

Title: Geometry with B-fields

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Abstract:

GEOMETRY WITH B-FIELDS

Nigel Hitchin (Oxford)

Coxeter Lectures – Perimeter Institute
November 17th 2004

No Signal

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GEOMETRY WITH B-FIELDS

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- NJH: *Generalized Calabi-Yau manifolds*
math.DG/0209099 (QJM 54 (2003) 281–308)
- Marco Gualtieri: *Generalized complex geometry*
math.DG/0401221
- Frederik Witt (D Phil student)
- Gil Cavalcanti (D Phil student)

- A Kapustin, *Topological strings on noncommutative manifolds* **hep-th/0310057**
- D Huybrechts, *Generalized Calabi-Yau structures, K3 surfaces, and B-fields* **math.AG/0306162**
- U Lindström, R Minasian, A Tomasiello, M Zabzine, *Generalized complex manifolds and supersymmetry*, **hep-th/0405085**
- O Ben-Bassat, *Mirror Symmetry and Generalized Complex Manifolds*, **math.AG/0405303**

BASIC SCENARIO

- manifold M^n
- replace T by $T \oplus T^*$
- inner product of signature (n, n)

$$(X + \xi, X + \xi) = -i_X \xi$$

- skew adjoint transformations:

$$\text{End} T \oplus \Lambda^2 T^* \oplus \Lambda^2 T$$

- in particular $B \in \Lambda^2 T^*$

TRANSFORMATIONS

- exponentiate B :

$$X + \xi \mapsto X + \xi + i_X B$$

- $B \in \Omega^2$... *the B-field*
- natural group $\text{Diff}(M) \ltimes \Omega^2(M)$

DIFFERENTIAL FORMS = SPINORS FOR $T \oplus T^*$

- Invariant bilinear pairing (*Mukai pairing*) with values in $\Lambda^n T^*$:

$$\langle \varphi, \psi \rangle = [\varphi \wedge \sigma(\psi)]_n$$

$$\sigma(\varphi_0 + \varphi_2 + \varphi_4 + \dots) = \varphi_0 - \varphi_2 + \varphi_4 - \dots$$

- $\langle e^B \varphi, e^B \psi \rangle = \langle \varphi, \psi \rangle$

- cf. Chern character $\text{ch} : K(M) \rightarrow H^{ev}(M)$

$$\text{ch}(E^*) = \sigma \text{ch}(E)$$

$$\text{ch}(LE) = e^{c_1(L)} \text{ch}(E)$$

Apply a 2-form B ...

- $d \mapsto e^{-B}de^B = d + dB$
- $[X + \xi, Y + \eta] \mapsto [X + \xi, Y + \eta] - 2i_X i_Y dB$
- ... more generally *twist* by a closed 3-form H :

$$d \mapsto d_H = d + H : \Omega^{od} \rightarrow \Omega^{ev}$$

$$[X + \xi, Y + \eta] \mapsto [X + \xi, Y + \eta] - 2i_X i_Y H$$

GENERALIZED GEOMETRIC STRUCTURES

- $SO(n, n)$ compatibility
- integrability $\sim d$ or Courant bracket
- $\sim d + H$ or twisted Courant bracket
- transform by $\text{Diff}(M) \ltimes \Omega_{closed}^2(M)$

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DIFFERENTIAL FORMS AS SPINORS

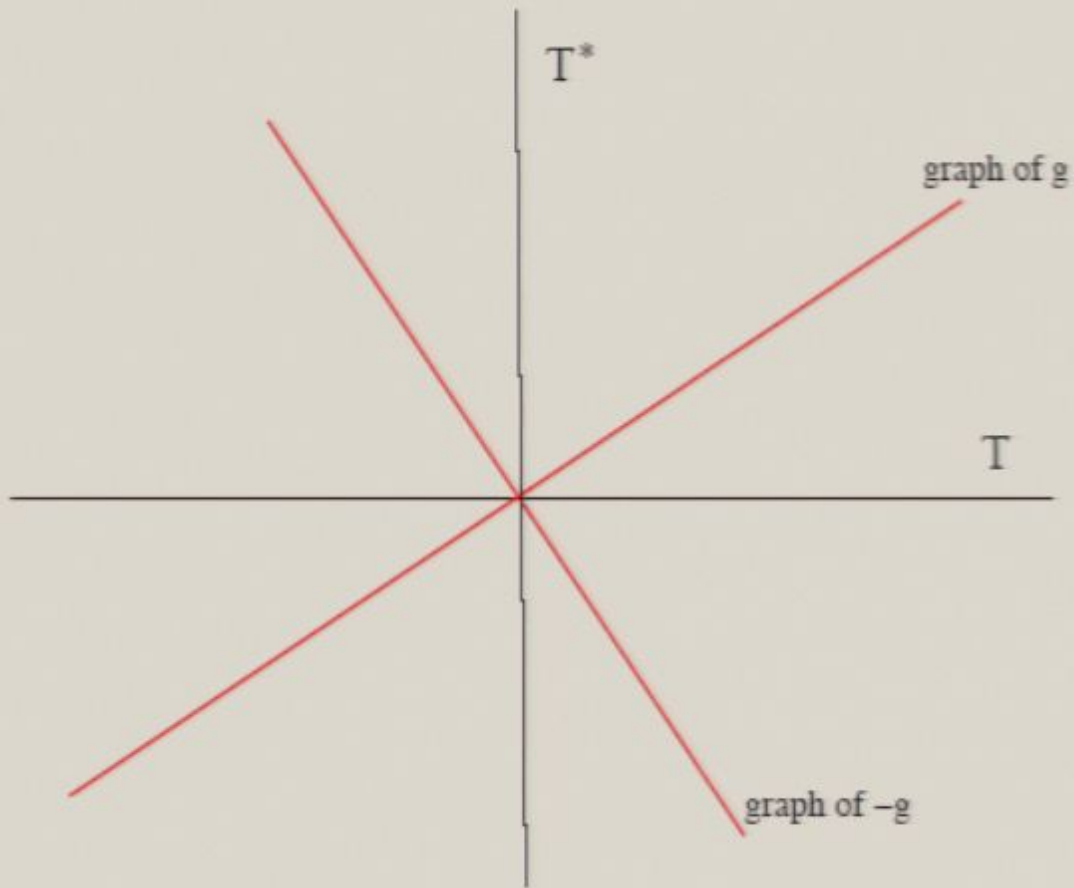
- No exterior product — *use Mukai pairing*
- No interior product with vector fields — *use Clifford product*
- No \mathbb{Z} -grading — *only even/odd*
- Use exterior differential d or $d + H$

RIEMANNIAN METRIC

- Riemannian metric g_{ij}
- $X \mapsto g(X, -) : g : T \rightarrow T^*$
- *graph* of $g = V \subset T \oplus T^*$
- $T \oplus T^* = V \oplus V^\perp$

GENERALIZED RIEMANNIAN METRIC

- $V \subset T \oplus T^*$ positive definite rank n subbundle
- = graph of $g + B : T \rightarrow T^*$
- $g + B \in T^* \otimes T^*$: g symmetric, B skew
- = B-field transform of a metric

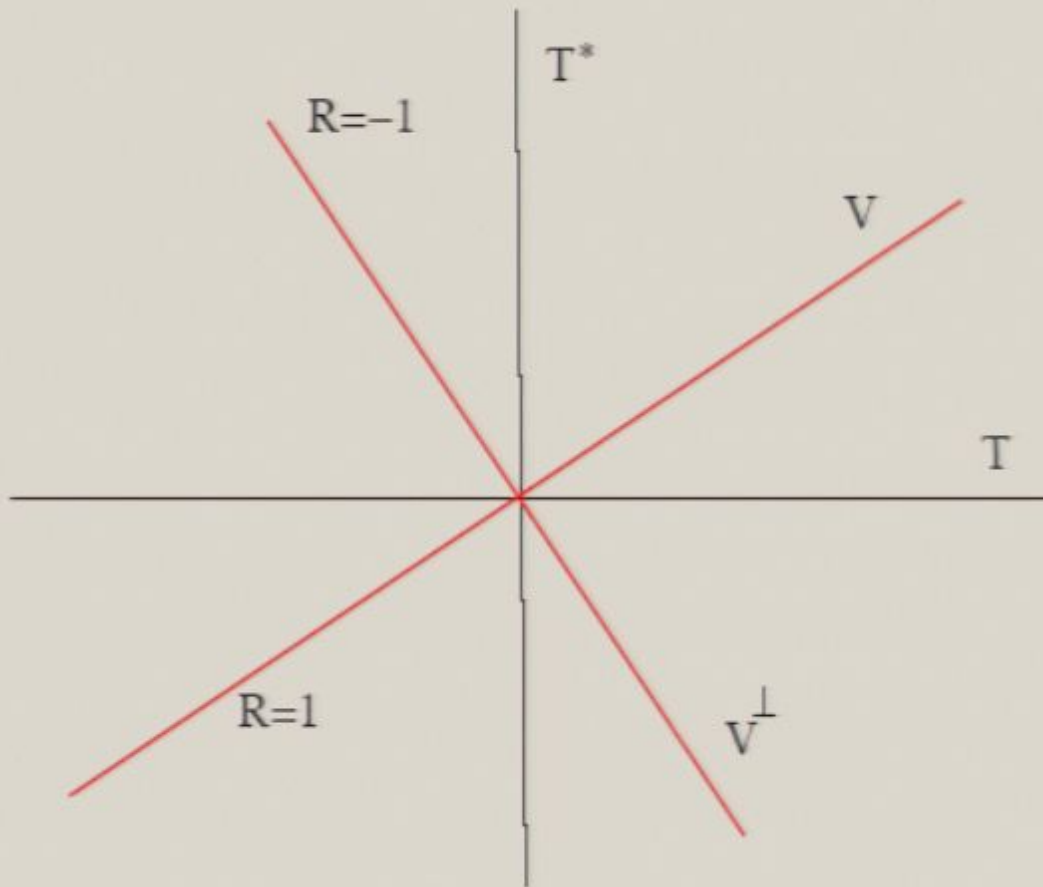


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... OR ALTERNATIVELY

- Define $R \in \text{End}(T \oplus T^*)$, $R^* = R$, $R^2 = 1$ by...
- $R(v) = v$ on V , $R(v) = -v$ on V^\perp



- lift $R \in O(n, n)$ to $\tilde{R} \in Pin(n, n)$
- $\tilde{R} : \Omega^* \rightarrow \Omega^*$ *generalized Hodge star operator*
- $(B = 0, \tilde{R} = \sigma^*)$



EXAMPLES OF GENERALIZED GEOMETRIES

- Generalized G_2
- Generalized complex structures
- Generalized Kähler structures

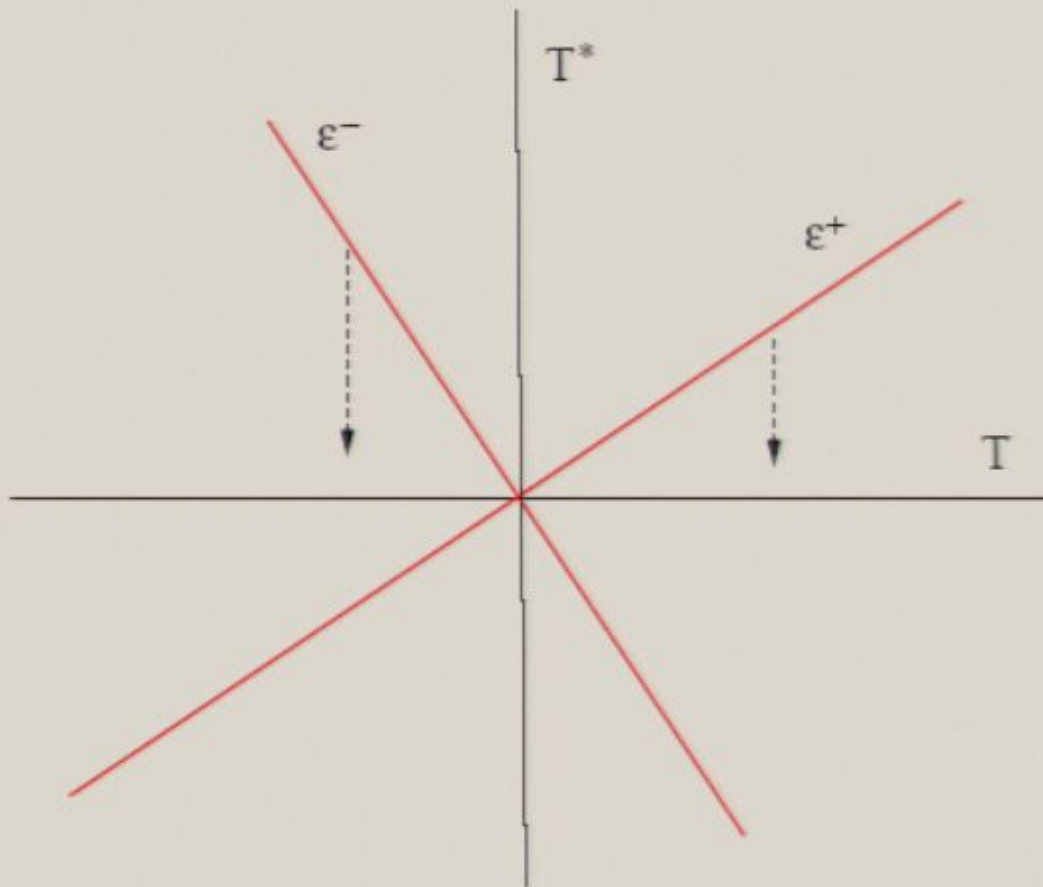
Superstrings with Intrinsic Torsion, J P Gauntlett, D Martelli and D Waldram, Phys.Rev. **D69** (2004)

"...A type II geometry will preserve supersymmetry if and only if there is at least one ϵ^+ or ϵ^- satisfying"

$$\begin{aligned}\nabla_M^\pm \epsilon^\pm &\equiv \left(\nabla_M \pm \frac{1}{8} H_{MNP} \Gamma^{NP} \right) \epsilon^\pm = 0, \\ \left(\Gamma^M \partial_M \Phi \pm \frac{1}{12} H_{MNP} \Gamma^{MNP} \right) \epsilon^\pm &= 0,\end{aligned}$$

....locally the three-form is given by $H = dB$

- Riemannian connections ∇^\pm with *skew torsion* $\pm dB \in \Omega^3$
- unit spinors ϵ^\pm , covariant constant $\nabla^\pm \epsilon^\pm = 0$
- function $\Phi : (d\Phi \pm dB) \cdot \epsilon^\pm = 0$



7 DIMENSIONS

- Clifford multiplication $\Lambda^{od/ev}T^* \cong \text{End}(S) \cong S \otimes S$

- $\rho = e^\Phi e^b(\epsilon^+ \otimes \epsilon^-) \in \Lambda^{od}$


- $\Phi : 1 \quad b : 21 \quad g : 28 \quad \epsilon^+ : 7 \quad \epsilon^- : 7$

$$1 + 21 + 28 + 7 + 7 = 64 = \dim \Lambda^{od}$$

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 $1 + 21 + 28 + 7 + 7 = 64 = \dim \Lambda^{od}$

- open $SO(7, 7) \times \mathbf{R}^*$ orbit
- stabilizer $G_2 \times G_2$ ($\dim 1 + 14 \cdot 13/2 - 64 = 28$)

EQUATIONS:

- $d\rho = 0$

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unfortunately..... if M is compact, then

- $H = 0$
- $\Phi = \text{const.}$
- $\nabla^+ = \nabla^- = \text{Levi-Civita connection}$
- standard G_2 manifold \dagger closed B-field.

GENERALIZED COMPLEX STRUCTURES

A *generalized complex structure* is:

- $J : T \oplus T^* \rightarrow T \oplus T^*, J^2 = -1$
- $(JA, B) + (A, JB) = 0$
- if $JA = iA, JB = iB$ then $J[A, B] = i[A, B]$ (*Courant bracket*)
- $U(m, m) \subset SO(2m, 2m)$ structure on $T \oplus T^*$

EXAMPLES

(*i*-eigenspace of $J = E \subset (T \oplus T^*) \otimes \mathbb{C}$)

- complex manifold $J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$

$$E = [\dots \partial/\partial z_i \dots, \dots d\bar{z}_i \dots]$$

- symplectic manifold $J = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$

$$E = [\dots, \partial/\partial x_j + i \sum \omega_{jk} dx_k, \dots]$$

GENERALIZED COMPLEX SUBMANIFOLDS

Submanifold $Y \subset M$,

- $0 \rightarrow TY \rightarrow TM|_Y \rightarrow NY \rightarrow 0$
- $TY \oplus (TY)^\circ = TY \oplus N^*Y \subset (TM \oplus T^*M)|_Y$
- J preserves $TY \oplus N^*Y \Leftrightarrow$ generalized complex submanifold

THE COMPLEX CASE

Generalized complex submanifold = *complex submanifold*

THE SYMPLECTIC CASE

$$\begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = \begin{pmatrix} -\omega^{-1}\xi \\ \omega X \end{pmatrix}$$

- $\omega^{-1}\xi \in TY \Rightarrow Y$ coisotropic $\omega X \in N^* \Rightarrow Y$ isotropic
- $\Rightarrow Y$ is *Lagrangian submanifold*

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Generalized complex submanifold = *complex submanifold*

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- a generalized complex submanifold is not necessarily a generalized complex manifold
- the intersection of two generalized complex submanifolds is not necessarily generalized complex
- the whole manifold may not be a generalized complex submanifold
- a point may not be a generalized complex submanifold

EXAMPLE: COMPLEX SYMPLECTIC MANIFOLD

- complex manifold M
- holomorphic symplectic form $\omega = \omega_1 + i\omega_2$
- $\omega_2 \Rightarrow$ symplectic form \Rightarrow generalized complex structure
- **B-field** $B = \omega_1$
- a generalized complex submanifold is a **coisotropic complex submanifold** (e.g. any complex hypersurface)

EXAMPLE: K3 SURFACE

Generalized complex submanifolds: complex coisotropic

- the complex manifold M itself
- any holomorphic curve
- **NO** points

EXAMPLE: CP^2 AS A COMPLEX POISSON MANIFOLD

Holomorphic section of $\Lambda^2 T \sim$ cubic curve $C \Rightarrow$ generalized complex structure.

Generalized complex submanifolds:

- the complex manifold M itself
- any holomorphic curve
- all points on C
- cf *non-commutative complex plane*

COHOMOLOGY

DECOMPOSITION OF SPINORS

- spin representation for $U(m, m)$
- $S^+ = (\Lambda^{0,0} + \Lambda^{0,2} + \dots + \Lambda^{0,2m})K^{1/2}$
 $S^- = (\Lambda^{0,1} + \Lambda^{0,3} + \dots + \Lambda^{0,2m-1})K^{1/2}$
- differential forms = spinors for $SO(2m, 2m)$
- $\Lambda^{ev/od}$ decomposes into eigenspaces under the Lie algebra action of J ($J^* = -J$)
- $U_{m-k} = \Lambda^{0,k}K^{1/2}$ has eigenvalue $i(m - k)$

THE GENERALIZED dd^c -LEMMA

- define $d^c = J^{-1}dJ$ with the Lie group action ($J^* = J^{-1}$) of J
- dd^c -lemma: $d\alpha = 0, \alpha = d^c\beta \Rightarrow \alpha = dd^c\gamma$
- complex structure: usual dd^c -lemma (e.g. Kähler) \Rightarrow generalized dd^c -lemma
- symplectic structure: strong Lefschetz theorem (e.g. Kähler) \Rightarrow generalized dd^c -lemma
- generalized Kähler: generalized dd^c -lemma holds (Gualtieri)

HODGE DECOMPOSITION

If the generalized dd^c -lemma holds, the cohomology has a decomposition

$$H^*(M, \mathbb{R}) = \bigoplus_{-m}^m U_k$$

- complex structure:

$$U_k = \bigoplus_{p-q=k} H^{p,q}(M)$$

- symplectic structure:

$$U_k \cong H^{m+k}(M)$$

THE SYMPLECTIC CASE

$$J = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

Lie algebra action on spinors is

$$(L - \Lambda)\alpha = \omega \wedge \alpha - i_{\omega^{-1}}\alpha$$

L, Λ generate an $SL(2, \mathbb{R})$ action on forms:

$$[\Lambda, L] = H, \quad [H, L] = -2L, \quad [H, \Lambda] = 2\Lambda$$

$$H = (m - p) \text{ on } p\text{-forms}$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \Lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

•

$$P = \begin{pmatrix} 1 & i/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} = \exp i\Lambda/2 \exp iL$$

$$L - \Lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = P^{-1}iHP$$



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HOMOLOGY CLASSES DEFINED BY GENERALIZED COMPLEX SUBMANIFOLDS

- **Prop:** *A compact generalized complex submanifold $Y^n \subset M^{2m}$ is Poincaré dual to a class in $U_0 \cap H^{2m-n}(M, \mathbb{Z})$*
- symplectic: $L\alpha = 0, \Lambda\alpha = 0 \Rightarrow H\alpha = [\Lambda, L]\alpha = 0$
- $\Rightarrow \alpha \in H^m(M^{2m}, \mathbb{R})$ primitive class

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QUESTIONS

- Is there a generalized Hodge conjecture?
- For what classes $a \in H^*(Y)$ is $i_*a \in H^*(M)$ in U_0 ?
- What is a generalized holomorphic bundle?
- What has all this to do with noncommutative geometry?
....etc.

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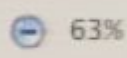
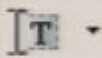
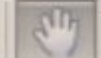
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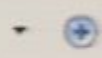
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