge v_{S^1}
\end{tikzcd}
$$

Theorem [BEM]: The H-twisted K-theory of P is isomorphic to the $\tilde{H}$-twisted K-theory of $\tilde{P}$.

Theorem [Cavalcanti, MG]: Let $F = u \wedge \tilde{u}$. The map

$$\mathcal{F} = g_* \circ e^F \circ f^*$$

induces an isomorphism of Courant algebroids

$$\left(\frac{T_P \oplus T_P^*}{S^1}, H \right) \longrightarrow \left(\frac{T_{\tilde{P}} \oplus T_{\tilde{P}}^*}{S^1}, \tilde{H} \right)$$

Transport of geometric structure

The theorem can be used to transport generalized geometries from one manifold to the T-dual side.

Examples:

- Let G be a generalized metric on P. Then $g_* \circ e^F \circ f^*(G)$ is a generalized metric $(\tilde{g} + \tilde{b})$ on $\tilde{P}$. These are known as the Buscher rules
- Let D be a Dirac structure on P. Then $g_* \circ e^F \circ f^*(D)$ is a Dirac structure on $\tilde{P}$. Operations f^*, f_* defined by Weinstein.
- Let $\mathcal{J}$ be a generalized complex structure on P. Then $g_* \circ e^F \circ f^*(\mathcal{J})$ is a generalized complex structure $\tilde{\mathcal{J}}$ on $\tilde{P}$, with $\text{type}(\tilde{\mathcal{J}}) = \text{type}\mathcal{J} \pm 1$. E.g. in 6 dimensions, 3 T-dualizations are required to go from type $= 3$ to type $= 0$.
- Compare with gerbe T-duality of Donagi and Pantev.

Transport of generalized Kähler structure

$$(\mathcal{J}_A, \mathcal{J}_B) \overset{T}{\Longleftrightarrow} (\tilde{\mathcal{J}}_A, \tilde{\mathcal{J}}_B)$$

$$H_c^{r,s}(M) \Longleftrightarrow \tilde{H}_c^{r,s}(M)$$

If $(\mathcal{J}_A, \mathcal{J}_B)$ were a Kähler $2n$-manifold and we could perform n T-dualities,

$$(\mathcal{J}_J, \mathcal{J}_\omega) \Longleftrightarrow (\mathcal{J}_{\tilde{\omega}}, \mathcal{J}_{\tilde{J}})$$

$$H^{p,q} \Longleftrightarrow H^{n-p,q}$$

First step: T-duality between principal S^1-bundles

Data:

$$\begin{cases} \pi : P \longrightarrow B \text{ a principal } S^1\text{-bundle} \\ u \in \Omega^1(P) \text{ a connection, with curvature } du = F \\ H \in \Omega^3_{cl}(P)^{S^1} \text{ a NS 3-flux, } [H] \in H^3(M, \mathbb{Z}). \end{cases}$$

write $H = u \wedge \tilde{F} + h, \;\; h \in \Omega^3(B).$

Dual data:

$$\begin{cases} \pi_*[H] \in H^2(B, \mathbb{Z}) \text{ determines a dual principal } S^1\text{-bundle } \tilde{P} \\ \text{choose a connection } \tilde{u} \text{ on } \tilde{P} \\ \tilde{H} := \tilde{u} \wedge F + h \end{cases}$$

Exchanging flux for topology

$$
\begin{tikzcd}
 & P \times_B \tilde{P} \arrow[dl, "f"'] \arrow[dr, "g"] & \\
P \arrow[dr] & & \tilde{P} \arrow[dl] \\
H & B & \tilde{H}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & S^3 \times S^1 \arrow[dl, "f"'] \arrow[dr, "g"] & \\
S^3 \arrow[dr] & & S^2 \times S^1 \arrow[dl] \\
H = 0 & S^2 & \tilde{H} = v_{S^2} \wedge v_{S^1}
\end{tikzcd}
$$

Theorem [BEM]: The H-twisted K-theory of P is isomorphic to the $\tilde{H}$-twisted K-theory of $\tilde{P}$.

Theorem [Cavalcanti, MG]: Let $F = u \wedge \tilde{u}$. The map

$$\mathcal{F} = g_* \circ e^F \circ f^*$$

induces an isomorphism of Courant algebroids

$$\left(\frac{T_P \oplus T_P^*}{S^1}, H \right) \longrightarrow \left(\frac{T_{\tilde{P}} \oplus T_{\tilde{P}}^*}{S^1}, \tilde{H} \right)$$

Any questions?