Title: Topological Quantum Computation and Electrons in Solids

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Abstract:

Topological Quantum Computation and Electrons in Solids

M. Freedman, C.N., K. Shtengel, K. Walker, Z. Wang

Outline

- Introduction: a simple example.
- Generalizing the example: a class of P,T-invar. topological phases.
  Topological Quantum Computation.
- How do we get into these phases?
  Microscopic Models.
  Effective Field Theories.
- Conclusions, Open Problems

What is the physics of a Hamiltonian such as

$$H = \sum_{i,j} J_{ij}\, \mathbf{S}_i \cdot \mathbf{S}_j$$

Large number of microscopic d.o.f. coupled together.

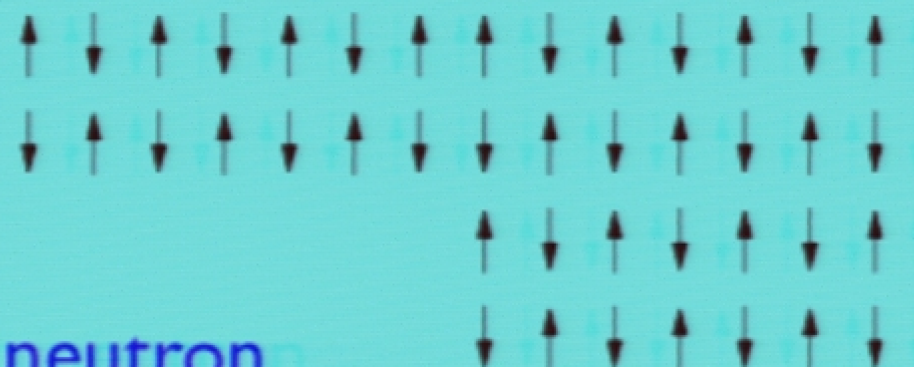
How do they react when you shine light, neutrons, etc. on them?

Starting point of analysis:

What *phase* is the system in?

Within a phase, certain properties are determined (e.g. by symmetry) and others can be obtained quant. by perturbing from a soluble model.

e.g. an antiferromagnet



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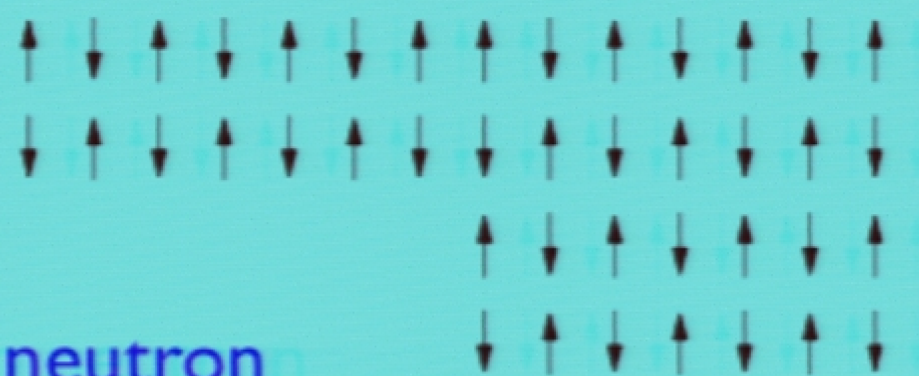

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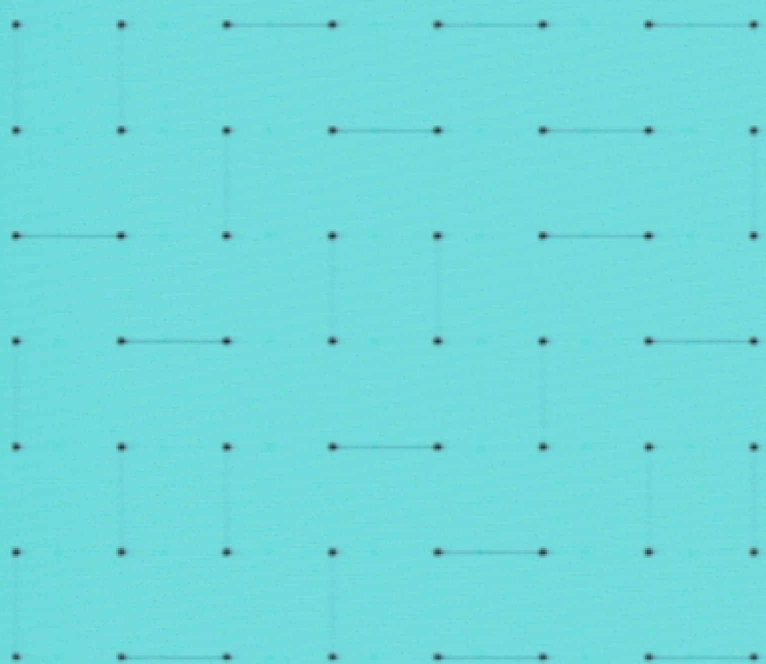
They could form near-neighbor singlets which resonate -- i.e. superpositions of different singlet configs. (Anderson):



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Need to make it precise.

What is the long-wavelength, universal physics of such a state?

What is its effective field theory?

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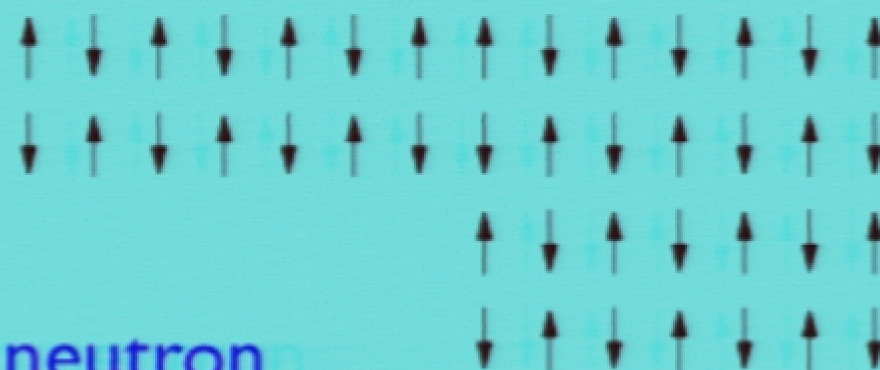

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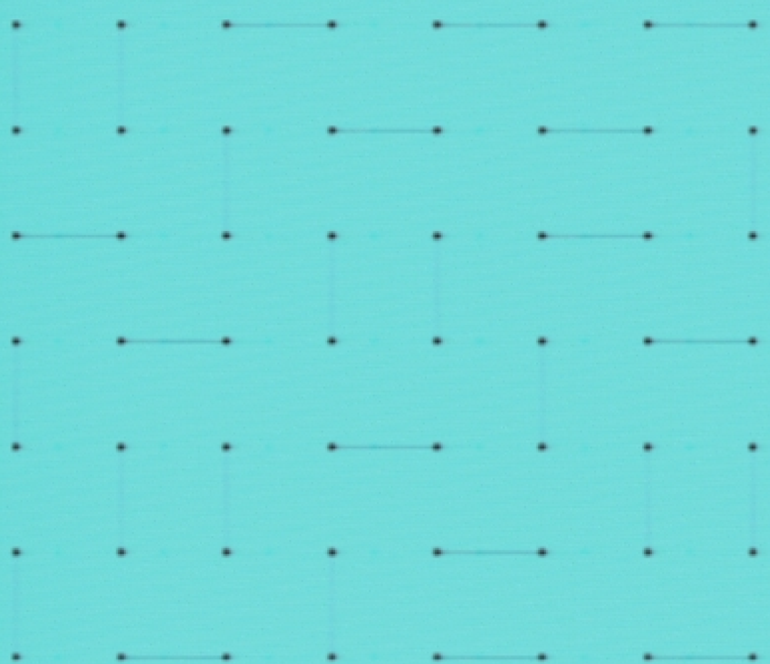

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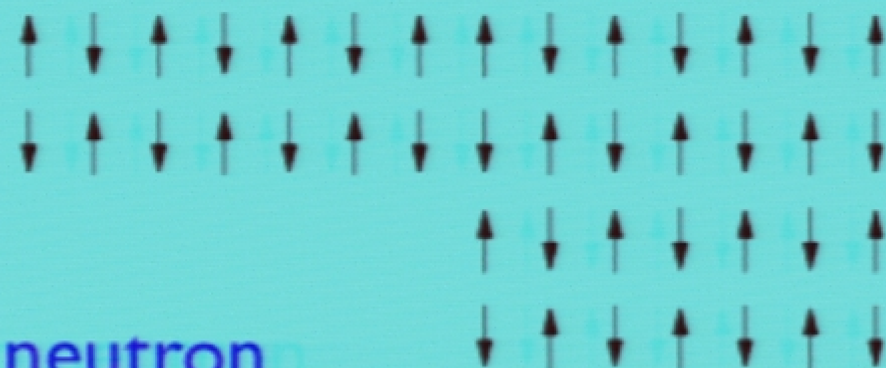

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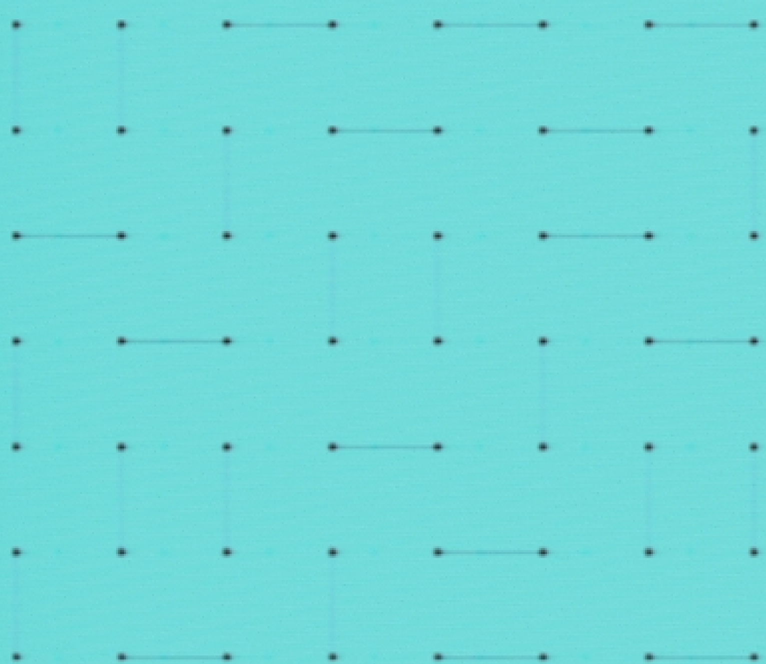

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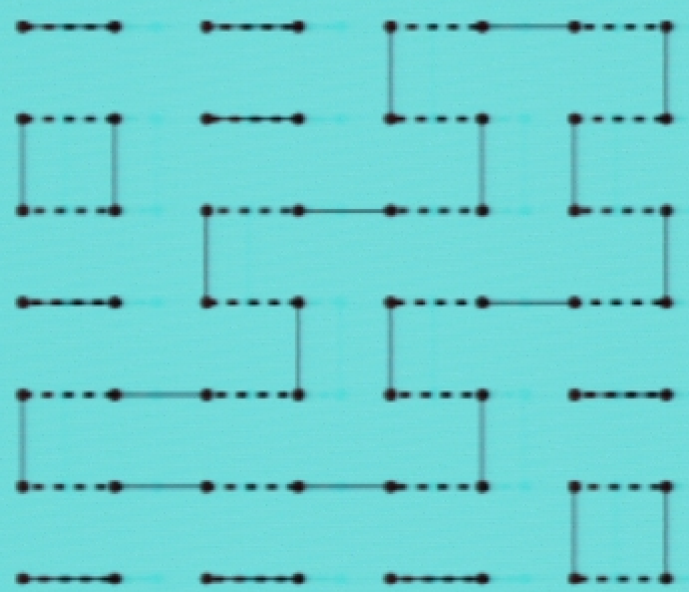
Consider the 'transition' graph (Rokhsar and Kivelson):



Fluctuating loops

Details of H determine how they fluctuate.

This, in turn, determines the physics of such a state.

Goal: understand possible states using this perspective.

A Simpler Model

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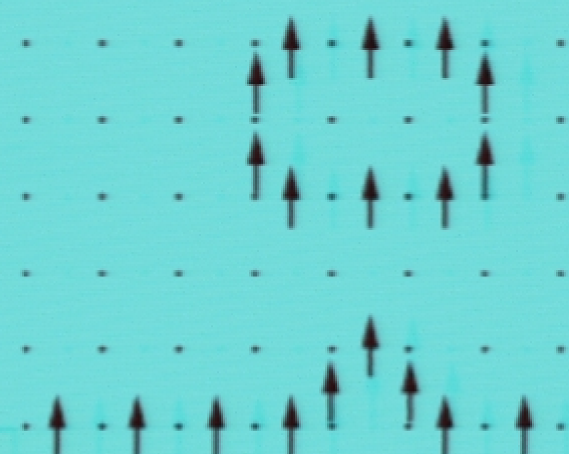
$$A_v \equiv \Pi_{\alpha \in \mathcal{N}(v)} \sigma^z_\alpha \,, \;\; F_p \equiv \Pi_{\alpha \in p} \sigma^x_\alpha$$

$$[A_v, F_p] = 0$$

Represent up-spins by colored bonds, down spins by uncolored bonds.



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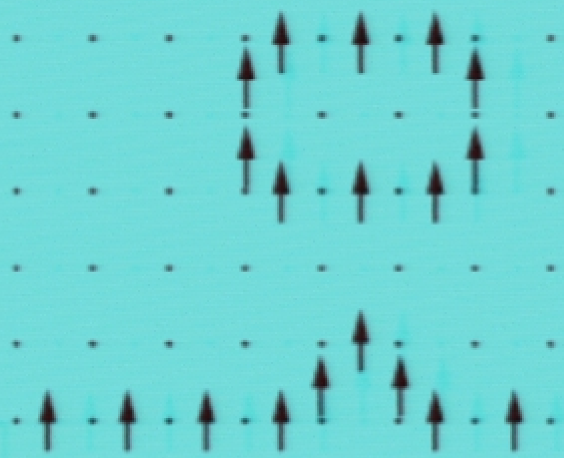

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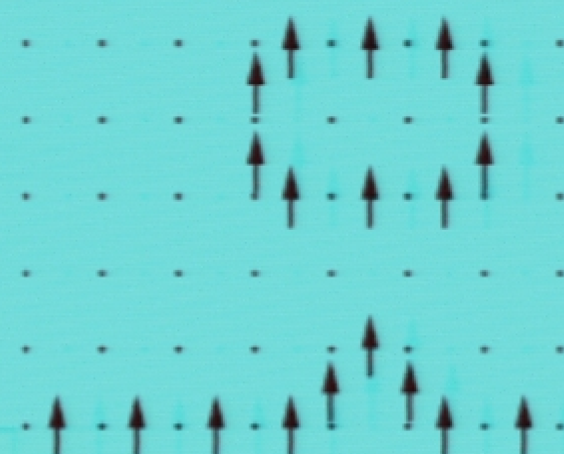

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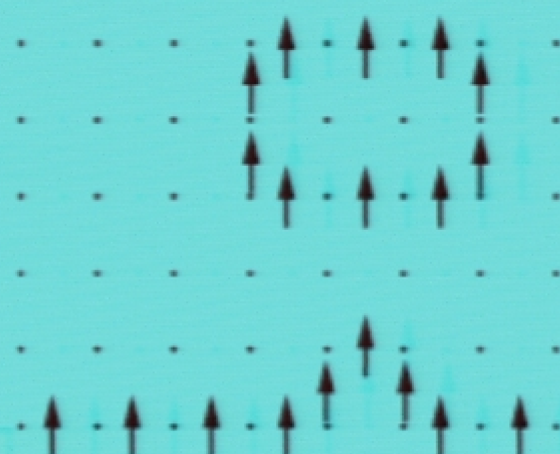

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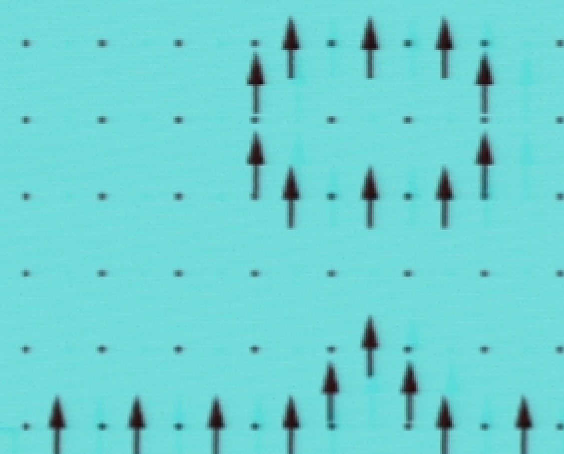

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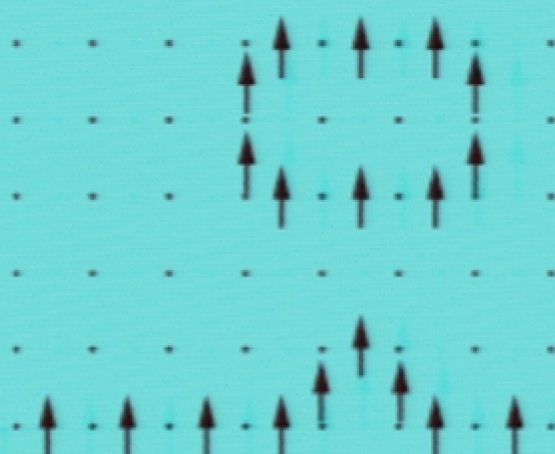

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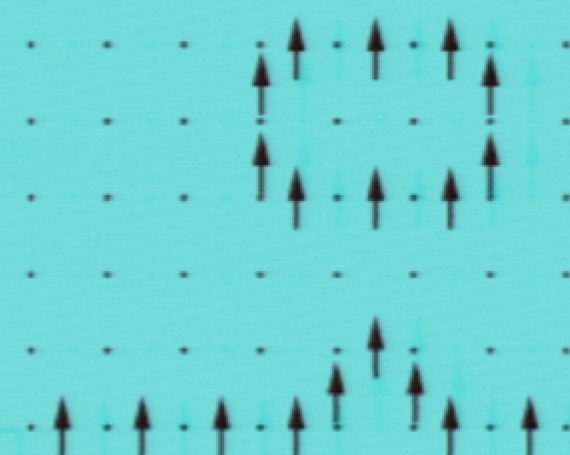

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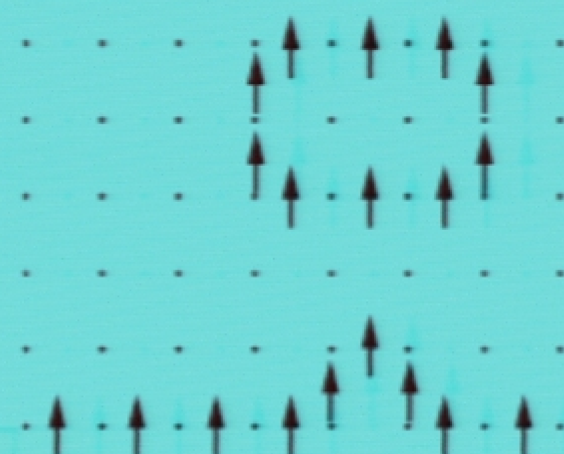

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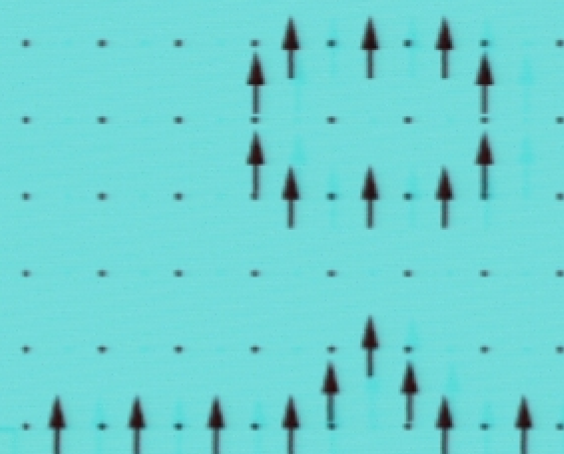

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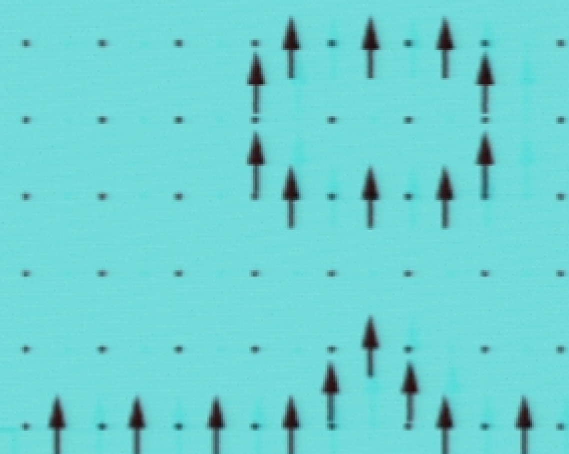

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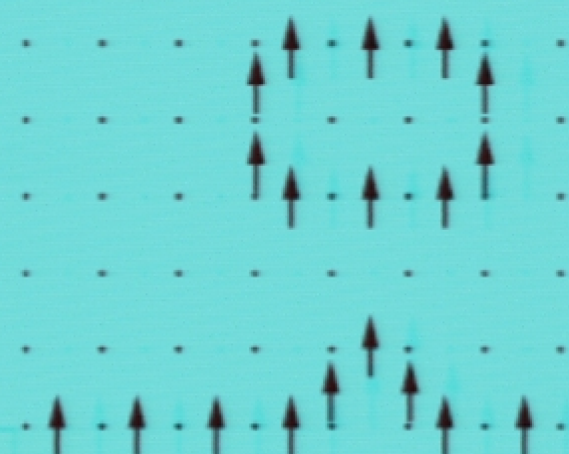

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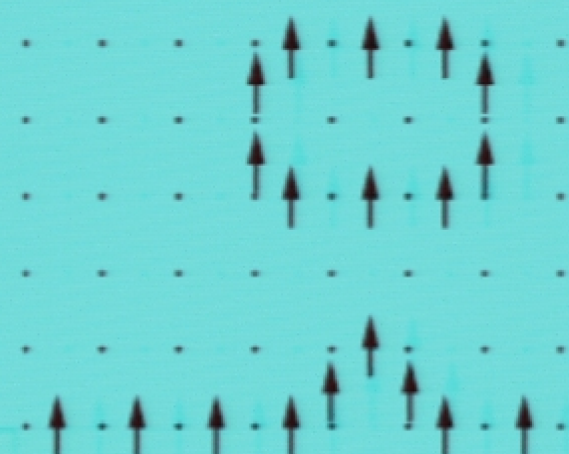

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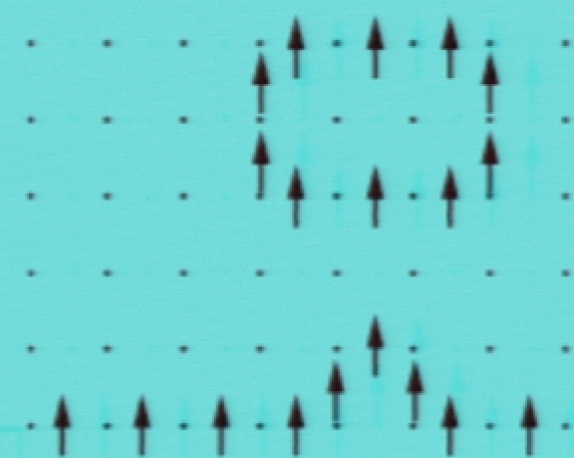

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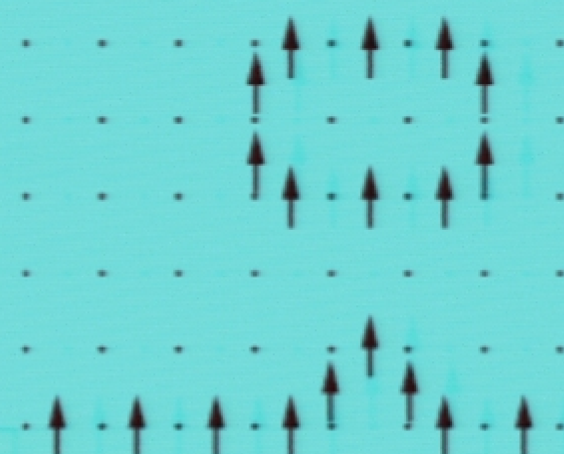

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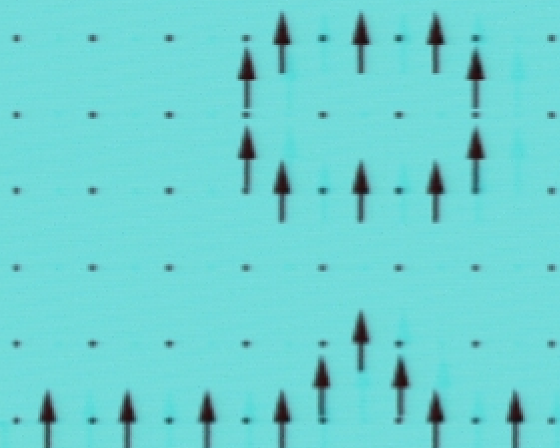

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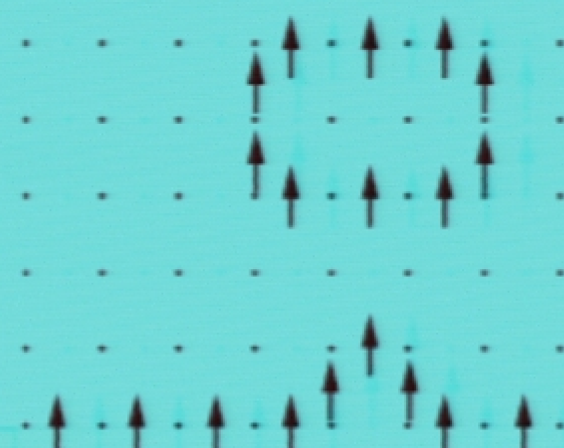

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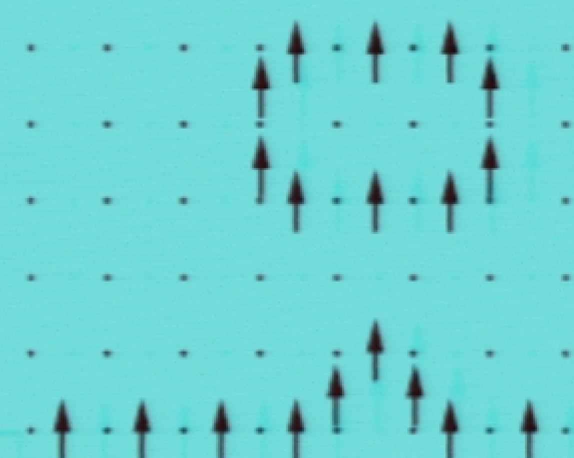

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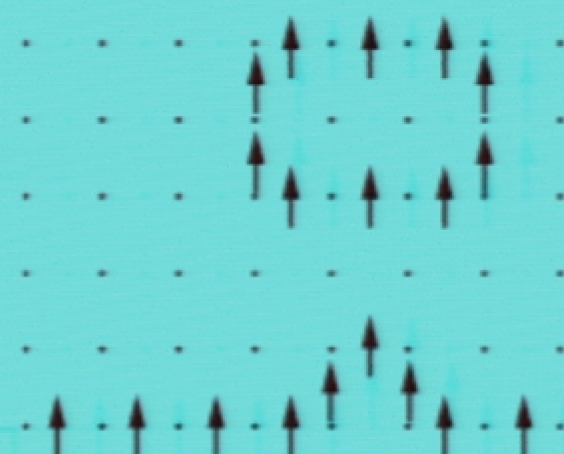

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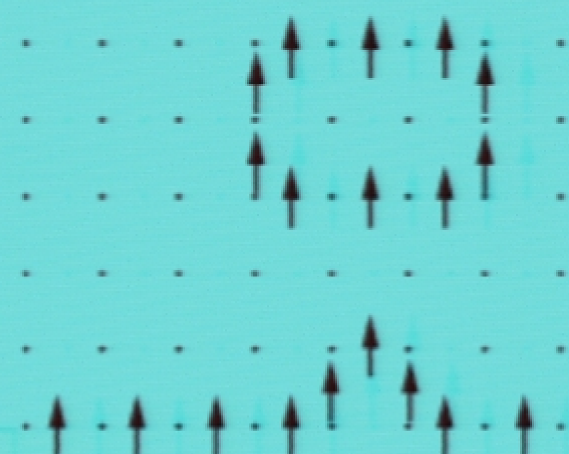

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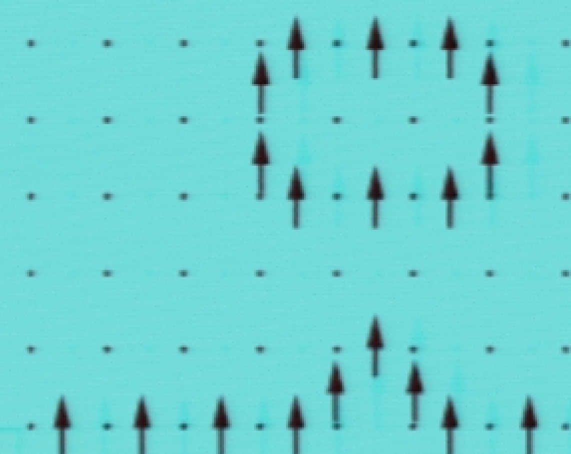

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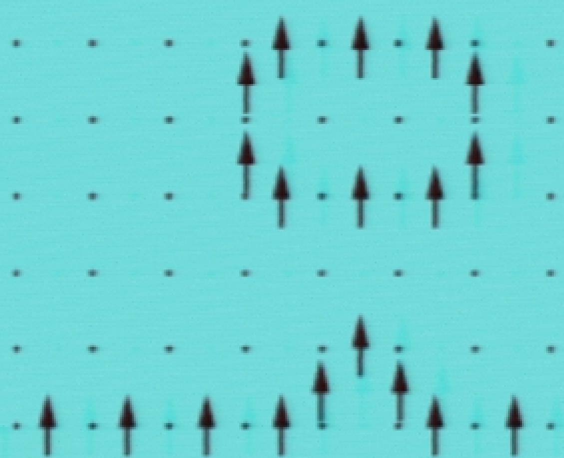

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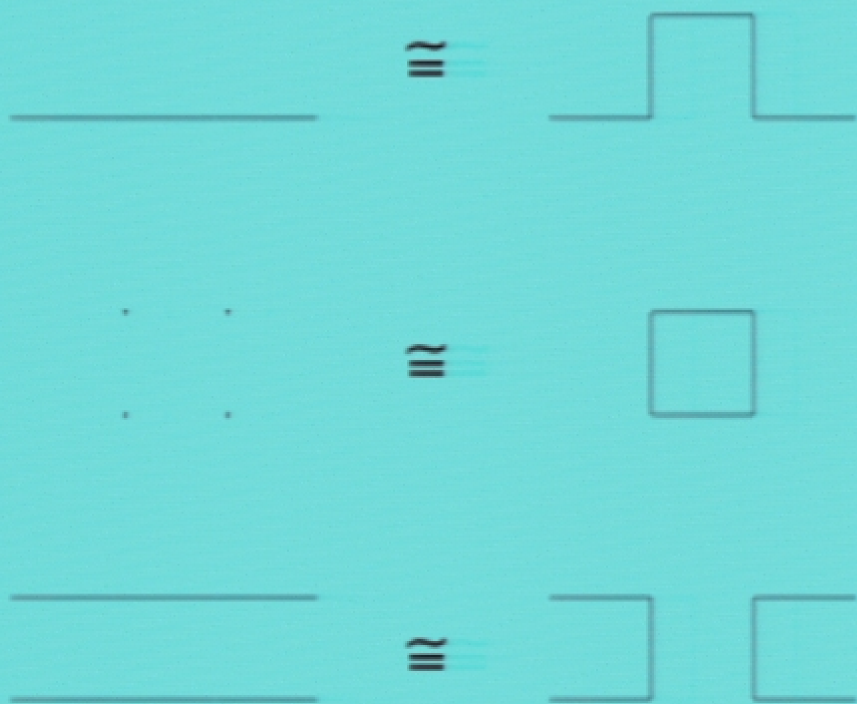
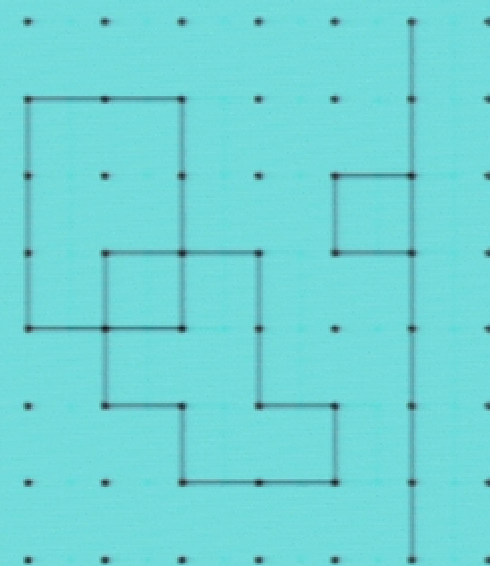




Ground state: equal amplitude superposition of all loop configurations

Topological Properties

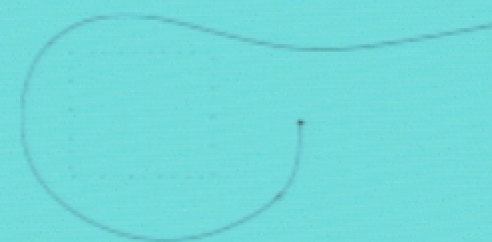
There are two inequivalent g.s. on the annulus, corresponding to even/odd winding numbers.

They cannot be distinguished by local measurements.

(Topologically protected qubits.)

Excitations: vertices or plaquettes at which

$A_v = -1$ or $F_p = -1$



When one is taken around another, a minus sign results.

(Wen: 'Topological Order')

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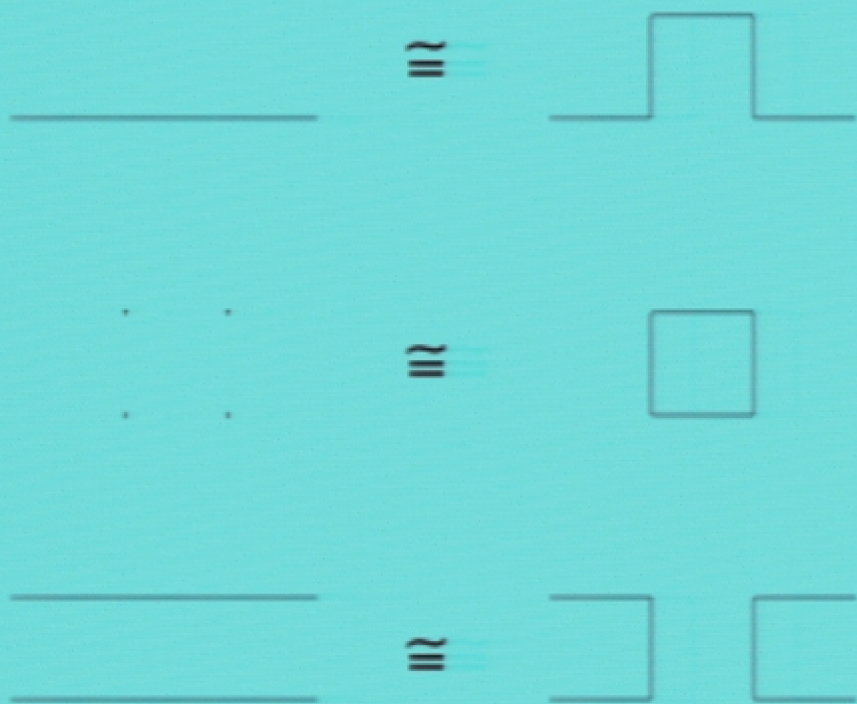

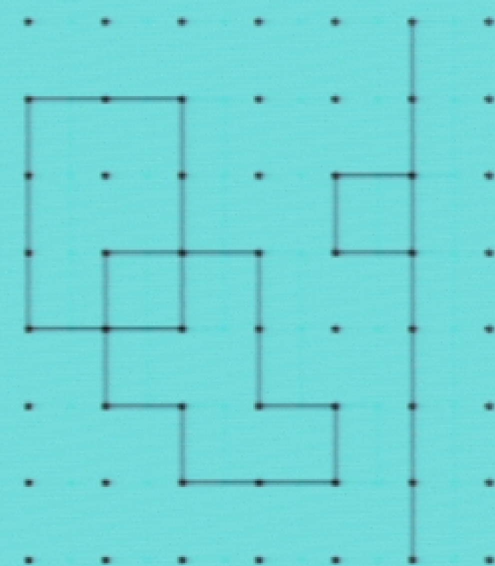

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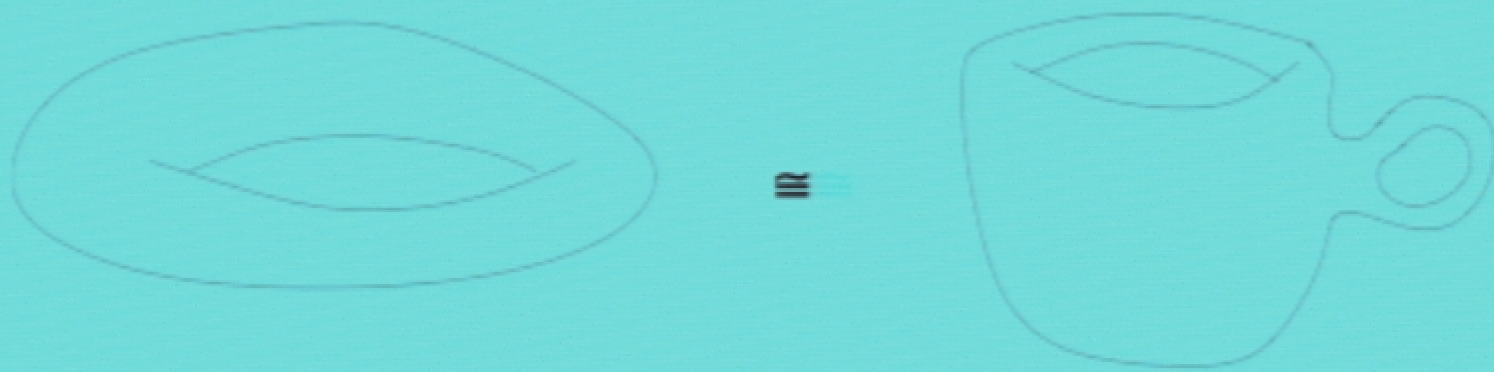
(Wen: 'Topological Order')

Topological Quantum Computation



Donut = Coffee Cup $\Longrightarrow$ Rigidity Against Perturb.

(Fault Tolerance)

Local perturbations can neither *measure* nor *change* topological degrees of freedom.

(to exp. in size accuracy)

Effective Field Theory

These properties are encapsulated in a long-wavelength field theory which is <u>topological</u>.

$$S = \frac{1}{2\pi}\int d^2x\, d\tau\, \epsilon^{\mu\nu\lambda} E_\mu F_{\nu\lambda}$$

E_μ, A_μ coupled to plaquettes (vortices) and sites (particles).

Aharonov-Bohm $\longrightarrow$ exotic braiding statistics (Wilczek)

Transition to a non-topol. phase = confinement transition of Z_2 gauge theory (Senthil-Fisher)

A simple model of interacting spins can have a topological phase.

0. Topological Phases occur in the QHE, but energy scales are lower than in spin sys.

1. Low-energy Hilbert space = loop configs.

2. Topologically-degenerate ground states which are locally indistinguishable. (protected qubits)

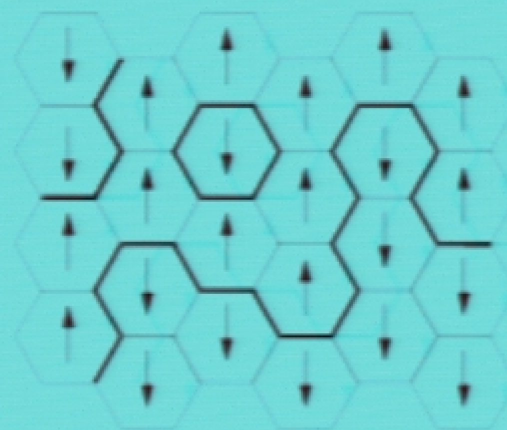
3. Exotic Braiding Statistics of QPs. (need non-Abelian to do non-trivial ops., read-out)

4. Low-energy EFT is a TQFT

Generalizing this Structure

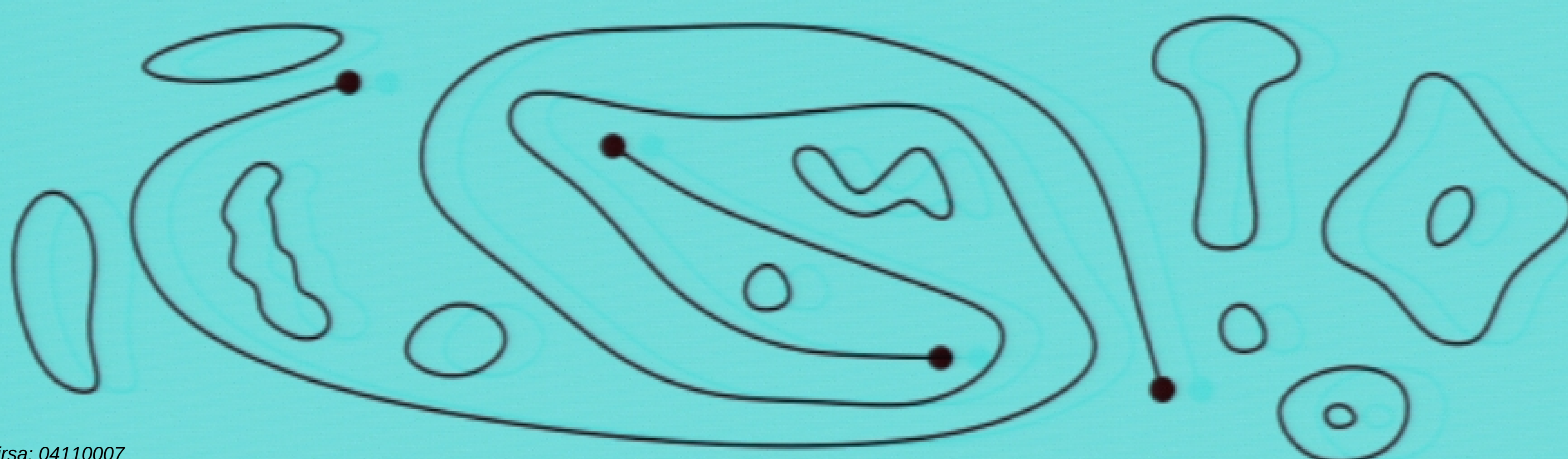
(Freedman, Turaev-Viro)

- Loops may arise as domain walls, dimers, chains of up-spins.



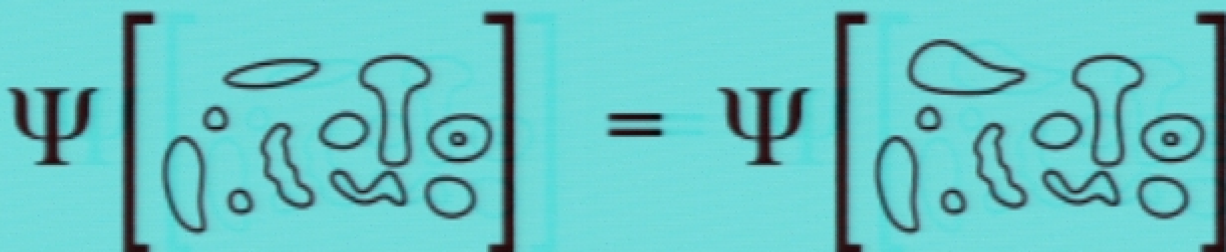
- *H*: rules satisfied by loops

- Excitations will be violations of the g.s. conditions, such as broken loops



Ground State Conditions/Hamiltonian

1. Wavefunctions on multi-loops, invariant under smooth deformations of the loops.



Expected for any topological phase.

Can be imposed by including a term in H such as A_v, suitable plaquette terms.

2. A `fugacity' for small, contractible loops

Ψ  $=$ d $\cdot$ Ψ 

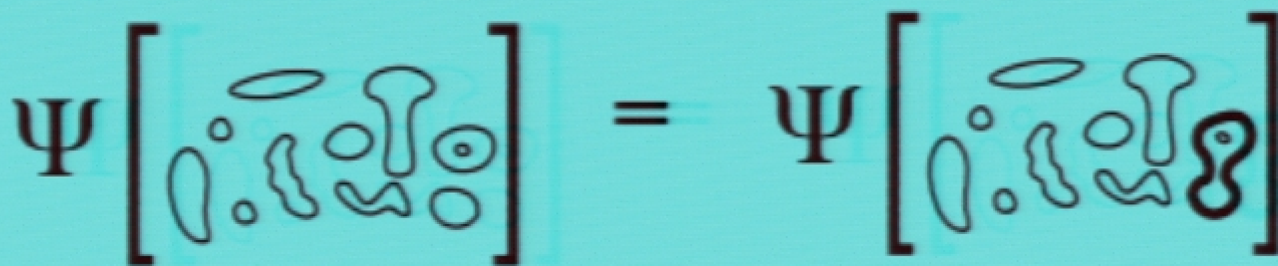
d=1 in Kitaev's model

Such a relation is needed to have finite g.s. degeneracy on the sphere.

Can be imposed with a plaquette term in H

3. Invariance of the wavefcn. under a 'surgery' rel.

e.g. in Kitaev's model

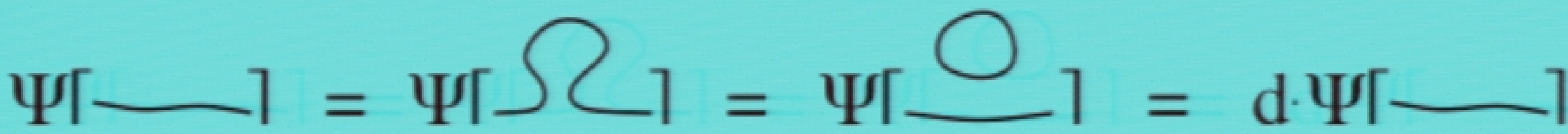


Without such a rel., infinite g.s. degeneracy even on the torus.

Generalizing 2 and 3, we will obtain a family of topological states, corresponding Hamiltonians

Consistency Conditions for Quantum Loop Gases

If $d \neq 1$ the surgery rel. must be modified



Hence, we must look for surgery relations involving 3, 4, ... curves

Important Mathematical Result: for almost all d, there is no consistent surgery relation.

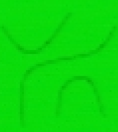
Jones-Wenzl Projectors

Consistent surgery relations can only be found for

$$d = 2\cos\left(\frac{\pi}{k+2}\right)$$

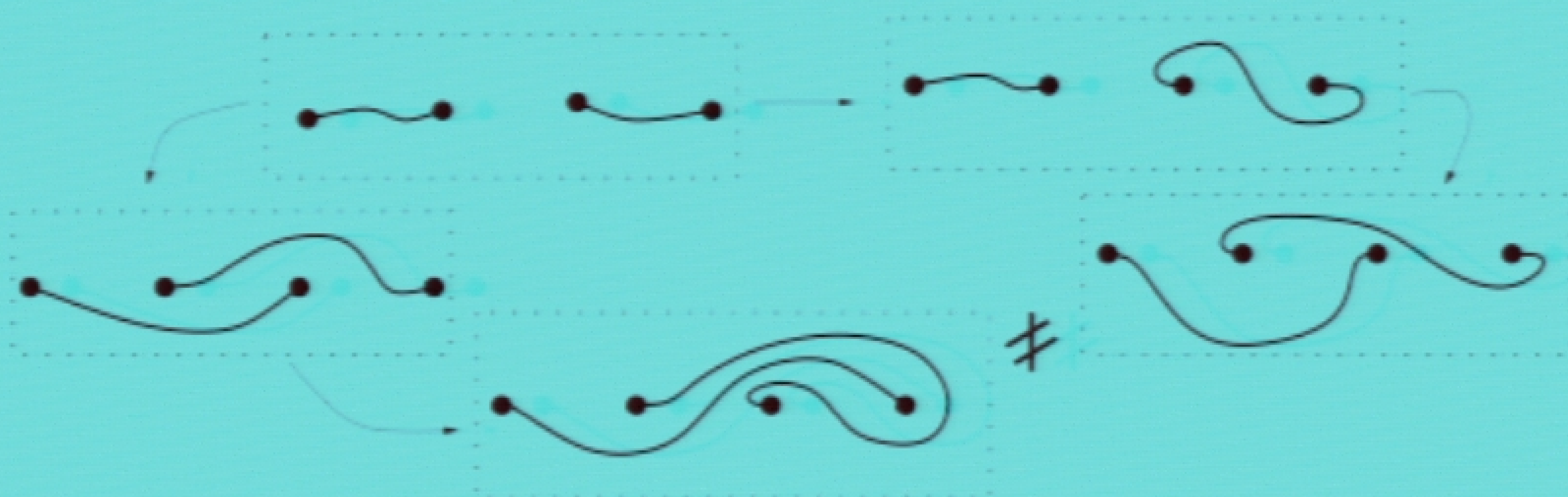
e.g. for $d = \sqrt{2}$

$\Psi[$  $] - \sqrt{2}\,\Psi[$  $] - \sqrt{2}\,\Psi[$  $]$

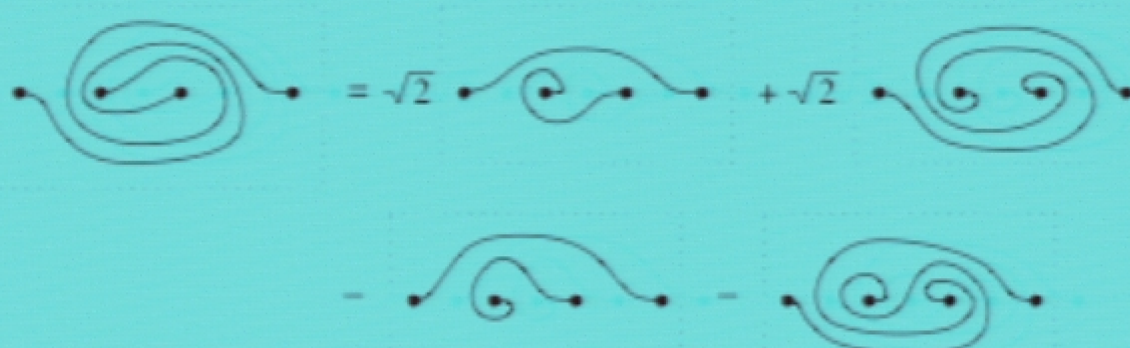
$+\ \Psi[$  $]\ +\ \Psi[$  $] = 0$

Non-Abelian Statistics Made Easy

For $d \neq 1$ the order of braiding ops. matters



e.g. $k=2$:



1. Given n quasiparticles at fixed positions, there is an exponentially-large set of degenerate states ψ_a with $a = 1, 2, \ldots, g$

2. Braiding particles i and j transforms:

$$\psi_a \to M_{ab}\psi_b$$

3. Braiding particles j and k: N_{ab} which need not commute with M_{ab}

4. For a large class of states, braiding operations implement all of U(g) to desired accuracy.

Effective Field Theory

- The effective field theories are gauge theories.
- Topological properties from the generalized Aharonov-Bohm effect.
- Relation to combinatorics of curves: Wilson loop operators.
- Unoriented curves: SU(2). Other gauge groups: oriented, labeled curves in trivalent graphs.

Doubled Chern-Simons/BF Theory

- P,T-invariant action:

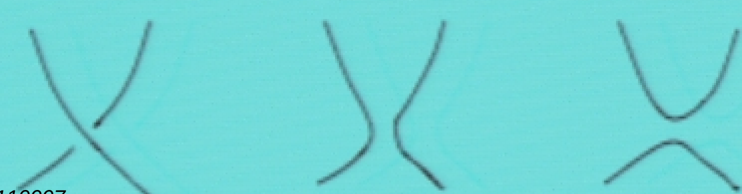
$$S = \frac{k}{\pi}\int d^2x d\tau \, \mathrm{tr}\left(e \wedge (da + a \wedge a) + \frac{1}{3} e \wedge e \wedge e\right) = \frac{k}{4\pi}\Big(S_{\mathrm{CS}}(a+e) - S_{\mathrm{CS}}(a-e)\Big)$$

$$W_{\pm}[\gamma] \equiv \mathrm{tr}\left(\mathcal{P} e^{i\oint_\gamma (\mathbf{a}^c \pm \mathbf{e}^c) T^c \cdot d\mathbf{l}}\right)$$

- Wilson loop operators act pictorially:

$$W_+[\gamma]\,\Psi[\beta] = \Psi[\beta \star \gamma] \qquad \Psi[\alpha] = A\Psi[\alpha'] + A^{-1}\Psi[\alpha'']$$

$$A = i \exp\left(\pi i / 2(k+2)\right)$$



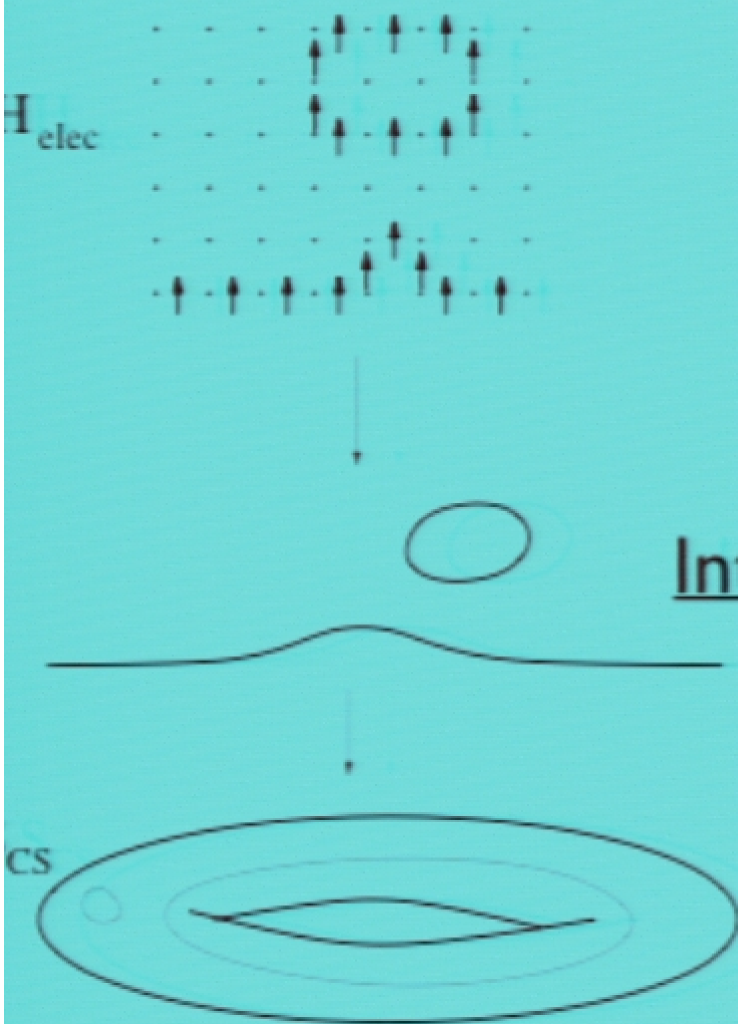
α α' α''

- We now have some understanding of the **what** (are top. phases) and **why** (are they useful).

- The open question is **when** (do they occur).

Where should experimentalists look for such phases?

When Will Such Phases Occur?



Short Scales: electrons/spins at points (0-D)

↓ Simple spin/dimer Hamiltonians

Intermed. Scales: fluctuating loops (1-D)

↓ **How?**

Long scales: deg. g.s. on genus-g surfaces (2-D)

Fluctuating Loops and d-isotopy

$$H_{d-\mathrm{iso}} = \sum_v \Big(1 + \prod_{i\in\mathcal{N}(v)} \sigma_i^z\Big) + \sum_p \Big[\frac{1}{d^2}(F_p^0)^\dagger F_p^0 + F_p^0(F_p^0)^\dagger - \frac{1}{d}F_p^0 - \frac{1}{d}(F_p^0)^\dagger$$

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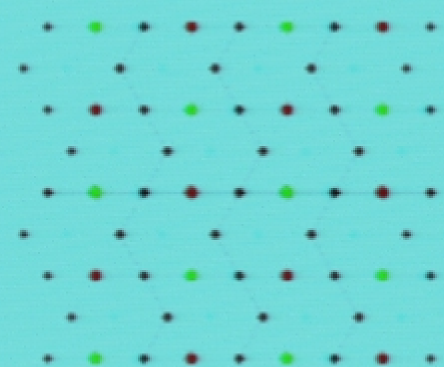
with $F_p^0 = \sigma_1^- \sigma_2^- \sigma_3^- \sigma_4^- \sigma_5^- \sigma_6^-$ etc.

$$H = \sum_i \Delta_i n_i + U_0 \sum_i n_i^2$$

$$+U \sum_{(i,j)\in\mathrm{hex.}} n_i n_j + \sum_{(i,j)\in\mathrm{bowtie}} V_{ij} n_i n_j$$

$$-\sum_{\langle i,j\rangle} t_{ij}(c_i^\dagger c_j + c_j^\dagger c_i) + \mathrm{Ring.}$$



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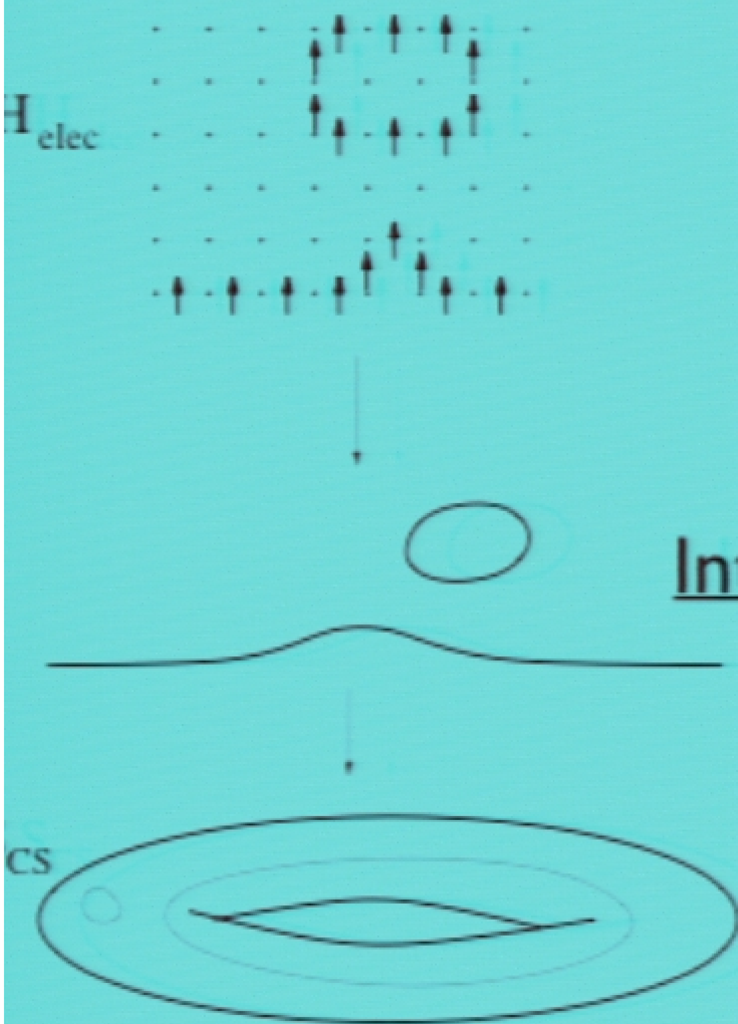

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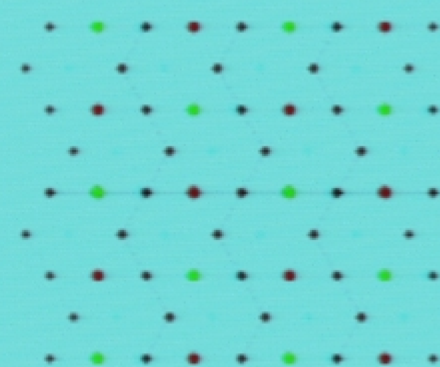

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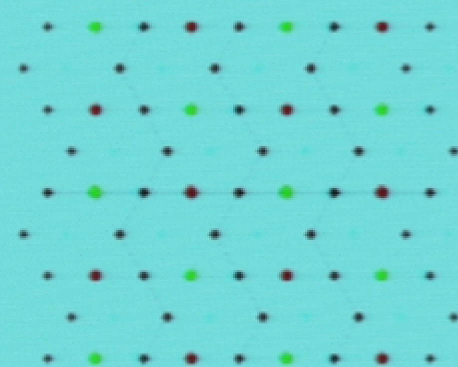

- It is relatively easy to construct models which impose the first two defining conditions for these topological phases (d-isotopy).

- The third condition (JW) is complicated; it is unlikely that a real Hamiltonian will impose precisely this condition. *How much margin for error is there? Are these phases experimentally relevant?*

- As we will see, d-isotopy is *critical.* Answering these questions involves understanding the *instabilities* of and the *phase diagram* near this critical point.

d-isotopy: from here to criticality

The Hamiltonians of the previous transparency describe fluctuating loops controlled by d-isotopy.

For $d \leq \sqrt{2}$ these Hamiltonians are *critical*

To see this, note that at $n = d^2$

$$\sum_\alpha |\Psi[\alpha]|^2 = \sum_\alpha d^{2n_\alpha} = Z_{\mathrm{O(n)}}(x = n)$$

$$Z_{\mathrm{O(n)}}(x) = \int \prod_i d\hat{S}_i \prod_{\langle i,j \rangle} (1 + x\hat{S}_i \cdot \hat{S}_j) = \sum_\alpha \left(\frac{x}{n}\right)^{\ell_\alpha} n^{n_\alpha}$$

This stat. mech. model is in its low-T phase, which is *critical*.

Loops meander over long-distances parameterized by exponents

$$\eta_k = \frac{g}{4}k^2 - \frac{1}{g}(1-g)^2 \quad \text{where} \quad n = -2\cos(\pi g)$$

Consequently, we can define `long' so that the trial wavefunction

$$|\Psi_1\rangle = \sum_{\alpha \in X} d^{n_\alpha} |\alpha\rangle - \sum_{\alpha \in Y} d^{n_\alpha} |\alpha\rangle$$

where X = configs. with long loops; Y = without, satisfies $\langle \Psi_0 | \Psi_1 \rangle = 0$

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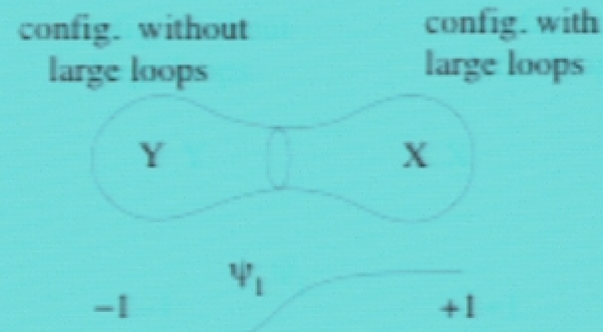

Stability of this Critical Line

- How many relevant perturbations?
- To what phase(s), apart from the desired topolgical phase, can they lead, i.e. ***what else should we be looking for in experiments***?

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To answer these questions, we would like an effective field theory for this critical line.

Critical SU(2) gauge theory, $\omega \sim k^2$

Continuously-varying exponents

Combining these requirements, we guess

$$S = \frac{1}{g^2} \int d^2x \, d\tau \left(E_i^a \partial_\tau A_i^a + A_0^a D_i E_i^a + \frac{1}{2} E_i^a D^2 E_i^a + \frac{1}{2} B^a B^a \right)$$

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But is this interacting theory actually critical?

At one-loop,

$$\boxed{\frac{dg}{d\ell} = 0}$$

Conclusions

- Hamiltonian ⟹ Topological Conditions ⟹ Ground State ⟹ Statistical Mechanics ⟹ Low-Energy Excitations.
- Perturbing from tractable models to realistic ones. (short-distance physics).
- Instabilities of the d-isotopy critical line (long-wavelength physics - field theory).
- Application to quantum computation.

Foundations

- A.Y. Kitaev, Ann. Phys. **303**, 2 (2003), quant-ph/9707021.
- M.H. Freedman, Commun. Math. Phys. **234**, 129 (2003), quant-ph/0110060.
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Quantum Loop Gases -- Effective Field Theories to Microscopic Models

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- A. Vishwanath, L. Balents, and T. Senthil, cond-mat/0311085.

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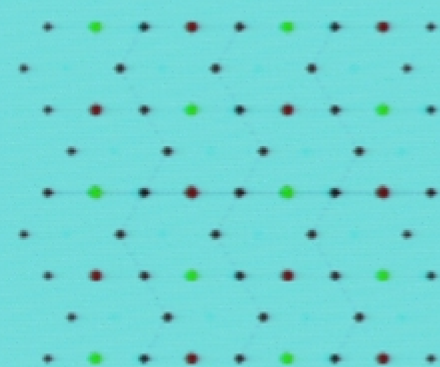

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