Title: Pre- and Post-Selection Paradoxes, Measurement-Disturbance and Contextuality in Quantum Mechanics

Date: Nov 10, 2004 04:00 PM

URL: http://pirsa.org/04110008

Abstract:


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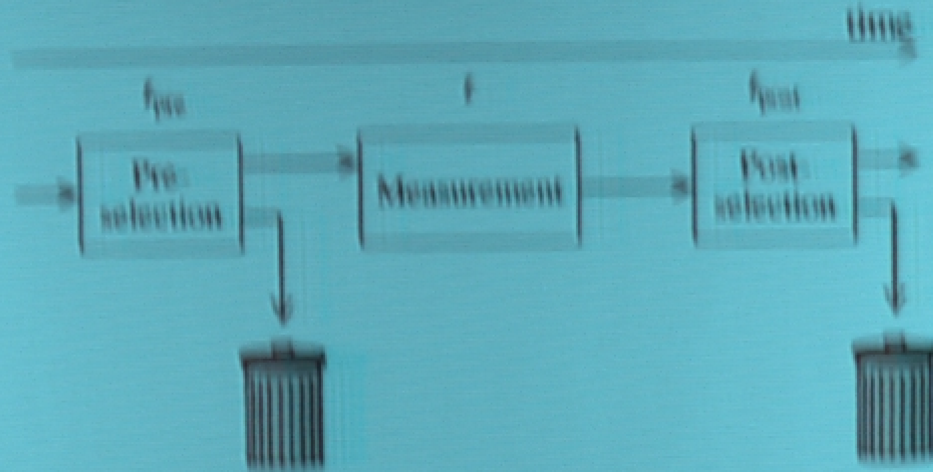
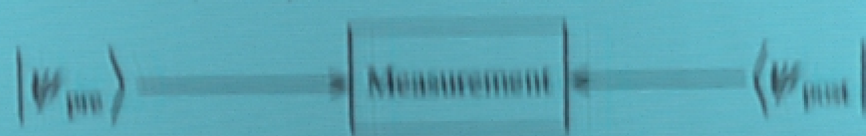
# Background and Motivation

- Pre- and Post-Selection was introduced by Aharonov, Bergmann and Lebowitz (ABL) in 1964.




- Two-State-Vector Formalism (TSVF) (see Aharonov & Vaidman, 2001)




- Foundational motivation : Time symmetry in the quantum measurement process.

## Background and Motivation

- Suggests possibilities for interpretation of QM.
  - Reality of both states - Aharonov (2001):
    - "Even at present, before the 'future' measurements, the backward evolving quantum state [...] exists! It exists in the same way as the quantum state evolving from the past exists. An element of arbitrariness: 'Why this particular outcome and not some other?' might discourage, but the alternative [...] – the collapse of the quantum wave – is clearly worse than that."
  - Counterfactual interpretation - Mohrhoff (2000):
    - "ABL probabilities are based on a complete set of facts, and are therefore objective, only if none of the measurements to the possible results of which they are assigned is actually performed (that is, only if between the 'preparation' or preselection and the 'retroparation' or postselection no measurement is performed)."

$$|\psi_{\text{pre}}\rangle \longrightarrow \boxed{\text{Measurement}} \longleftarrow \langle\psi_{\text{post}}|$$

$$|\psi_{\text{pre}}\rangle \text{———————} \langle\psi_{\text{post}}|$$

## Background and Motivation

- The counterfactual interpretation has several difficulties
  - We will see that "Logical PPS paradoxes" would imply that reality is contextual.
  - Further, a similar interpretation of "classical" probability would also imply that reality is contextual.
    - Reason - It does not take the possibility that measurements might disturb the system into account.
  - There is no compelling reason to treat the ABL probabilities counterfactually.
- See *Kastner Phil. Sci. 70, 145 (2003)* for refs.
- Main results:
  - Logical PPS paradoxes can be explained in noncontextual hidden variable theories, provided measurements disturb the values of the hidden variables.
  - However, for every logical PPS paradox there is a related proof of contextuality, which can be constructed from the same set of measurements.

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## Why should we care about PPS systems?

- Despite the controversy over interpretation, the TSVF agrees with QM for actual measurements.
- TSVF has yielded results of interest to quantum information theorists:
  - The Mean King's Problem (Vaidman et. al., 1987)
  - Cryptography protocols (Bub, 2000, Botero & Reznik, 1999)
- The formalism has been extended to more general types of pre- and post-selection (Aharovov & Reznik, 1995).
  - e.g. correlated pre- and post-selections of the type used in standard quantum cryptography (BB84, B92, etc.)
  - There is an analog of entanglement, which might be regarded as a resource for qinfo.
- It seems likely that many PPS effects could be exploited in quantum information protocols.
  - To find out if this is the case, we need to know whether these effects can be simulated "classically" and, if so, how efficiently? This means that we should re-examine the foundational questions with this in mind.

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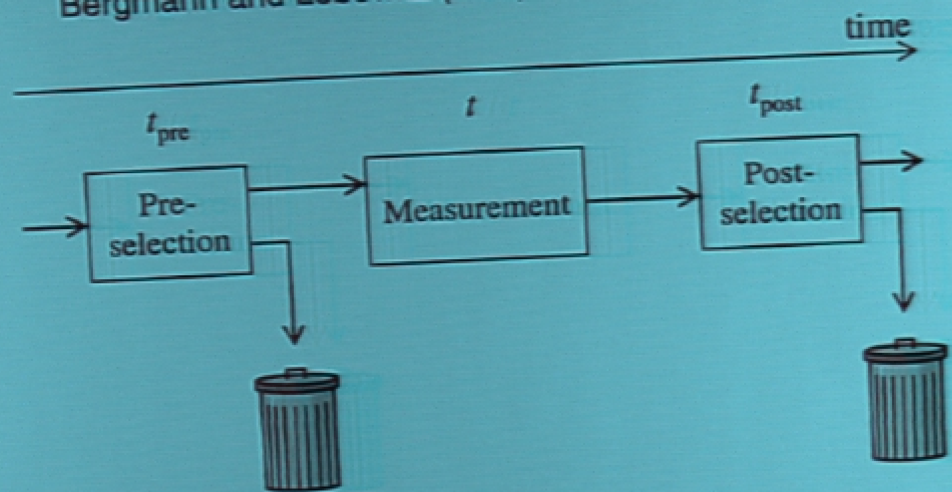


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## Outline

1) Operational Notation

2) Pre- and Post-Selection in Quantum Theory
   1) Quantum Measurement Formalism
   2) The ABL Rule
   3) The "Three-Box Paradox"

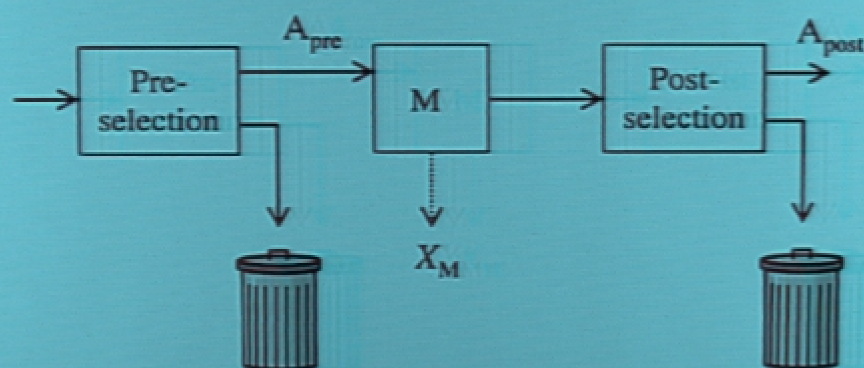
3) Hidden Variable Theories
   1) The "Partitioned-Box Paradox"
   2) General Formalism of HVTs
   3) Pre- and Post-Selection in HVTs
   4) Noncontextuality
   5) Noncontextuality, PPS and Disturbance

4) A Surprising Theorem
   1) Example: "Three-box" and Clifton
   2) Example: "Failure of the Product Rule"

5) Conclusions and Open Questions

## 1) Operational Notation

- M - A measurement.
- $X_M \in \{1, 2, \ldots, n\}$ - The outcome of M.
- pre/post - Refers to pre/post-selection.
- $A_{pre}/A_{post}$ - The occurrence of successful pre/post-selection.




- Goal: Compute $p(X_M = j \mid A_{pre}, A_{post}, M)$

## 2) Pre- and Post-Selection in Quantum Theory

### 2.1) Quantum Measurement Formalism

- States: Density operators, $\rho$, on a $d$-dimensional H. S.
- Sharp quantum measurements
  - Statistical aspect of M given by a PVM $\{P_{M,j}\}$

$$P_{M,j}^2 = P_{M,j} \qquad \sum_j P_{M,j} = I$$

Born rule: $p_\rho(X_M = j) = \mathrm{Tr}(P_{M,j}\rho)$

  - Transformation aspect given by CP-maps $\{\mathcal{E}_{M,j}\}$

$$\mathcal{E}_{M,j}^{\#}(I) = P_{M,j} \quad \text{where} \quad \mathrm{Tr}(\mathcal{E}^{\#}(A)B) = \mathrm{Tr}(A\mathcal{E}(B))$$

  - On obtaining $X_M = j$:

$$\rho \to \mathcal{E}_{M,j}(\rho) / \mathrm{Tr}(\mathcal{E}_{M,j}(\rho))$$

  - Lüders Rule (projection postulate)

$$\mathcal{E}_{M,j}(\rho) = P_{M,j}\,\rho\, P_{M,j}$$

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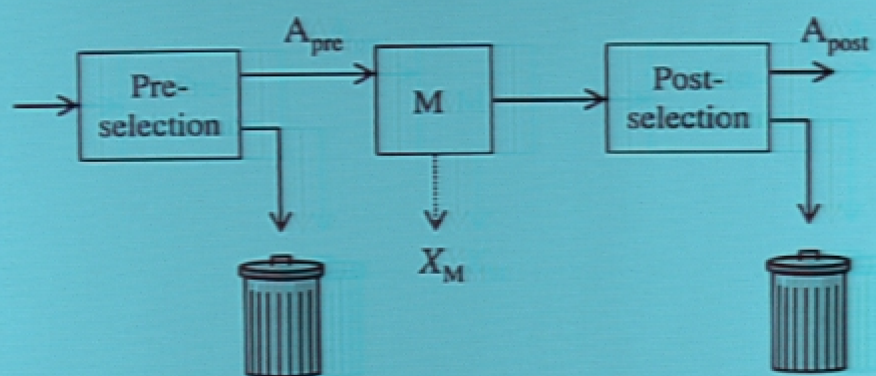


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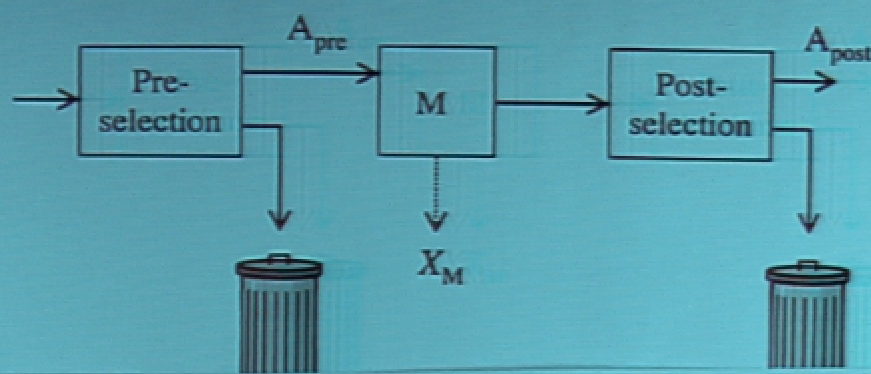


## 2.2) The ABL Rule

- Assumptions
  - Density matrix is $I/d$ prior to pre-selection.
  - Pre-selection measurement obeys Lüders Rule.
- Notation
  - $A_{pre}$ ($A_{post}$) corresponds to a projector $\Pi_{pre}$ ($\Pi_{post}$).
  - The measurement M corresponds to a PVM $\{P_{M,j}\}$ and CP-maps $\{\mathcal{E}_{M,j}\}$.

$$p\left(X_M = j \middle| A_{pre}, A_{post}, M\right)$$

$$= \frac{p\left(A_{post} \middle| A_{pre}, X_M = j, M\right) p\left(X_M = j \middle| A_{pre}, M\right)}{\sum_k p\left(A_{post} \middle| A_{pre}, X_M = k, M\right) p\left(X_M = k \middle| A_{pre}, M\right)}$$

$$\mathrm{Tr}\left(\Pi_{post} \mathcal{E}_{M,j}\left(\Pi_{pre}\right)\right)$$

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$$p\left(X_{\mathrm{M}} = j \middle| \mathrm{A}_{\mathrm{pre}}, \mathrm{A}_{\mathrm{post}}, \mathrm{M}\right) = \frac{\mathrm{Tr}\left(\Pi_{\mathrm{post}} P_{\mathrm{M},j} \Pi_{\mathrm{pre}} P_{\mathrm{M},j}\right)}{\sum_k \mathrm{Tr}\left(\Pi_{\mathrm{post}} P_{\mathrm{M},k} \Pi_{\mathrm{pre}} P_{\mathrm{M},k}\right)}$$

  - Rank-1 pre- and post-selection projectors

$$\Pi_{\mathrm{pre}} = |\,\psi_{\mathrm{pre}}\rangle\langle\psi_{\mathrm{pre}}\,| \qquad \Pi_{\mathrm{post}} = |\,\psi_{\mathrm{post}}\rangle\langle\psi_{\mathrm{post}}\,|$$

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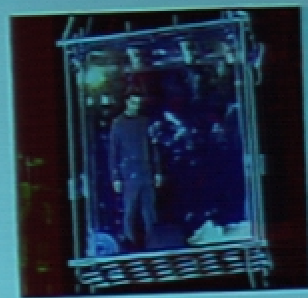
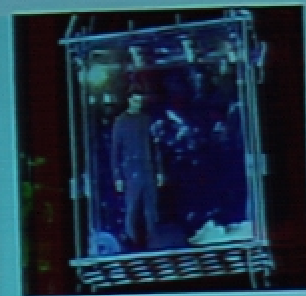
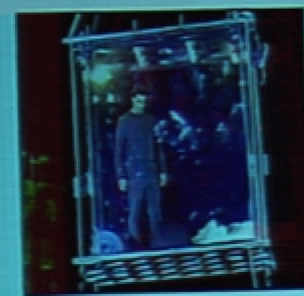
## 2.3) The "Three-Box Paradox"



$|1\rangle$



$|2\rangle$



$|3\rangle$

Pre-selection: $|\psi_{\text{pre}}\rangle = |1\rangle + |2\rangle + |3\rangle$

Post-selection: $|\psi_{\text{post}}\rangle = |1\rangle + |2\rangle - |3\rangle$

Intermediate measurements:

M: $\{P_{M,1} = |1\rangle\langle 1|,\ P_{M,2} = |2\rangle\langle 2| + |3\rangle\langle 3|\}$

N: $\{P_{N,1} = |2\rangle\langle 2|,\ P_{N,2} = |1\rangle\langle 1| + |3\rangle\langle 3|\}$

$$p(X_M = 1 \mid A_{\text{pre}}, A_{\text{post}}, M) = p(X_N = 1 \mid A_{\text{pre}}, A_{\text{post}}, N) = 1$$

Reason: $\langle \psi_{\text{post}} | P_{M,2} | \psi_{\text{pre}} \rangle = \langle \psi_{\text{post}} | P_{N,2} | \psi_{\text{pre}} \rangle = 0$

Counterfactual interpretation: David Blaine is in both boxes at the same time!

- Special cases:
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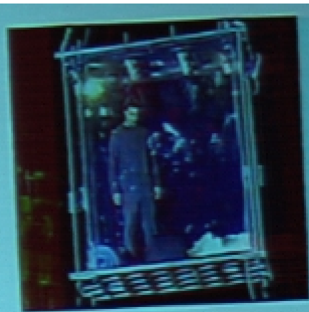

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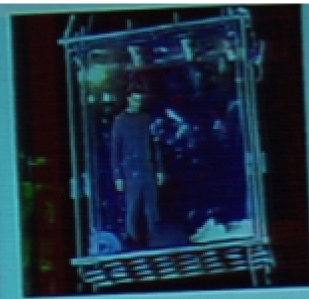

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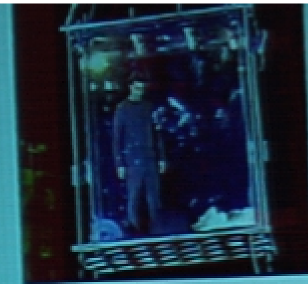

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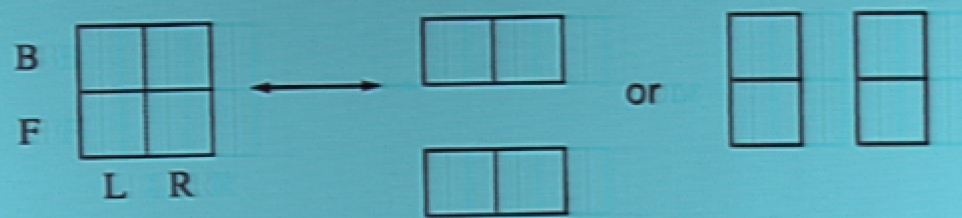
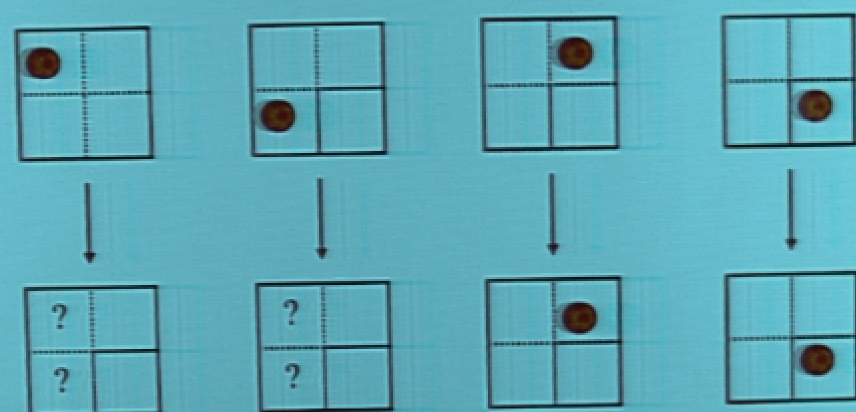
# 3) Hidden Variable Theories

## 3.1) The "Partitioned-Box Paradox"




- Left verification



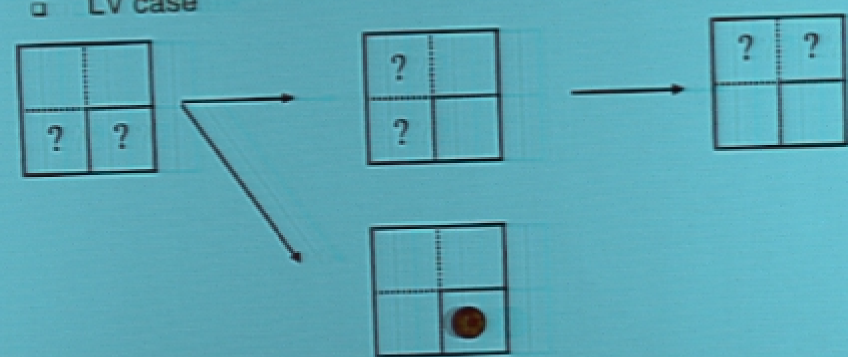
- Similarly for Right, Front and Back verification.

## 3.1) The "Partitioned Box Paradox"

- Pre-selection: Successful Front Verification.
- Post-selection: Successful Back Verification
- Possible Intermediate measurements:

  Left Verification or Right Verification

  - LV case




  - Counterfactual interpretation: The ball is both on the left and the right.
  - Correct interpretation: Different measurements disturb the ball in different ways.

## 3.2) General Formalism of HVTs

- Set of ontic states $\lambda \in \Omega$.
- States correspond to probability distributions $\mu(\lambda)$.
- Assumption: *Outcome determinism for sharp measurements.*
- Measurement M corresponds to:
  - Statistical aspect: indicator functions $\{\chi_{M,j}\}$

$\chi_{M,j}(\lambda) = 0$ or $1$ $\qquad \sum_j \chi_{M,j}(\lambda) = 1$

$p_\mu(X_M = j) = \int_\Omega \chi_{M,j}(\lambda)\, \mu(\lambda)\, d\lambda$

  - Transformation aspect: Stochastic transformations $\{D_{M,j}\}$

$D_{M,j}(\lambda,\omega)$ is prob. of transition from $\omega$ to $\lambda$.

  - Define transition matrices $\Gamma_{M,j}(\lambda,\omega) = D_{M,j}(\lambda,\omega)\, \chi_{M,j}(\lambda)$

$$\int_\Omega \Gamma_{M,j}(\lambda,\omega)\, d\lambda = \chi_{M,j}(\omega)$$

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$$\int_\Omega \Gamma_{M,j}(\lambda,\omega)\, d\lambda = \chi_{M,j}(\omega)$$

## 3.2) General Formalism of HVTs

- Update rule $\mu(\lambda | X_M = j) = \frac{\int_\Omega \Gamma_{M,j}(\lambda,\omega)\mu(\omega)d\omega}{\int_\Omega \Gamma_{M,j}(\lambda,\omega)\mu(\omega)d\omega d\lambda}$

## 3.3) Pre- and Post-Selection in HVTs

- Pre-selection prepares $\mu_{pre.}$
- Post-selection corresponds to $\chi_{post.}$
- Intermediate measurement M: $\{\chi_{M,j}\}$, $\{\Gamma_{M,j}\}$
- By Bayes' Theorem:

$$p(X_M = j | A_{pre}, A_{post}, M) = \frac{\int_\Omega \chi_{post}(\lambda)\Gamma_{M,j}(\lambda,\omega)\mu_{pre}(\omega)d\lambda\, d\omega}{\sum_k \int_\Omega \chi_{post}(\lambda)\Gamma_{M,k}(\lambda,\omega)\mu_{pre}(\omega)d\lambda\, d\omega}$$

## 3.4) Noncontextuality

- If there is an outcome $j$ of M and an outcome $k$ of N that have the same probability for all preparations then

$$\chi_{M,j} = \chi_{N,k}$$

- In QM equivalence $\Rightarrow$ both outcomes associated to the same projector $P$.

$$\chi_{M,j} = \chi_{N,k} = \chi_P$$

- $\Rightarrow$ projectors associated to unique properties.

- $\Rightarrow$ for *any* distribution $\mu$ can assign probabilities to projectors via $p(P) = \int_\Omega \chi_P(\lambda)\mu(\lambda)d\lambda$.

- Algebraic constraints

$$0 \leq p(P) \leq 1$$
$$p(I-P) = 1 - p(P)$$
$$p(I) = 1, \quad p(P_{\text{null}}) = 0$$

For $[P,Q]=0$

$$p(PQ) \leq p(P), \quad p(PQ) \leq p(Q)$$
$$p(P+Q-PQ) = p(P) + p(Q) \nleq p(PQ)$$

## 3.5) Noncontextuality, PPS and Disturbance

- We can also try to apply these constraints to ABL probabilities.
- Definition: We have a *Logical PPS paradox* whenever the ABL rule assigns probability 1 or 0 to the outcomes of measurements, such that associating the same probabilities to the projectors corresponding to those outcomes would violate the algebraic constraints.
  - The Three-Box paradox is an example of this.
- You *could* explain Logical PPS paradoxes via contextuality. However, this is not required as our "partitioned box" example shows.
- Reproducing ABL predictions *does* place nontrivial constraints on the transitions $\Gamma_{M,j}$.
  - They must be nontrivial $\Gamma_{M,j}(\lambda,\omega) \neq \delta(\lambda,\omega)\,\chi_{M,j}(\omega)$.
  - Outcomes associated with the same projector, but different CP-maps, must be associated with different transitions.

- If there is an outcome ... that have the same probability for all preparations then

$$\chi_{M,j} = \chi_{N,k}$$

- In QM equivalence ⇒ both outcomes associated to the same projector $P$.

$$\chi_{M,j} = \chi_{N,k} = \chi_P$$

- ⇒ projectors associated to unique properties.
- ⇒ for *any* distribution μ can assign probabilities to projectors via $p(P) = \int_\Omega \chi_P(\lambda)\mu(\lambda)d\lambda$.
- Algebraic constraints

$$0 \leq p(P) \leq 1$$
$$p(I-P) = 1 - p(P)$$
$$p(I) = 1, \quad p(P_{\text{null}}) = 0$$

For $[P,Q]=0$

$$p(PQ) \leq p(P), \quad p(PQ) \leq p(Q)$$
$$p(P+Q-PQ) = p(P) + p(Q) - p(PQ)$$

probabilities.

- Definition: We have a *Logical PPS paradox* whenever the ABL rule assigns probability 1 or 0 to the outcomes of measurements, such that associating the same probabilities to the projectors corresponding to those outcomes would violate the algebraic constraints.
  - The Three-Box paradox is an example of this.
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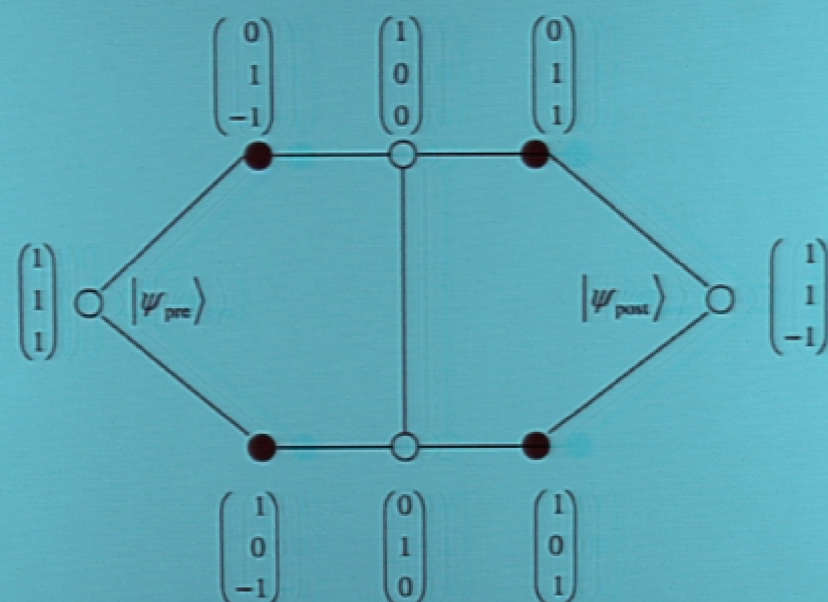
  - They must be nontrivial $\Gamma_{M,j}(\lambda,\omega) \neq \delta(\lambda,\omega)\, \chi_{M,j}(\omega)$.
  - Outcomes associated with the same projector, but different CP-maps, must be associated with different transitions.

- Whenever a set of projectors have ABL probabilities that give rise to a logical PPS paradox, and the pre- and post-selection projectors are not orthogonal, one can use fine grainings of the projectors, together with the pre- and post-selection projectors to show that a MNHVT is impossible.

## 4.1) Three box paradox and Clifton's proof

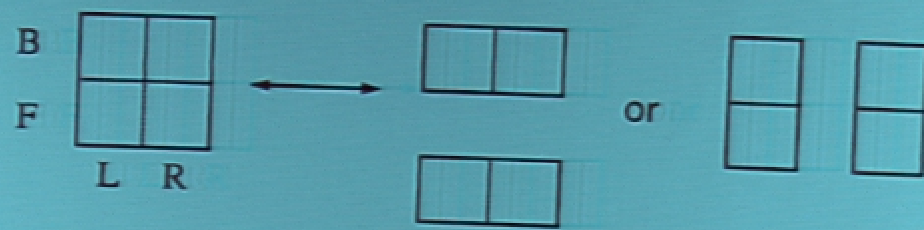


## 5) Conclusions and Open Questions

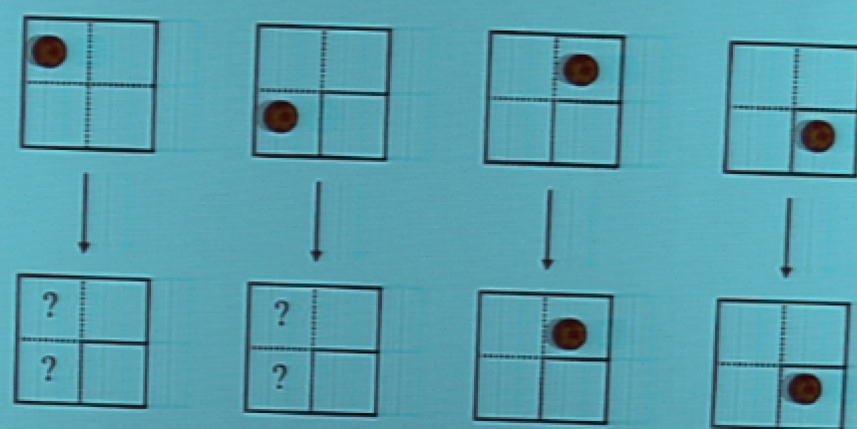
- We have shown that:
  - The existence of Logical PPS paradoxes in a theory does not imply contextuality.
  - Nonetheless, each Logical PPS paradox is related to a proof of contextuality.
- Possible reason for discrepancy:
  - HVTs have to satisfy additional constraints in order to reproduce quantum predictions.
  - Conjecture: Existence of Logical PPS paradoxes + some suitable analog of the uncertainty principle implies contextuality.
    - Justification: Toy theory of Spekkens has such an analog, is noncontextual and seems to be devoid of logical PPS paradoxes.
- Applications of logical PPS paradoxes in quantum information theory?
- Use of our results in axiomatics of quantum theory?

# 3) Hidden Variable Theories

## 3.1) The "Partitioned-Box Paradox"




- Left verification



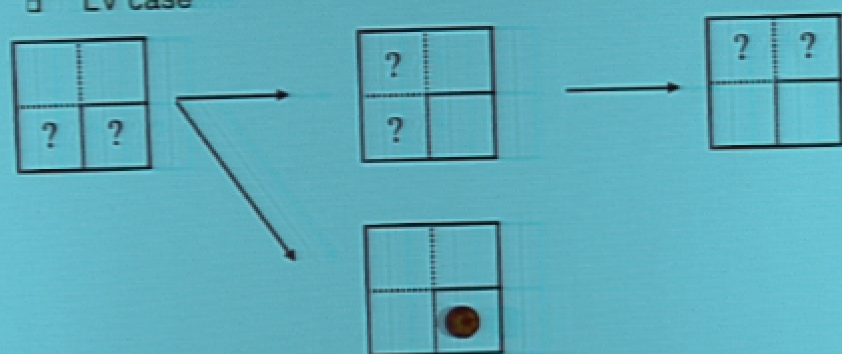

- Similarly for Right, Front and Back verification.

- We have shown that:
  - The existence of Logical PPS paradoxes in a theory does not imply contextuality.
  - Nonetheless, each Logical PPS paradox is related to a proof of contextuality.
- Possible reason for discrepancy:
  - HVTs have to satisfy additional constraints in order to reproduce quantum predictions.
  - Conjecture: Existence of Logical PPS paradoxes + some suitable analog of the uncertainty principle implies contextuality.
    - Justification: Toy theory of Spekkens has such an analog, is noncontextual and seems to be devoid of logical PPS paradoxes.
- Applications of logical PPS paradoxes in quantum information theory?
- Use of our results in axiomatics of quantum theory?

- Pre-selection: Successful Front Verification.
- Post-selection: Successful Back Verification
- Possible Intermediate measurements:

  Left Verification or Right Verification

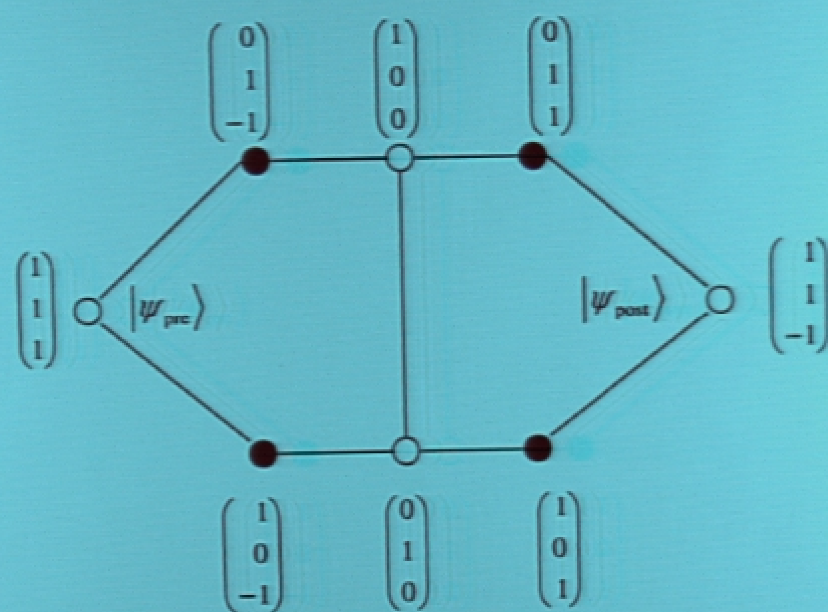
  - LV case




  - Counterfactual interpretation: The ball is both on the left and the right.
  - Correct interpretation: Different measurements disturb the ball in different ways.

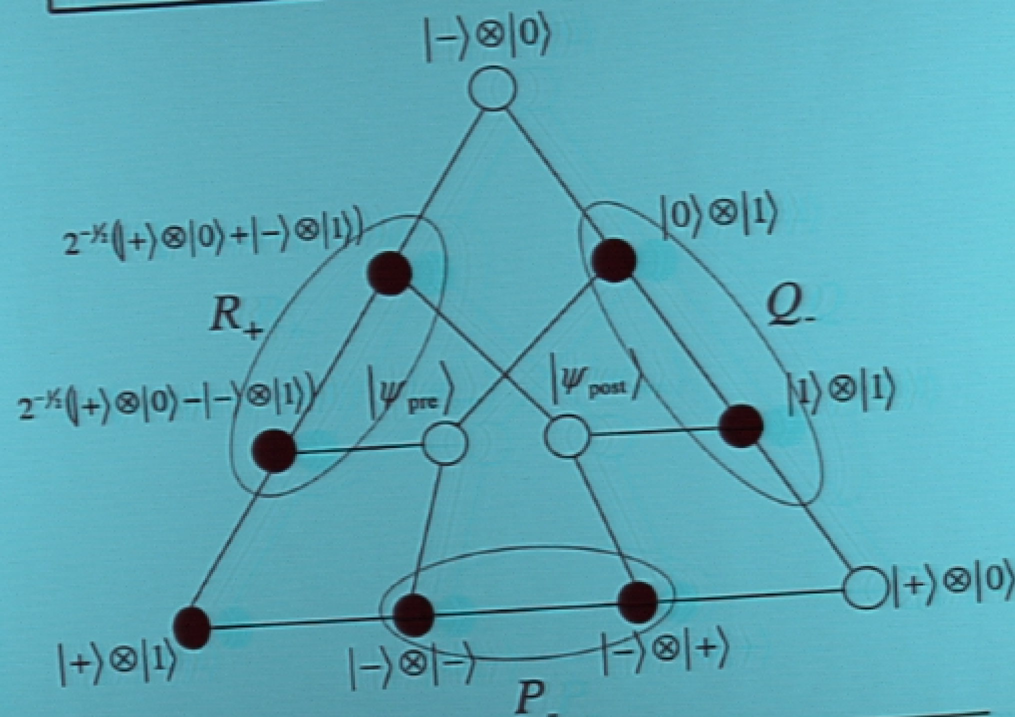
- Whenever a set of projectors have ABL probabilities that give rise to a logical PPS paradox, and the pre- and post-selection projectors are not orthogonal, one can use fine grainings of the projectors, together with the pre- and post-selection projectors to show that a MNHVT is impossible.

## 4.1) Three box paradox and Clifton's proof



## 4.2) Failure of the Product Rule

| Observable | Projectors onto Eigenvalues | |
|---|---|---|
| | +1 | -1 |
| $X\otimes I$ | $P_+$ | $P_-$ |
| $I\otimes Z$ | $Q_+$ | $Q_-$ |
| $X\otimes Z$ | $R_+ = P_+Q_+ + P_-Q_-$ | $R_+ = P_+Q_- + P_-Q_+$ |




WHITE