Title: Motives and Strings. (no audio)

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Abstract: Please Note: There is no audio for most presentations from November 18th due to technical issues. We appologize to presenters and interested parties for this inconvenience.

Perimeter Institute, 19 November 2004
Workshop on Mirror Symmetry

Motives and Strings

Jan Stienstra

'Does the *Theory of Everything* include Numbers?'

'The M in *M-theory* seems to stand for
magic or *mystery* or *mother of all strings*,
but could it also mean *Motive*?'

What is .... a Motive ?

Barry Mazur in Notices of the AMS november 2004:

Algebraic topology: one cohomology theory, characterized by the Eilenberg-Steenrod axioms, representable by Eilenberg-MacLane spaces and Postnikov towers.

Algebraic geometry: a profusion of different cohomology theories, no axioms, no representing objects:

- Hodge cohomology
- algebraic De Rham cohomology
- crystalline cohomology (for every prime number p)
- étale ℓ-adic cohomology (for every prime number ℓ)

and comparison isomorphisms between these.

Cohomology theories assign to varieties vector spaces (some with additional structure).
Inverse problem: *given a collection of vector spaces (+ additional structure, comparison isomorphisms), is this produced by the various cohomology theories from one common source?*
The common source is called a MOTIVE.

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More precisely:
a variety X is defined over some field K, $\mathrm{char}(K) = 0$, often with positive transcendence degree over $\mathbb{Q}$.

Every embedding σ of K into $\mathbb{C}$ leads to a complex variety to which algebraic topology and differential geometry with their cohomology theories apply.

Every embedding σ of K into a p-adic field $\widehat{\overline{\mathbb{Q}}}_p$ leads to crystalline cohomology spaces.

As the embeddings σ vary, with fixed target $\mathbb{C}$ or $\widehat{\overline{\mathbb{Q}}}_p$, the variation in the cohomology spaces is described
(partly by the action of Galois groups and partly)
by *Picard-Fuchs differential equations* and *Gauss-Manin connections.*

As yet no theory to mix the targets $\mathbb{C}$ and/or the various $\widehat{\overline{\mathbb{Q}}}_p$!

Why look for MOTIVE-STRING relation ?

Computations in Type IIb string theory proceed by manipulating solutions of certain differential equations. During the computations there are many denominators. In the end these drop out and true integers remain.

Many differential equations in Type IIb string can be recognized as Picard-Fuchs equations in De Rham cohomology of families of varieties.

The integrality statements can be recognized as consequences of theorems about the crystalline cohomology of families of ordinary varieties.

Challenge for Motive people:
Crystalline cohomology deals with only one prime p at a time and puts out statements about p-adic integrality.

What mechanism synchronizes the primes and leads to true integers?

Challenge for String people:
Crystalline cohomology implies extra symmetries in the differential equations.

Where are these extra symmetries in Nature?

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Line & Additive Group

arclength $\int_a^b dx = b - a$

invariant differential form dx
invariant derivation $\frac{d}{dx}$

$$dx = dz \Rightarrow z = x + y, \qquad y \text{ constant}$$

Circle

equations for the unit circle in the plane:

$$\begin{aligned} r &= 1 && \text{polar coordinates} \\ x^2+y^2 &= 1 && \text{Cartesian coordinates} \end{aligned}$$

arclength $\int_a^b \frac{dx}{\sqrt{1-x^2}} = \arcsin(b) - \arcsin(a)$

$$\begin{aligned} \arcsin(x) &= \sum_{n\geq 0} (-1)^n \binom{-\frac{1}{2}}{n} \frac{x^{2n+1}}{2n+1} \\ &= 2\sum_{n\geq 0} \frac{(2n)!}{n!^2} \frac{(x/2)^{2n+1}}{2n+1} \end{aligned}$$

better

$$\frac{1}{2}\arcsin(2x) = \sum_{n\geq 0} \frac{(2n)!}{n!^2} \frac{x^{2n+1}}{2n+1}$$

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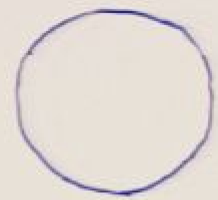

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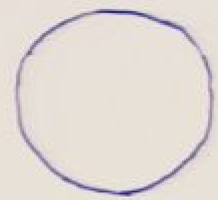

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invariant differential form $\dfrac{dx}{\sqrt{1-x^2}}$

invariant derivation $\sqrt{1-x^2}\,\dfrac{d}{dx}$

Addition law for trigonometric functions:

$$\frac{dx}{\sqrt{1-x^2}} = \frac{dz}{\sqrt{1-z^2}} \quad \Rightarrow$$

$$\begin{aligned} z &= \sin(\arcsin(x)+\arcsin(y)) \qquad\qquad y \text{ constant} \\ &= x\sqrt{1-y^2}+y\sqrt{1-x^2} \\ &= x+y+\sum_{n\geq 1}(-1)^n\binom{\frac{1}{2}}{n}(x\,y^{2n}+y\,x^{2n}) \end{aligned}$$

Better, with $\quad z=2w,\ x=2u,\ y=2v,$

$$w = u+v-4\sum_{n\geq 1}\frac{(2n-3)!}{n!(n-2)!}(u\,v^{2n}+v\,u^{2n})$$

The coefficients are integers:

$$-4\frac{(2n-3)!}{n!(n-2)!} = 2\binom{2n-2}{n} - 4\binom{2n-3}{n-1}$$

So, this is a formal group law over $\mathbb{Z}$ *!*

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Multiplicative Group

arclength $\int_a^b \frac{dx}{x} = \log(b) - \log(a)$

invariant differential form $\dfrac{dx}{x}$

invariant derivation $x\,\dfrac{d}{dx}$

$$\frac{dx}{x} = \frac{dz}{z} \quad \Rightarrow \quad z = xy\,, \quad y \text{ constant}$$

Coordinate change $\quad z = 1 + w,\ x = 1 + u,\ y = 1 + v$
gives

$$w = u + v + uv\,,$$

the standard multiplicative formal group law over $\mathbb{Z}$.

Multiplicative Group

integrate $\int_{x_0}^{x_1} \frac{dx}{x} = \ln(x_1) - \ln(x_0)$

invariant differential form $\frac{dx}{x}$

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Circle & Multiplicative Group

Substitution $\quad x \to ix,\ y \to iy,\ z \to iz$
transforms the addition law for trigonometric sine:

$$z = x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

into the addition law for hyperbolic sine:

$$z = x\sqrt{1+y^2} + y\sqrt{1+x^2}$$

i.e.
the two formal group laws are isomorphic over $\mathbb{Z}[i]$.

The series

$$\begin{aligned}\sinh(\log(1+u)) &= \frac{1}{2}\left((1+u)-(1+u)^{-1}\right)\\ &= u - \frac{1}{2}\sum_{n\geq 2}(-u)^n\end{aligned}$$

establishes an isomorphism, defined over $\mathbb{Z}[\frac{1}{2}]$, between the multiplicative group law and the addition law for the hyperbolic sine.

The addition law for the sine function and the multiplicative group law are isomorphic over $\mathbb{Z}[i][\frac{1}{2}]$.

Lemniscate & its group law

equations Bernouilli's Lemniscate in the plane:

$$r^2 = \cos(2\phi) \qquad \text{polar coordinates}$$
$$(x^2+y^2)^2 = x^2 - y^2 \qquad \text{Cartesian coordinates}$$

arclength $\displaystyle\int_a^b \frac{dx}{\sqrt{1-x^4}}$

$$\int \frac{dx}{\sqrt{1-x^4}} = \sum_{n\geq 0} (-1)^n \binom{-\frac{1}{2}}{n} \frac{x^{4n+1}}{4n+1}$$

$$= \sqrt{2} \sum_{n\geq 0} \frac{(2n)!}{n!^2} \frac{(x/\sqrt{2})^{4n+1}}{4n+1}$$

Coefficients can be made integral by substitution $x \to x\sqrt{2}$

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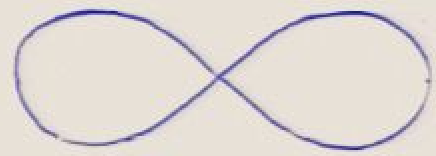

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Euler

$$\frac{dx}{\sqrt{1-x^4}} = \frac{dz}{\sqrt{1-z^4}} \quad \Rightarrow$$

$$z = \frac{x\sqrt{1-y^4}+y\sqrt{1-x^4}}{1+x^2y^2}, \qquad y \text{ constant}$$

Euler's result was the first example of an addition law for elliptic integrals.

This marked the beginning of the theory of elliptic curves !

The elliptic curve in this case is, in homogeneous coordinates in the weighted projective plane $\mathbb{P}^{[1,1,2]}$,

$$X^4+Y^4+Z^2=0$$

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One example of a Del Pezzo surface

dP_7 is:

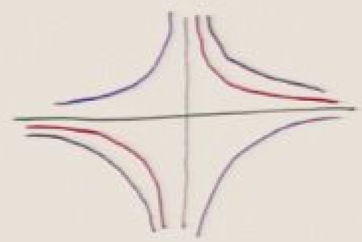
$$W^4 + X^4 + Y^4 + Z^2 = 0 \quad \text{in } \mathbb{P}^{[1,1,1,2]}$$

This is a branched double cover of $\mathbb{P}^{[1,1,1]}$;

On $\mathbb{P}^{[1,1,1]} = \mathbb{P}^2$, the usual projective plane,

there is the pencil of conics

$$W^2 - v XY = 0$$



This pencil of conics on $\mathbb{P}^2$ pulls back to a pencil of elliptic curves on dP_7:

$$v^2 X^2 Y^2 + X^4 + Y^4 + Z^2 = 0$$

Substitution $Z \mapsto Z - iv XY$ gives

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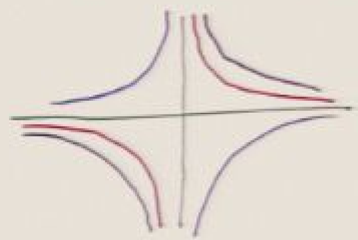

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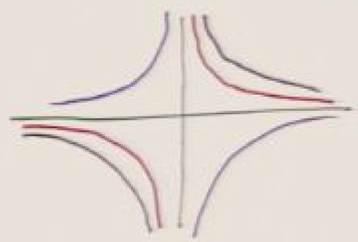

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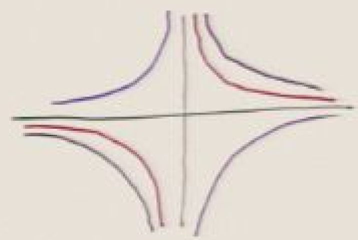

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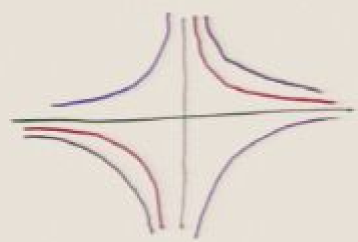

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$$X^4 + Y^4 + Z^2 + tXYZ = 0$$

with t a complex (deformation) parameter.

Viewed in $\mathbf{C}^3$ this equation has for $t^4 \neq 64$ only one singular point, namely $(0,0,0)$.

This is a so-called simple elliptic singularity known as the $\tilde{E}_7$ singularity.

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Viewed in $\mathbb{P}^{[1,1,2]}$ this equation describes for $t^4 \neq 64$ a smooth elliptic curve.

We have a family of elliptic curves with singular fibres at $t = \pm 2\sqrt{2},\ \pm 2\sqrt{-2},\ \infty$

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The elliptic curve

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a rank 2 *period lattice.*

One period (of a suitably normalized 1-form along a suitable closed path) can be computed via the residue theorem:

with $u = t^{-1}$ and $|u|$ sufficiently small:

$$
\begin{aligned}
f_0 &= \frac{1}{2\pi i} \oint_{\gamma_0} \frac{-t\,dx}{2z - tx}\bigg|_{x^4+1+z^2=txz,\ |z|<1} \\
&= \left(\frac{1}{2\pi i}\right)^2 \oint \oint \frac{-t\,dx\,dz}{x^4 + 1 + z^2 - txz} \\
&= \left(\frac{1}{2\pi i}\right)^2 \oint \oint \frac{1}{1 - ux^{-1}z^{-1}(x^4 + 1 + z^2)} \frac{dx}{x} \frac{dz}{z} \\
&= \sum_{n\geq 0} u^n \left(\frac{1}{2\pi i}\right)^2 \oint \oint \left(\frac{x^4 + 1 + z^2}{xz}\right)^n \frac{dx}{x} \frac{dz}{z} \\
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&= \sum_{n\geq 0} u^n \left(\frac{1}{2\pi i}\right)^2 \oint\oint \left(\frac{x^4 + 1 + z^2}{xz}\right)^n \frac{dx}{x}\frac{dz}{z} \\
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The elliptic curve

$$X^4 + Y^4 + Z^2 - tXYZ = 0$$

a rank 2 *period lattice.*

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A second period

$$f_1 = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{-t\,dx}{2z - tx}\bigg|_{x^4+1+z^2=txz,\ |z|<1}$$

can be determined from the formula (with $\epsilon^2 = 0$)

$$f_0 + f_1\epsilon \equiv \sum_{m\geq 0} \frac{(1+4\epsilon)_{4m}}{(1+\epsilon)_m^2\,(1+2\epsilon)_{2m}} u^{4m+4\epsilon}$$

using the rising Pochhammer symbol: for $k \geq 0$

$$(a)_k := a(a+1)\cdot\cdots\cdot(a+k-1)$$

(so $(1)_k = k!$)

Note

$$f_1 = 4f_0 \log u + g_1$$

where g_1 is a power series in u^4 with constant term 0.

Thus if we define τ and q by

$$\tau := \frac{f_1}{f_0}, \qquad q := \exp(\tau) = u^4 \exp(\frac{g_1}{f_0})$$

Then $q^{1/4}$ is another local coordinate on the u-line near $u = 0$.

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14

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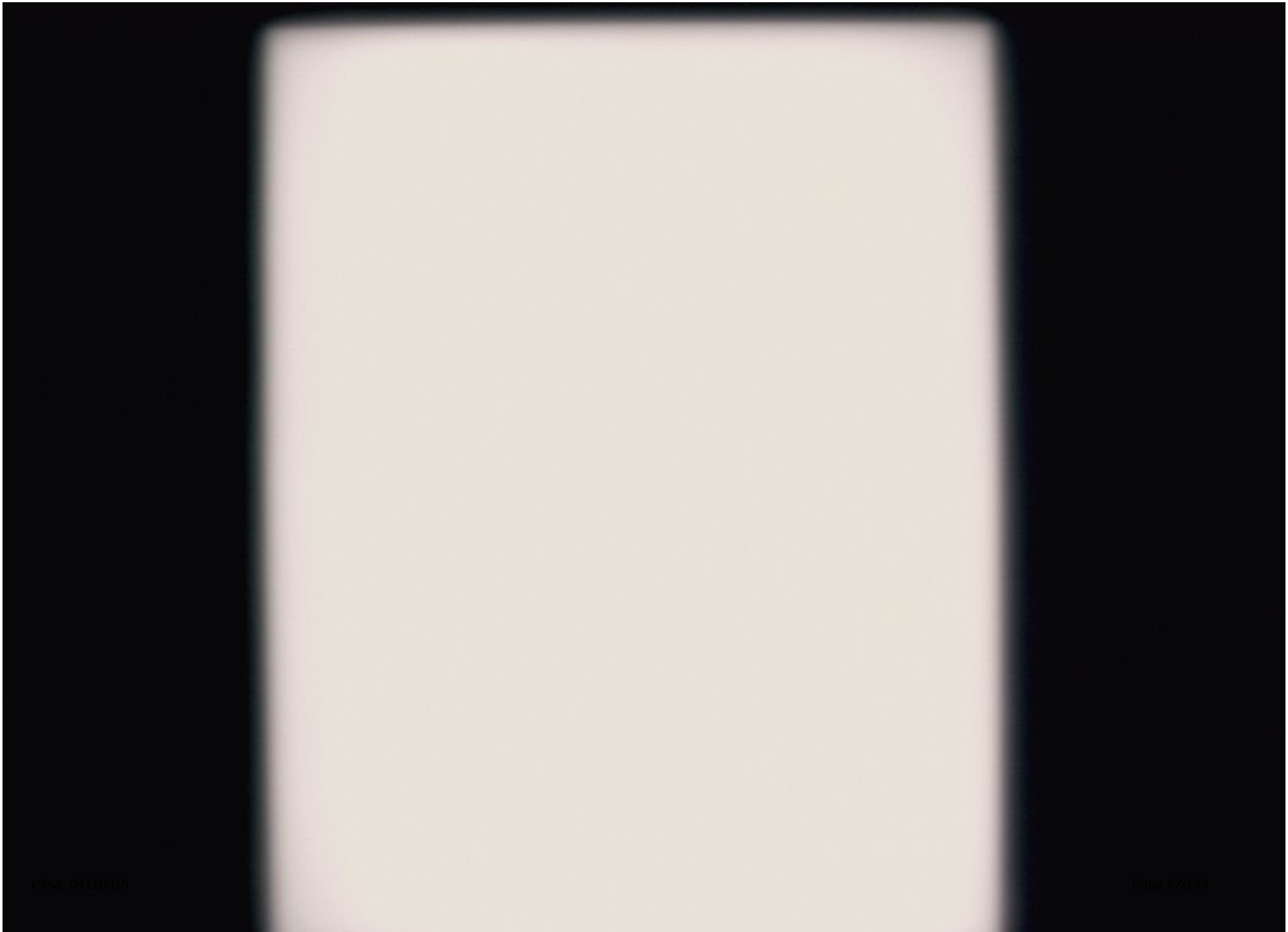

More formal group laws.

Put

$$\ell(u) = \int f_0\, du = \sum_{m \geq 0} \frac{(4m)!}{m!^2 (2m)!} \frac{u^{4m+1}}{4m+1}$$

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Then

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$$L^{-1}(L(x) + L(y))$$

are two formal group laws over $\mathbb{Z}$ and they are isomorphic over $\mathbb{Z}[i]$.

Moreover $L^{-1}(L(x) + L(y))$ is the integer version of the addition law for the lemniscate.

i.e. the base of the elliptic pencil

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carries in the neighborhood of $t = \infty$, $u = 0$ a formal group law over $\mathbb{Z}$, in the coordinate u, which is over $\mathbb{Z}[i]$ isomorphic to the formal group law of the fiber at $t = 0$, $u = \infty$

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Seiberg-Witten

"Electric-Magnetic Duality, Monopole Condensation, and Confinement in $N = 2$ Supersymmetric Yang-Mills Theory"

illustrate their general theory with a first example starring the functions

$$\mathfrak{a} = \int f_0 \, dt$$

$$\mathfrak{a}_D = \int f_1 \, dt$$

They show that $\mathfrak{a}$ and $\mathfrak{a}_D$ give the periods of some *meromorphic* 1*-form without residues* on the elliptic curve $X^3 + Y^3 + Z^3 - tXYZ = 0$.

Modulo ϵ^2:

$$\mathfrak{a} + \mathfrak{a}_D \epsilon \equiv -\sum_{m \geq 0} \frac{(1-4\epsilon)_{4m}}{(1-\epsilon)_m^4 (1-2\epsilon)_{2m}} \frac{u^{4m-1-4\epsilon}}{4m-1+4\epsilon}$$

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They show that a and a_D are the periods of some meromorphic 1-form [illegible] on the elliptic curve [illegible]

Modulus [illegible]

$$a(u) = \sum [illegible]$$

The functions $\mathfrak{a}$ and $\mathfrak{a}_D$ are used to construct the potential

$$K := \frac{1}{2i}(\bar{\mathfrak{a}}\,\mathfrak{a}_D - \mathfrak{a}\,\bar{\mathfrak{a}}_D)$$

for a so-called *rigid special Kähler metric* on the base space (Moduli space) of the pencil: the metric is

$$\begin{aligned}
\frac{\partial^2 K}{\partial\mathfrak{a}\,\partial\bar{\mathfrak{a}}}\, d\mathfrak{a}\, d\bar{\mathfrak{a}} &= \frac{1}{2i}\left(\frac{\partial\mathfrak{a}_D}{\partial\mathfrak{a}} - \frac{\partial\bar{\mathfrak{a}}_D}{\partial\bar{\mathfrak{a}}}\right) d\mathfrak{a}\, d\bar{\mathfrak{a}} \\
&= \frac{1}{2i}\left(\frac{f_1}{f_0} - \frac{\overline{f_1}}{\overline{f_0}}\right) f_0\overline{f_0}\, dt\, d\bar{t} \\
&= -2i(\tau - \bar{\tau}) f_0\overline{f_0}\, dt\, d\bar{t}
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$$(\mathfrak{a} - \mathfrak{a}_D\epsilon)\overline{(\mathfrak{a} - \mathfrak{a}_D\epsilon)} = \mathfrak{a}\bar{\mathfrak{a}} - 2iK\epsilon$$

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A Glimpse of Mirror Symmetry

Golyshev looked at the system of differential equations

$$\frac{d}{dt}\Phi = (t - A)^{-1} P \Phi$$

where

$$A = \begin{pmatrix} 12 & 552 & 7488 \\ 1 & 40 & 552 \\ 0 & 1 & 12 \end{pmatrix}, \qquad P = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

He showed that it has a solution

$$\Phi = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ \mathfrak{a}_D & \mathfrak{a} & 1 \end{pmatrix}$$

and he showed that the entries of A count how many curves of certain kinds there exist on the DelPezzo surface dP_7, i.e. the blow up of $\mathbb{P}^2$ at seven points, more precisely A is the matrix for multiplication with the anti-canonical class in the quantumcohomology of dP_7.

Note: classical theory relates this DelPezzo to the root system E_7.

Notice the coordinate translation $t \to t - \lambda$ has the same effect as replacing A by $A - \lambda$.

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$$A = \begin{pmatrix} 12 & 752 & 7488 \\ 1 & 60 & 752 \\ 0 & 1 & 12 \end{pmatrix} \qquad P = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

He showed that it has a solution

$$\Phi = \begin{pmatrix} [illegible] & [illegible] & 0 \\ [illegible] & [illegible] & 0 \\ \phi_2 & \phi & 1 \end{pmatrix}$$

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	(A,B,λ)	A
1.	$(432, 0, 60)$	$\begin{pmatrix} 60 & 20520 & 1339200 \\ 1 & 312 & 20520 \\ 0 & 1 & 60 \end{pmatrix}$
2.	$(64, 0, 12)$	$\begin{pmatrix} 12 & 552 & 7488 \\ 1 & 40 & 552 \\ 0 & 1 & 12 \end{pmatrix}$
3.	$(27, 0, 6)$	$\begin{pmatrix} 6 & 108 & 756 \\ 1 & 15 & 108 \\ 0 & 1 & 6 \end{pmatrix}$
4.	$(16, 0, 4)$	$\begin{pmatrix} 4 & 40 & 192 \\ 1 & 8 & 40 \\ 0 & 1 & 4 \end{pmatrix}$
5.	$(11, -1, 3)$	$\begin{pmatrix} 3 & 20 & 75 \\ 1 & 5 & 20 \\ 0 & 1 & 3 \end{pmatrix}$
6.	$(7, -8, 2)$	$\begin{pmatrix} 2 & 12 & 36 \\ 1 & 3 & 12 \\ 0 & 1 & 2 \end{pmatrix}$

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$$\begin{pmatrix} 6 & 108 & 756 \\ 1 & 15 & 108 \\ 0 & 1 & 6 \end{pmatrix}$$

4. $(16, 0, 4)$
$$\begin{pmatrix} 4 & 40 & 192 \\ 1 & 8 & 40 \\ 0 & 1 & 4 \end{pmatrix}$$

5. $(11, -1, 3)$
$$\begin{pmatrix} 3 & 20 & 75 \\ 1 & 5 & 20 \\ 0 & 1 & 3 \end{pmatrix}$$

6. $(7, -8, 2)$
$$\begin{pmatrix} 2 & 12 & 36 \\ 1 & 3 & 12 \\ 0 & 1 & 2 \end{pmatrix}$$

	u_m	$c_n = u_m$	
1.	$\dfrac{(6m)!}{m!\,(2m)!\,(3m)!}$	if $6 \mid n$	$m = \frac{n}{6}$
2.	$\dfrac{(4m)!}{m!^2\,(2m)!}$	if $4 \mid n$	$m = \frac{n}{4}$
3.	$\dfrac{(3m)!}{m!^3}$	if $3 \mid n$	$m = \frac{n}{3}$
4.	$\dfrac{(2m)!^2}{m!^4}$	if $2 \mid n$	$m = \frac{n}{2}$
5.	$\sum_{k=0}^{m} \binom{m}{k}^2 \binom{m+k}{k}$		$m = n$
6.	$\sum_{k=0}^{m} \binom{m}{k}^3$		$m = n$

$c_n = 0$ otherwise

c_n = constant term of $p(x,y)^n$

Recurrence relation

$$(m+1)^2 u_{m+1} - (Am^2 + Am + \lambda) u_m + B m^2 u_{m-1} = 0$$

	u_m	$c_n = u_m$	
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3.	$\dfrac{(3m)!}{m!^3}$	if $3 \mid n$	$m = \frac{n}{3}$
4.	$\dfrac{(2m)!^2}{m!^4}$	if $2 \mid n$	$m = \frac{n}{2}$
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Recurrence relation

$$(m+1)^2 u_{m+1} - (A m^2 + A m + \lambda) u_m + B m^2 u_{m-1} = 0$$

$u_0 = 1$

1. $X^2 + Y^3 + Z^6$

2. $X^2 + Y^4 + Z^4$

3. $X^3 + Y^3 + Z^3$

4. $(X+Y)\,(XY + Z^2)$

5. $(X+Y+Z)(X+Z)(Y+Z)$

6. $(X+Y)(Y+Z)(Z+X)$

$$\underbrace{\qquad\qquad\qquad}_{P(X,Y,Z)}$$

$$p(x,y) = x^{-1} y^{-1} P(x,y,1)$$

$$p:\ \mathbb{C}^* \times \mathbb{C}^* \longrightarrow \mathbb{C}$$

level sets are elliptic curves
minus a few points)*

critical values $\Longleftrightarrow$ singular fibres

physics: p is Landau - Ginzburg potential

interested in critical points of p

More formal group laws.

Recall

$$\mathfrak{a} = \int f_0\, dt = -\int f_0 \frac{du}{u^2} = -\sum_{m \geq 0} \frac{(4m)!}{m!^2 (2m)!} \frac{u^{4m-1}}{4m-1}$$

Note that the coefficients

$$\frac{(4m)!}{m!^2(2m)!}\frac{1}{4m-1} = 4\frac{(4m-3)!}{m!^2(2m-2)!}$$

$$= -2\frac{(4m-2)!}{m!^2(2m-2)!} + 8\frac{(4m-3)!}{m!(m-1)!(2m-2)!}$$

are sums of multinomial coefficients and, hence, are integers.

This can also be stated as: *The function*

$$\frac{1}{\mathfrak{a}} = u + 4u^5 + 76u^9 + 2224u^{13} + \dots$$

is the logarithm of a formal group law over $\mathbb{Z}$ in the coordinate u which is over $\mathbb{Z}$ isomorphic to the additive group law.

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Lerche, Mayr, Warner

'Non-critical Strings, Del Pezzo Singularities and Seiberg-Witten curves'

work in one of their examples with the functions

$$\mathsf{b} = 4\int f_0 \frac{du}{u}$$

$$\mathsf{b}_D = 4\int f_1 \frac{du}{u}$$

They relate these to periods of *multivalued 1-forms* on the elliptic curve $X^4 + Y^4 + Z^2 - tXYZ = 0$.

They also define a function $\mathcal{F}$ so that

$$\mathsf{b}_D = \frac{d\mathcal{F}}{d\mathsf{b}}.$$

The functions b and $\mathcal{F}$ are then used to construct the potential

$$K = -\log(\mathrm{Im}\mathcal{F} - \mathrm{Im}\mathsf{b}\,\mathrm{Re}\mathsf{b}_D)$$

for a so-called *local special Kähler metric* on the base space (Moduli space) of the pencil.

From curves to surfaces to threefolds.

The total space of our elliptic pencil can be embedded as the surface

$$\mathcal{S}: \qquad (X^4 + Y^4 + Z^2)U - XYZT = 0$$

of bidegree $(4,1)$ in $\mathbb{P}^{[1,1,2]} \times \mathbb{P}^{[1,1]}$.

The elliptic pencil arises via the projection from $\mathcal{S}$ onto the projective line $\mathbb{P}^{[1,1]}$. Projection onto the weighted projective plane $\mathbb{P}^{[1,1,2]}$ shows $\mathcal{S}$ as the blow up of this plane in 8 points.

One DelPezzo surface dP_7 is given by

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Lerche, Mayr, Warner model their work on the family of non-compact Calabi-Yau threefolds

$$V^{-4} + W^4 + X^4 + Y^4 + Z^2 + \mu VWXYZ = 0$$

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From

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\begin{aligned}
f_0 + f_1\epsilon &= \sum_{m\geq 0} \frac{(1+4\epsilon)_{4m}}{(1+\epsilon)_m^2\,(1+2\epsilon)_{2m}} u^{4m+4\epsilon} + \mathcal{O}(\epsilon^2) \\
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Then

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and thus

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$$

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$$
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A Glimpse of Mirror Symmetry

The function $\mathcal{F} - \frac{\mathfrak{b}^3}{6}$ is a function of $e^{\mathfrak{b}}$. It can be written as a series

$$\mathcal{F} = \frac{1}{6}\mathfrak{b}^3 + \sum_{k \geq 1} a_k \mathrm{Li}_3(e^{k\mathfrak{b}})$$

with *integers* a_k;
but no proof yet for the observed integrality.

Li_3 is the *trilogarithm function*;

more generally, for $m \in \mathbb{Z}_{\geq 1}$, the *m-logarithm function* is

$$\mathrm{Li}_m(x) := \sum_{n \geq 1} \frac{x^n}{n^m}$$

The integers a_k count *instantons*.

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$$\begin{aligned} \tau &:= \frac{f_1}{f_0} = \frac{u d\mathfrak{b}_D/du}{u d\mathfrak{b}/du} = \frac{d\mathfrak{b}_D}{d\mathfrak{b}} \\ &= \mathfrak{b} - \sum_{k\geq 1} k^2 a_k \mathrm{Li}_1(e^{k\mathfrak{b}}) \end{aligned}$$

Hence, setting $Q := e^{\mathfrak{b}}$, we find

$$q := \exp(\tau) = Q \prod_{k\geq 1} (1 - Q^k)^{-k^2 a_k}$$

So, the numbers a_k can be recovered directly from the relation between q and Q.

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It can be shown that the series expansion of Q

$$Q = e^{\mathfrak{b}} = u^4 \exp\left(\sum_{m\geq 1} \frac{(4m)!}{m!^2(2m)!} \frac{u^{4m}}{m}\right)$$

is a power series in u^4 with of all its coefficients in $\mathbb{Z}$.

So if we define $Q(x) = x\exp\left(\sum_{m\geq 1} \frac{(4m)!}{m!^2(2m)!} \frac{x^m}{m}\right)$

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\begin{aligned}
\mathfrak{b} &= 4\int f_0 \frac{du}{u} \\
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So $-\frac{1}{4}\mathrm{Re}\,\mathfrak{b}$ is the *logarithmic Mahler measure* of the Laurent polynomial

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gives examples in which the L-function of a K3-surface is computed from the Picard-Fuchs equation of an elliptic pencil on the K3-surface

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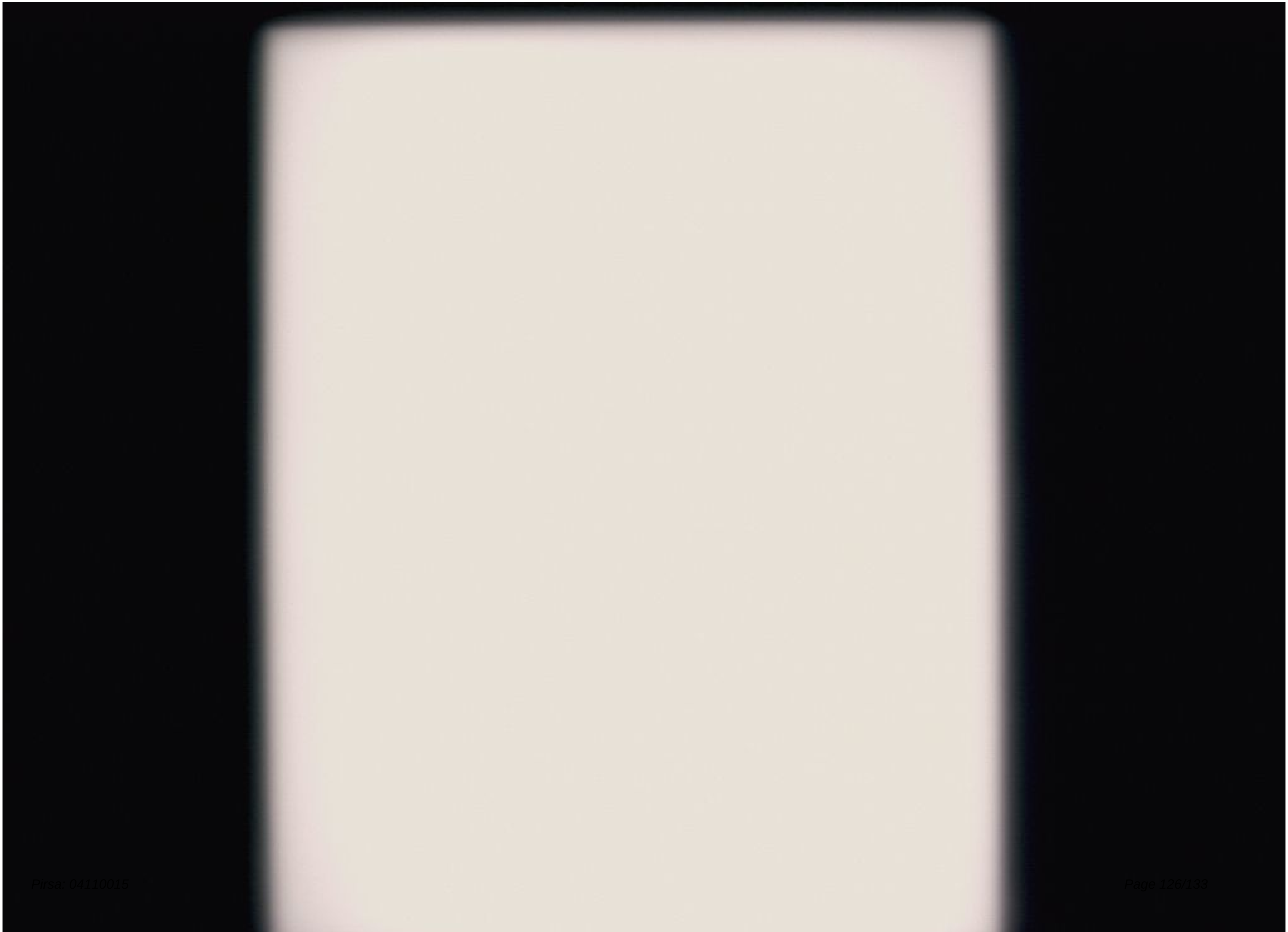

Modular Forms

$u(\tau)$ is a modular function

$f_0(u(\tau))$ is a modular form of weight 1

$\frac{du}{d\tau}$ is a modular form of weight 2

All integrands

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