Title: TBA

Date: Nov 04, 2004 01:10 PM

URL: http://pirsa.org/04110003

Abstract:

Einstein-Hilbert action

$$I = \frac{1}{16\pi G} \int_{\mathcal{M}} d^3x \sqrt{-g}\,(R - 2\Lambda)$$

$$I_{tot} = I + I_{GCS} \,,$$

Chern-Simons gravity term is

$$I_{GCS} = -\frac{1}{32\pi g \mu} \int_{\mathcal{M}} d^3x\, \epsilon^{\lambda\mu\nu} \Gamma^{\rho}_{\lambda\sigma} \left(\partial_\mu \Gamma^{\sigma}_{\rho\nu} + \frac{2}{3} \Gamma^{\sigma}_{\mu\tau} \Gamma^{\tau}_{\nu\rho}\right)$$

We consider classical solutions, which are spaces of constant curvature: (AdS case)

$$M \sim [0,1] \times \Sigma_{(g,n)}$$

genus, "holes" or, actually black holes

Fundamental variables: a local frame e^a_μ

$$\eta^{ab} e_{a\mu} e_{b\nu} = g_{\mu\nu} \,, \quad \eta_{ab} = \mathrm{diag}(-++)$$

a spin connection $\omega_\mu{}^{ab}$

1-forms $\quad e^a = e_\mu{}^a dx^\mu \,, \quad \omega^a = \frac{1}{2} \epsilon^{abc} \omega_{\mu bc} dx^\mu$

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Action becomes

$$I = -2\int_{\mathcal{M}} \left(e\wedge d\omega + \frac{1}{2} e\wedge\omega\wedge\omega + \frac{\Lambda}{6} e\wedge e\wedge e \right)$$

For $\Lambda = -1/\ell^2 < 0$, introducing the variables

$$A^{(\pm)a} = \omega^a \pm \frac{1}{\ell} e^a$$

$so(2,1) \times so(2,1)$ gauge potential with CS action

$$I[A^{(+)}, A^{(-)}] = I_{CS}[A^{(+)}] - I_{CS}[A^{(-)}]$$

$$I_{CS}[A] = \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{tr}\left(A\wedge dA + \frac{2}{3} A\wedge A\wedge A \right)$$

Poisson Brackets: $i,j = 1,2$

$$\{ A_i^{(\pm)a}(x), A_j^{(\pm)b}(x') \} = \pm \frac{1}{\ell} \epsilon_{ij} \eta^{ab} \delta^2(x-x')$$

Observables are holonomies, or geodesic functions :

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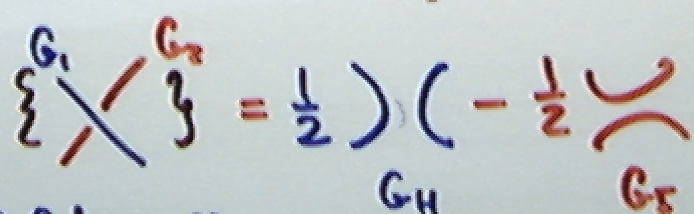
$$G_\gamma^{\pm} = P\exp\left\{ \int_\gamma A_i^{(\pm)a} T_a dx^i \right\}$$

Relations between G_γ :



Goldman '87

However . a topological theory, that is, the number of parameters is finite

A convenient parametrization is given by graph description of Teichmüller space of R.S. of genus g with n punctures (holes) due to Penner and Fock

We show that these coordinates admit a convenient and simple quantization procedure resulting in quantum Turaev-type relations for quantum geodesic functions. This supports this choice of quantization procedure constructed by Fock & Ch. (also by R.Kashaev)

Que: Space of observables = space of closed geodesics on R.S. Due to the skein relation (below), the basis in this space is constituted by multicurves (curves without self intersections)

However, 2+1 gravity is a topological theory, that is, the number of parameters is finite

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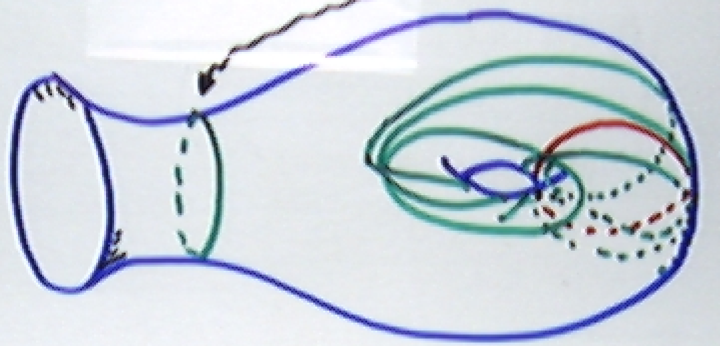
No closed geodesics intersect the horizon



Still we have infinitely many curves and in mathematics there was a construction (the train track theory, a part of Thurston theory) describing a proper topological completion of this set of curves
That is, the set of observables of quantum theory.

Problem The mapping class group action on train tracks makes their set nonseparable (non-Hausdorff)

Solution We introduce additional (gauge) parameters on the graph (graph lengths), which are also transformed under the mapping class group action and remain just classical variables upon quantization

Then, merely the ratio $l_{p.e.}/l_{g.l.}$ acquires nice transformation properties under the mapping class group action and is continuous (classically) or is weakly continuous (quantum case) w.r.t. the Thurston coordinates of the space of (generalized) multicurves

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Teichmüller space of surfaces with holes

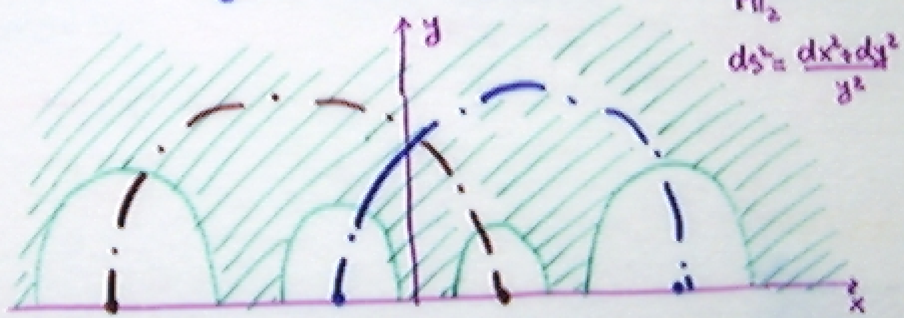
GRAPH DESCRIPTION

classical: Penner '88, Fock '93
quantum: Ch.-Fock 97-99
Kashaev 97-00

1. Explicit construction of the Fuchsian group $\Delta_{g,n}$ in the Poincaré uniformization

R.S.: $\Sigma_{g,n} = \mathbb{H}_2^+ / \Delta_{g,n}$

$\Delta_{g,n} \subset PSL(2,\mathbb{R})$



A fundamental domain is hatched; the problem is to find a proper description of elements of $\Delta_{g,n}$.

The mapping class group changes the parametrization

Observables must be m.c.g.-ivariant

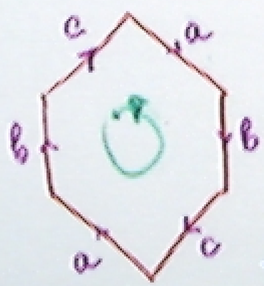
Those are lengths of closed geodesics

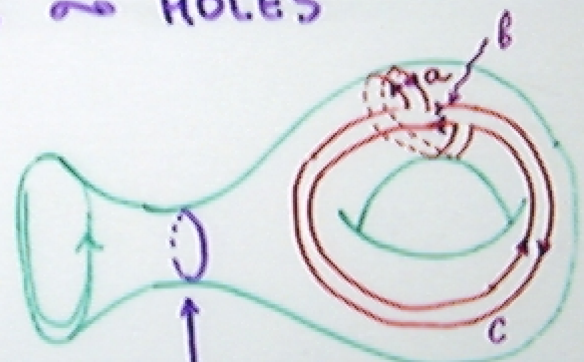
Importa[illegible] closed geodesics
are multi[illegible] minations :

Geodesic lamination is a set of multi- and self-nonintersecting curves. Each such set describes a folding of the corresponding R.S. These sets satisfy an algebra of geodesics to be derived below.

Representing graphs = FAT (or RIBBON) graphs on holed (punctured) R.S.

FACES ⇝ HOLES





$\Sigma_{1,1}$: torus with one hole

„Bottleneck" : no closed geodesics allowed thru
(in 2D gravity setting it is a black hole horizon)

For 3-valent graphs, # edges = $6g-6+\underline{3n}$

$\dim_{\mathbb{R}} \mathcal{M}_{g,n} = 6g-6+2n \Rightarrow$ n central elements

There are exactly geodesic lengths around holes.

Important subclass of closed geodesics are multicurves, or laminations:

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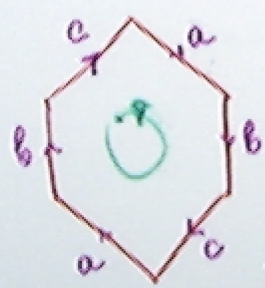

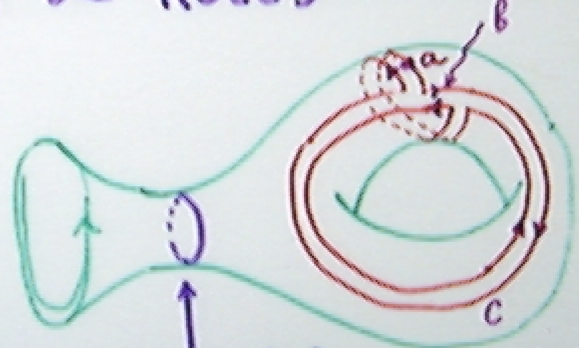

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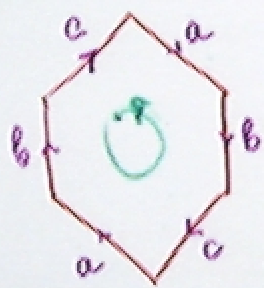

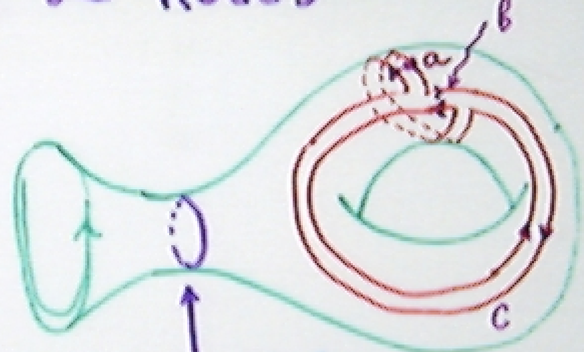

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Teichmüll... urfaces with holes

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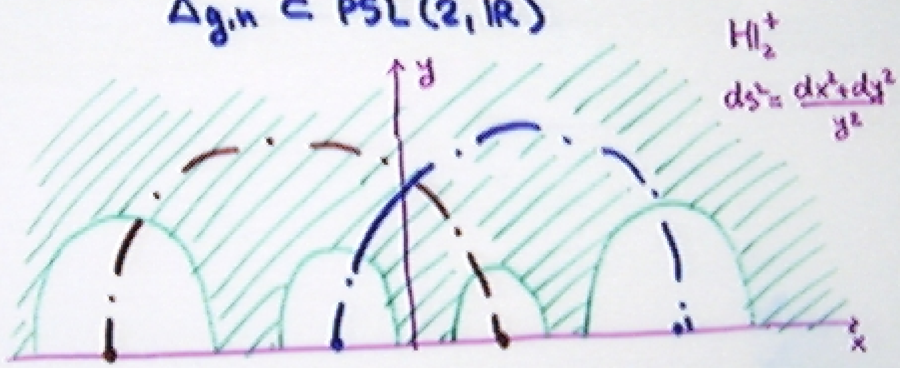

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$6g-6+2n$

$\mathrm{im}_{\mathbb{R}} \mathcal{M}_{g,n} + \mathbb{R}$

- We associate real number Z_i to each edge of the graph

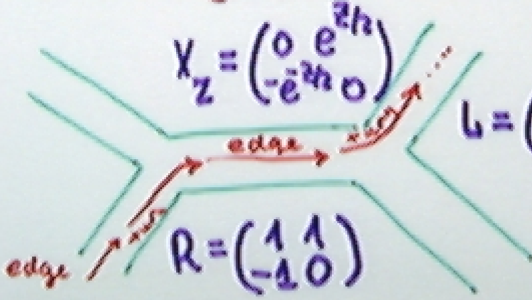
- the set of 1-1 correspondences

$$\left\{ \begin{matrix} \text{conjugacy classes} \\ \text{of } \Delta(\Sigma) \subset PSL(2,\mathbb{R}) \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{closed geodesics} \\ \gamma \text{ on } \Sigma_{g,n} \end{matrix} \right\}$$

$$\updownarrow \qquad\qquad\qquad\qquad \updownarrow$$

$$\left\{ \begin{matrix} \text{conjugacy classes} \\ \text{of } \pi_1(\Sigma_{g,n}) \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{closed paths on} \\ \text{(any) of graphs } \Gamma(\Sigma) \\ \text{representing } \Sigma_{g,n} \end{matrix} \right\}$$

- Explicit parameterisation of the PATH in the graph $\Gamma(\Sigma)$ as an element of $PSL(2,\mathbb{R})$



$$X_Z = \begin{pmatrix} 0 & e^{Z/2} \\ -e^{-Z/2} & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

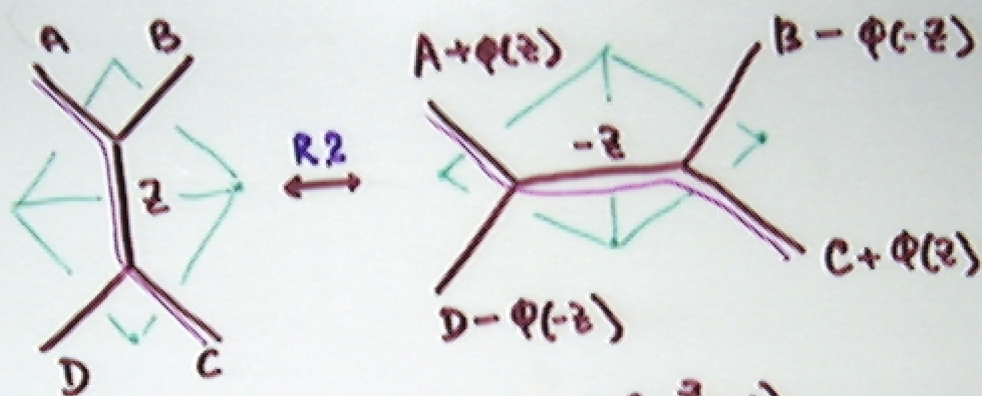
$$P_{Z_1 \ldots Z_n} = L X_{Z_1} R X_{Z_2} R \cdots L X_{Z_n}$$

The physical geodesic length $l(\gamma)$ is

$$\boxed{G_\gamma \equiv \mathrm{tr}\, P_{Z_1 \ldots Z_n} = 2 \cosh l(\gamma)/2}$$

Classical m... group transformation

[Purely geometrical origin]

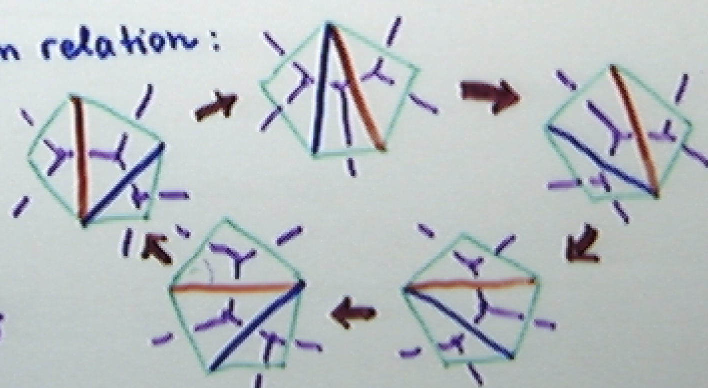


$$\varphi(z) = \log(e^z + 1)$$

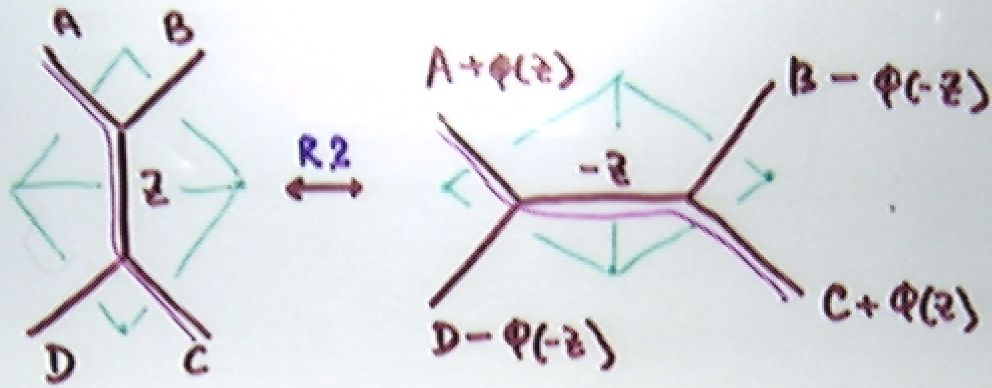
Lemma 1 The set of $G(\gamma)$ is invariant w.r.t. the flip transformation

Lemma 2. R.2 satisfies: α. $(R.2)^2 = id$

β. Pentagon relation:



identity



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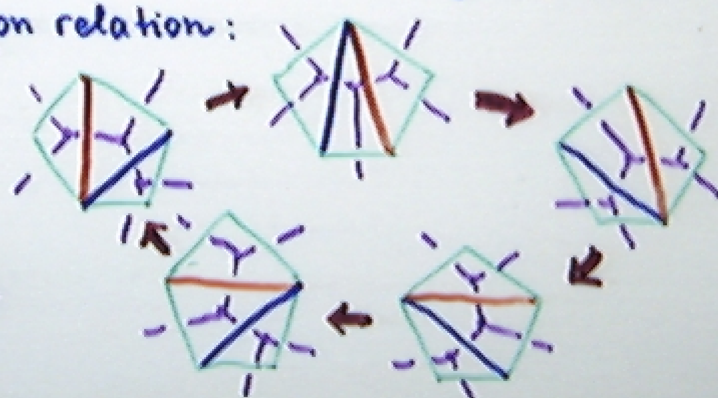

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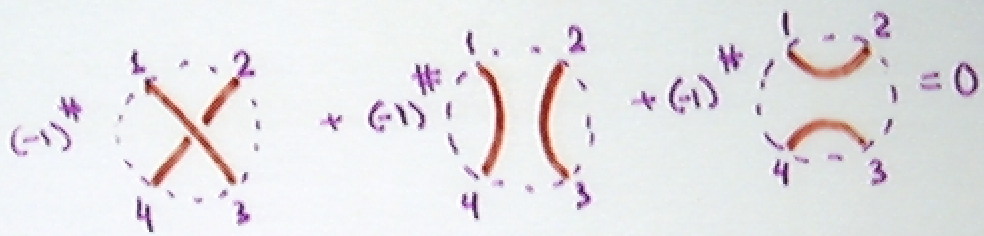
CLA ... A of G.L.

① SKEIN ...

This is a purely algebraic relation that holds for any 2×2 matrices A and B with unit determinants:

$$\mathrm{tr}A\,\mathrm{tr}B = \mathrm{tr}AB + \mathrm{tr}AB^{-1}$$

Then, graphically, in a local neighborhood



= total number of geodesics in the whole graph

We can therefore disentangle any crossing and always express any (nonmulticurve) geodesic function through G.L. The same is true for any product of G.L.'s:

$$G_{\{\gamma\}}G_{\{\gamma'\}} = \sum C^{\{\gamma''\}}_{\{\gamma\gamma'\}}\, G_{\{\gamma''\}}$$

Here, if $\{\gamma\}$ contains several geodesics (simple) $\gamma_1, \dots, \gamma_d \in \{\gamma\}$, then

$$G_{\{\gamma\}} = \prod_{i=1}^{d} G_{\gamma_i}\ .$$

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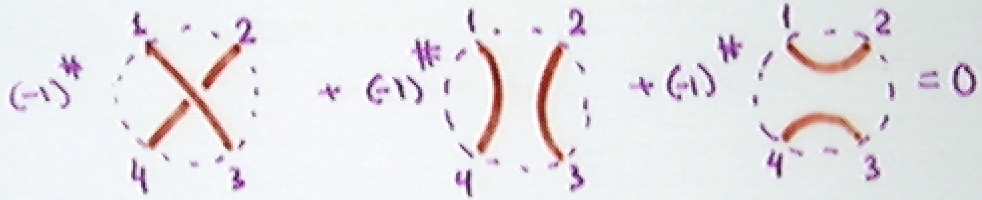

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First step …… tion

Poisson structure

We introduce the Poisson structure on the space of edge variables Z_i

$\{ Z_i , Z_{i+1} \} = 1$, $i = 1,2,3 \mod 3$

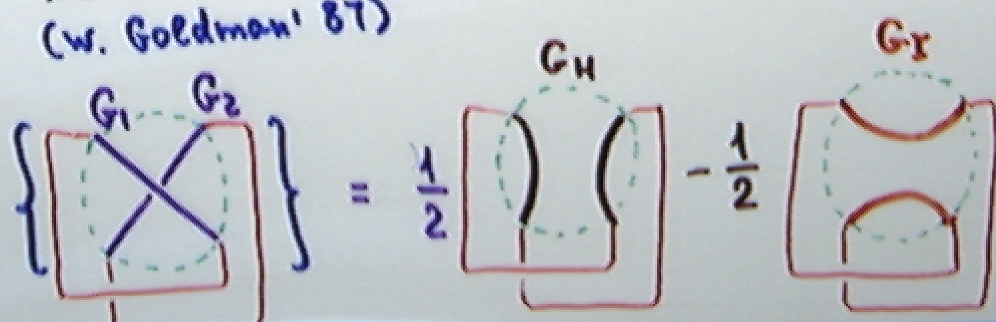
Motivations: comparison with 2+1 gravity approach by Nelson & Regge

Then: α. geodesic functions for geodesics around holes are central elements

$L_I = \sum_{i \in I} Z_i$

β. the Poisson algebra is m.c.g. invariant

Lemma 2. The above Poisson brackets generate the classical Poisson algebra of geodesics (W. Goldman '87)



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1. All algebras are isomorphic as linear spaces.
2. For $\hbar = 0$, the algebra is isomorphic as a $\mathcal{D}$-module to the $*$-algebra of complex-valued functions on the Poisson manifold.
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So
$$\boxed{[Z^\hbar_\alpha, Z^\hbar_\beta] = 2\pi i \hbar \{Z_\alpha, Z_\beta\}}$$
$$Z^{\hbar\,*}_\alpha = Z^\hbar_\alpha$$

Quantum flip morphisms with 
quantum function

$$\varphi^\hbar(z) = -\frac{\pi\hbar}{2} \int \frac{e^{-ipz}}{\sinh(\pi p)\sinh(\pi \hbar p)}\, dp$$

(Faddeev '95)

Th. 4 The family of algebras $\mathcal{T}^\hbar(S)$ is a quantization of the space $\mathcal{T}^H(S)$. Relation

$D(\Sigma)$: $\{A, B, C, D, Z\} \to \{A + \varphi^\hbar(Z), B - \varphi^\hbar(-Z), C + \varphi^\hbar(Z), D - \varphi^\hbar(-Z), -Z\}$

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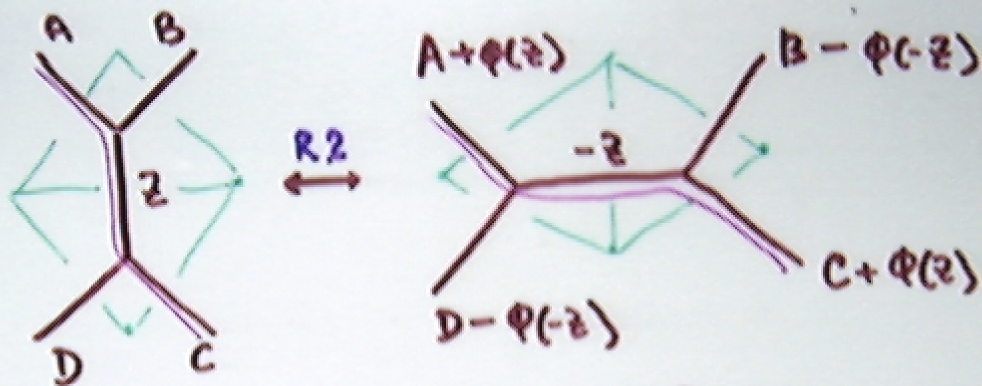
Classical group transformation

[Purely geometrical origin]



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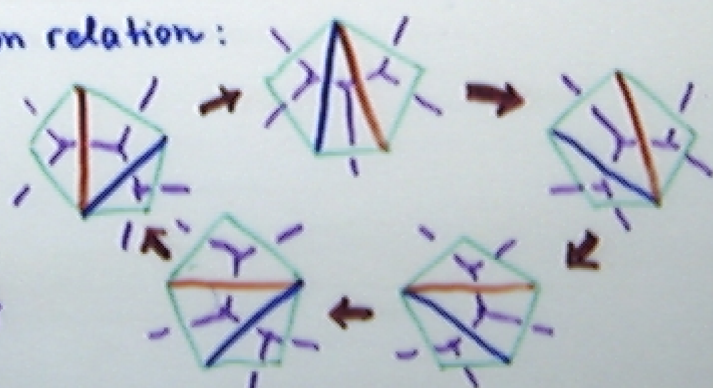

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- Thus, algebra does not depend on the choice of the basis graph.

Quantum geodesic length operators

We embed the geodesic algebra into a suitable completion of the algebra $\mathcal{T}^\hbar(S)$.
(must introduce the ordering)

$$G^\hbar_\gamma = {:}\,\mathrm{tr}\, P_{Z_1 \ldots Z_n}{:} = \sum_{\substack{j \in J \\ \varkappa \in \{j\}}} \exp\left\{ \frac{1}{2} \sum_{\alpha \in E(\Gamma)} \left(m_j(\gamma,\alpha) Z^\hbar_\alpha + 2\pi i \hbar C^{\varkappa}_j(\gamma,\alpha) \right) \right\}$$

$m_j(\gamma,\alpha)$ are exactly as in classical case,

but integer coefficients $C^{\varkappa}_j(\gamma,\alpha)$ are to be determined from the conditions below.

- The mapping class group action $\Delta(S)$ must preserve the set $\{G^\hbar_\gamma\}$: $\delta \in \Delta(S) \Rightarrow$
  $$\delta\, G^\hbar_\gamma = G^\hbar_{\delta\gamma}.$$
- Geodesic algebra. The product of two quantum geodesics is a linear combination of Quantum geodesic laminations (QGLs) with skein relation by Turaev
  QGL = set of non-(self)intersecting geodesics
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- $G^\hbar_{n\gamma} = 2T_n(G^\hbar_\gamma/2)$, as in the classical case, T_n are Chebyshev's polynomials
- $G^\hbar_\gamma$ always commute with $G^{1/\hbar}_{\gamma'}$.

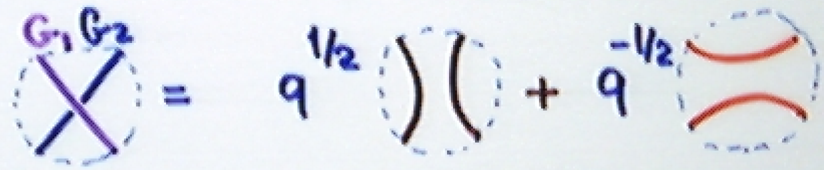
Quan elations

$$G_{\gamma_1} G_{\gamma_2} = \sum_{L} C^{\gamma_L}_{\gamma_1 \gamma_2} \cdot L_{\gamma_L}$$

L - laminations [well defined because nonintersecting G_γ commute]

Lemma

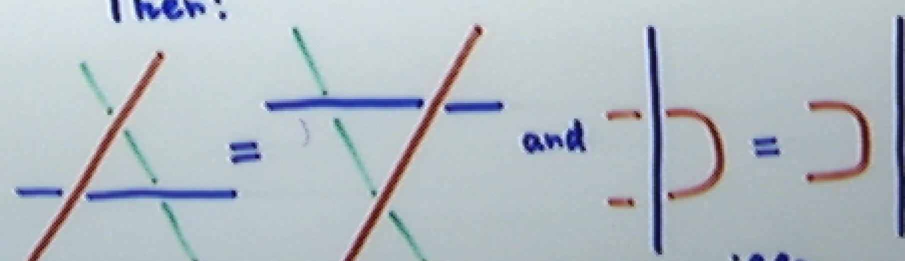
In Each intersection simultaneously we have quantum (Turaev) relations:



$$q = e^{2\pi i \hbar}$$

Lemma 2 If an empty loop ○ ≡ $q + q^{-1}$

Then:



Reidemeister moves are possible

$$G_{\gamma_1} G_{\gamma_2} = \sum_{L} C^{\gamma_L}_{\gamma_1 \gamma_2} \cdot L_{\gamma_L}$$

L - laminations [well defined because nonintersecting G_γ commute]

Lemma

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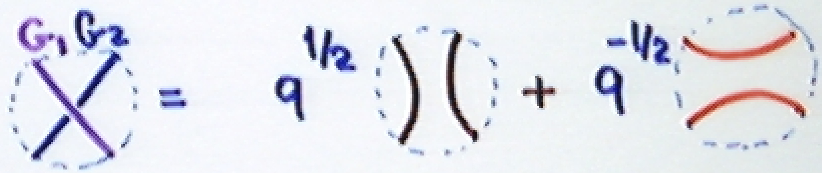

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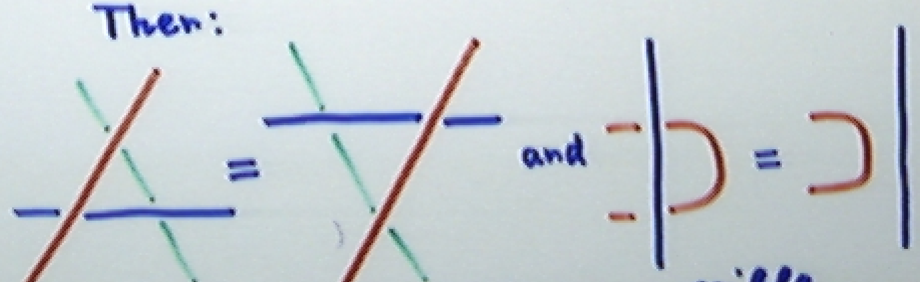

i.e., Riedemeister moves are possible

Claim For $\overset{\text{graph}}{\text{simple}}$ geodesics, the ordering

$\colon\,\colon$ coincides with the Weyl ordering,

i.e., all $c_j^{\varkappa} \equiv 0$ in this case.

Note in the general case this apparently not the true.

Quantum geodesics for torus,

an illustrative example

$$\tilde{G}_z = e^{-X_h - Y_h - 2} + e^{X_h - Y_h - 2} + e^{X_h - Y_h} \cdot 2\cos(\pi\hbar) + e^{Y_h - Y_h + 2} + e^{Y_h + Y_h + 2}.$$

But, again, Because

$$G_X G_Y = e^{i\pi\hbar/2}\tilde{G}_z + e^{-i\pi\hbar/2} G_z$$

and

$$G_Y G_X = e^{-i\pi\hbar/2}\tilde{G}_z + e^{i\pi\hbar/2} G_z \,,$$

We obtain finitely generated algebra with the lexicographic basis and relations:

Let $[A,B]_q \equiv q^{1/2}AB - q^{-1/2}BA$, $q = e^{-i\pi\hbar}$

$$\xi \equiv q - q^{-1}$$

Then:

$$\left.\begin{array}{l} [G_X, G_Y]_q = \xi G_Z \\ [G_Y, G_Z]_q = \xi G_X \\ [G_Z, G_X]_q = \xi G_Y \end{array}\right\} \; so_q(3) \text{ algebra}$$

$\xi = q - q^{-1}$

$\ast\ \ast$ the Weyl ordering,

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$\cancel{\xi = q + q^{-1}}$

quantum Markov relation

$$\mathcal{M} = G_X G_Y G_Z - q^{1/2}(G_X^2 + q^{-2}G_Y^2 + G_Z^2)$$

is the central element through geodesics

MC INTENSIONS

- We have a parameterization of "observable space basis" = the set of geodesic laminations, or multicurves, on a R.S. of genus g and # of punctures n. (In a particular case of torus with one puncture those are c.f.'s) $\{m_i\}$
- Coordinates on Teichmüller space to be quantized $\{Z_i^{(n)}\}$

UNIFICATION LOOKS JUST TRIVIAL :

GEOD. FUNCTION $G_\gamma = \mathrm{tr}\,\Delta_\gamma$ = Laurent polyn. in $e^{Z_i/2}$ for each (rational) $\{m_i\}$

⊙ PROBLEM : WE MUST ENSURE

- FINITE ANSWERS FOR INFINITE-LENGTH CURVES (IRRAT. m_i), WHICH NEEDS WE MUST HAVE NORMALIZATION ; THIS NORMALIZATION MUST REMAIN CLASSICAL, i.e., INDEPENDENT on Teichmüller space coordinates $\{Z_i\}$.
- WE MUST ENSURE CONTINUITY IN BOTH CLASSICAL (LITERAL) AND QUANTUM (WEAK OPERATORIAL) SENSES BOTH AT RATIONAL AND AT IRRATIONAL POINTS

- These conditions suffice for fixing

MOTIVATIONS & INTENSIONS

- We have a parameterization of "observable space basis" = the set of geodesic laminations, or multicurves, on a R.S. of genus g and # of punctures n. (In a particular case of torus with one puncture those are c.f.'s) $\{m_i\}$
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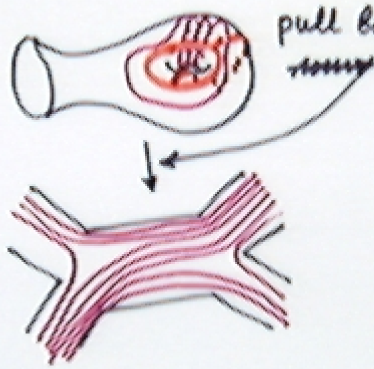
Thurs[illegible]

(from physical standpoint)
is "effective" theory for observables =
= lengths of closed geodesics

Closure of sets of foliations on the R.S.



pull back to FAT GRAPH:

Actual problem:
What happens if # of threads becomes large?

We have two types of variables:

(1) Z_α – coordinates of the Teichmüller space

(2) n_α – numbers describing the laminations

OLD tracktation: (2) = proj. limit of (1) as $|Z_\alpha| \to \infty$

OUR approach: We need (1) $\oplus$ proj. limit of (1)
to describe quantum Thurston theory =
quantum theory of observables of 3D gravity.

Thurston theory

(from physical standpoint)
is "effective" theory for observables =
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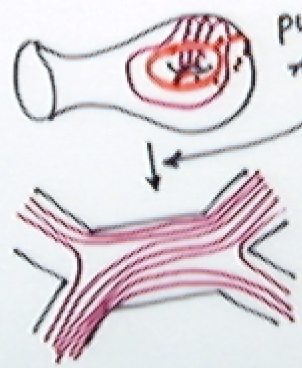

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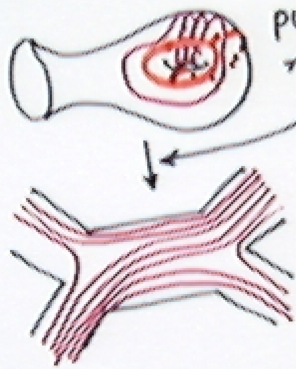

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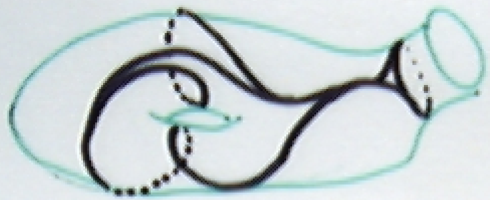
THURST[illegible] (with R. Penner)

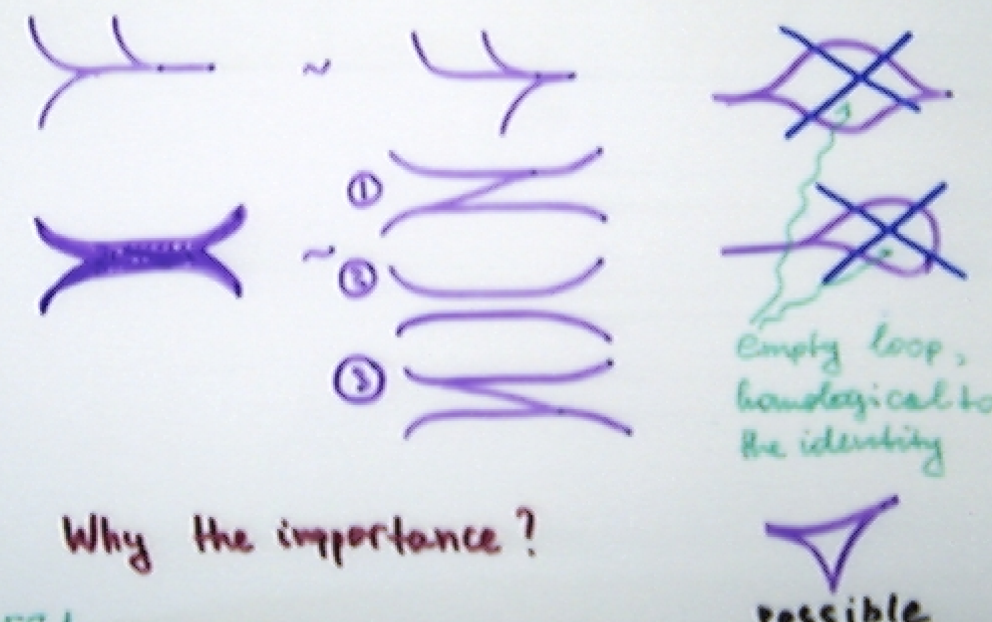
(TRAIN TRA[illegible] [illegible]RADAS)

DESCRIBING "LONG" MULTICURVES

A set of edges on a R.S. :



With identification properties:



Why the importance?

DEF. 1 A train track carrying a lamination

∃ a lamination that can be smoothly transformed into a (collection of) curve on a tr.tr. s.t. each edge of tr.tr. contains at least one (part of) curve.

THURS… (with R. Penner)

(TRAIN TRACKS, or AUTOSTRADAS)

DESCRIBING „LONG" multicurves

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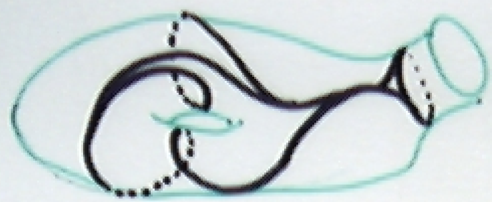

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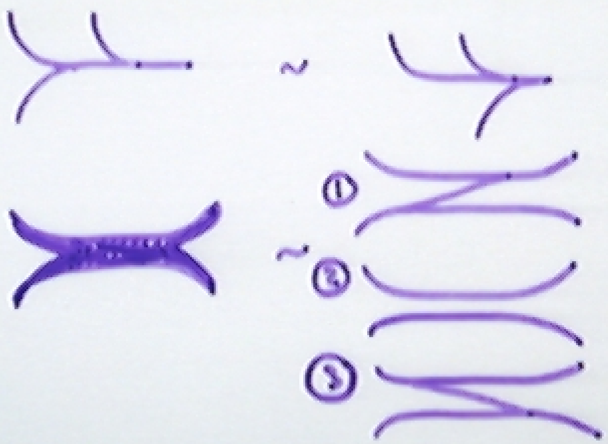

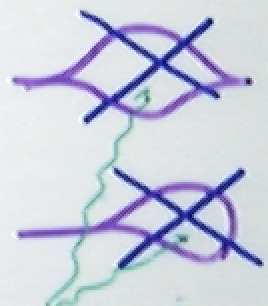

Empty loop, homological to the identity

Why the importance ?

possible

DEF. 1 A train track **carrying** a lamination

∃ a lamination that can be smoothly transformed into a (collection of) curve on a tr.tr. s.t. each edge of tr.tr.

THURSTON THEORY (with R. Penner)

(TRAIN TRACKS, or AUTOSTRADAS)

DESCRIBING „LONG" MULTICURVES

A set of edges on a R.S. :

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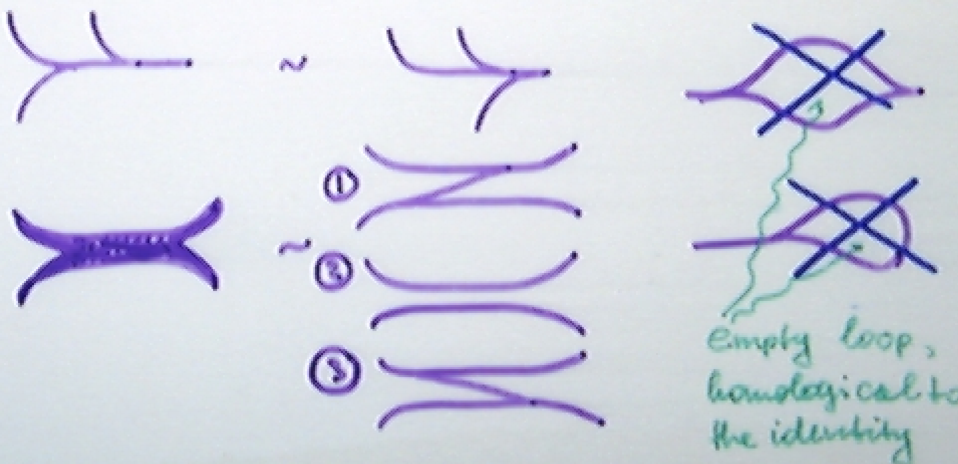

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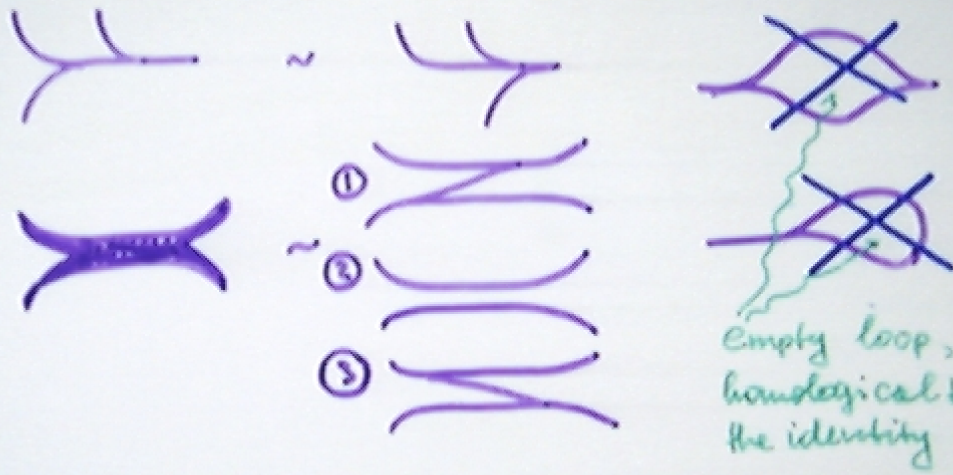
∃ a lamination that can be

DESCRIBING ... AVES

A set of edges ...

With identification properties:



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∃ a lamination that can be
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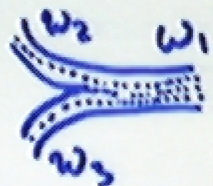
curve on a tr. tr. s.t. each edge of tr. tr.
contains at least one (part of) curve.
Tr. tr. is recurrent if it may carry a lamination

DEF. 2 TRANSVERSE MEASURES ON TR.TR.

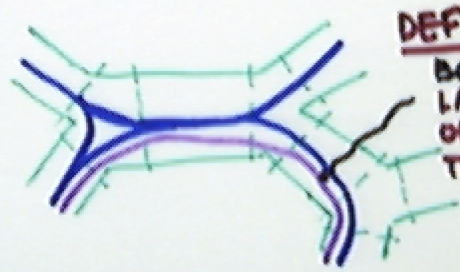
Positive real numbers w_i:



$w_1 = w_2 + w_3$

All tr.tr. supporting transverse measures are recurrent and vice versa.

CANONICAL TR.TR. : DRAWN ON A FAT GRAPH



DEF. 2'

BOUNDARY COMPONENT: LARGEST SEPARABLE PART OF TR. TR. HOMEOMORPHIC TO A BOUNDARY OF HOLE

DEF. 3 HAVING NONNEGATIVE REALS Γ_i ON EDGES OF FAT GRAPH, WE DEFINE A UNIQUE CANONICAL TR.TR. IFF Γ SATISFY Δ inequalities at each vertex.

(STILL $\{Z_\alpha\}$ and $\{\Gamma_\alpha\}$ are very different: Z_α are nonconstrained.)

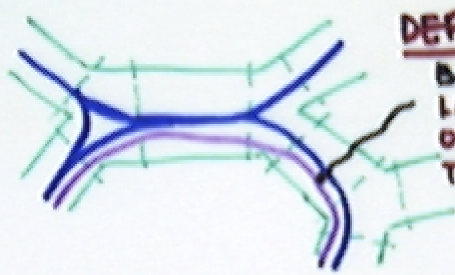
DEF. 4 BROKEN MEASURED TR.TR. = THE EQUIVALENCE ...

Positive

$\omega_1 = \omega_2 + \omega_3$

All tr.tr. supporting transverse measures are recurrent and vice versa.

CANONICAL TR.TR. : DRAWN ON A FAT GRAPH



DEF. 2'

BOUNDARY COMPONENT: LARGEST SEPARABLE PART OF TR.TR. HOMEOMORPHIC TO A BOUNDARY OF HOLE

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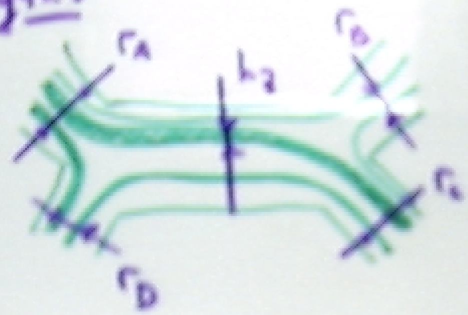
(STILL $\{Z_\alpha\}$ and $\{r_\alpha\}$ are very different: Z_α are nonconstrained.)

DEF. 4 BROKEN MEASURED TR.TR. = THE EQUIVALENCE CLASSES OF TR.TR. WITH MEASURES DIFFERENT BY BOUNDARY COMPONENTS.

PROJECTIVELY INV. TR.TR. $\{r_i\} \sim \{\gamma r_i\}$.

$\to$ ARE GIVEN BY A (PROJECTIVE) SET OF h-length

h-lengths

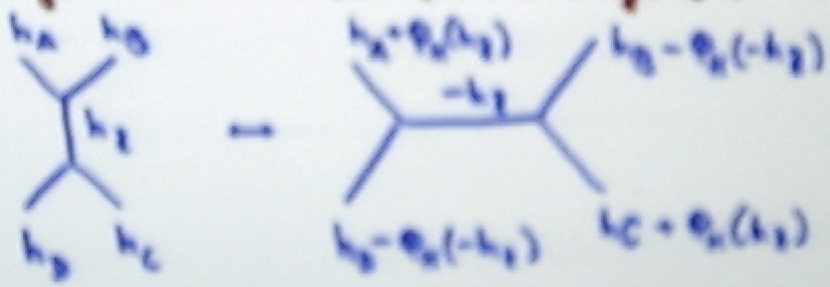


$$h_z = \frac{1}{2}(r_A - r_B + r_C - r_D)$$

h_z are now unrestricted

except $\sum_{z \in P_i} h_{z_i} = 0$

M.c.g. transformation for h-lengths:



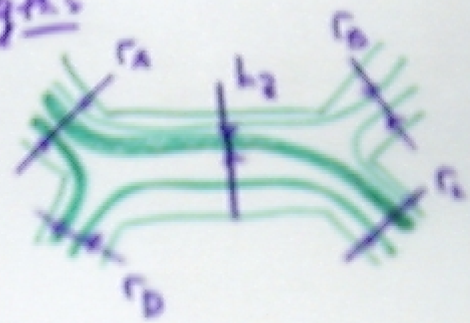
where

$$\Phi_\hbar(x) = (x + |x|)/2 = \lim_{\lambda \to \infty} \frac{1}{\lambda} \log(1 + e^{\lambda x})$$

This picture explains why h-lengths – the shear coordinates

We see that $\Phi(x)$ is just a

h-lengths



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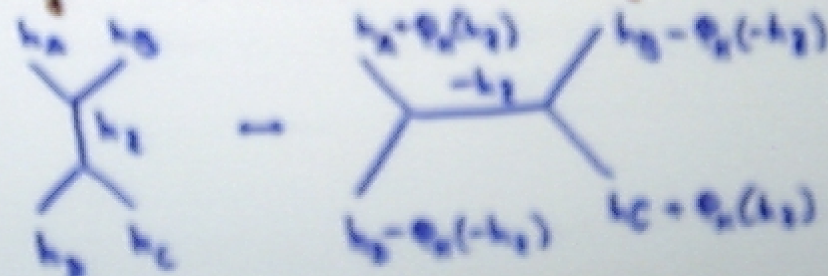

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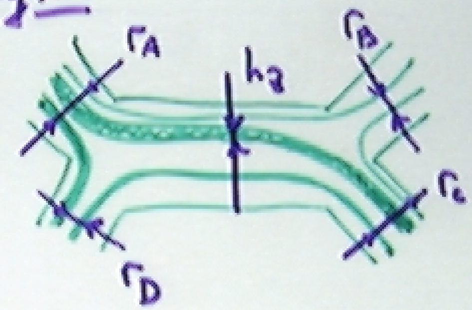

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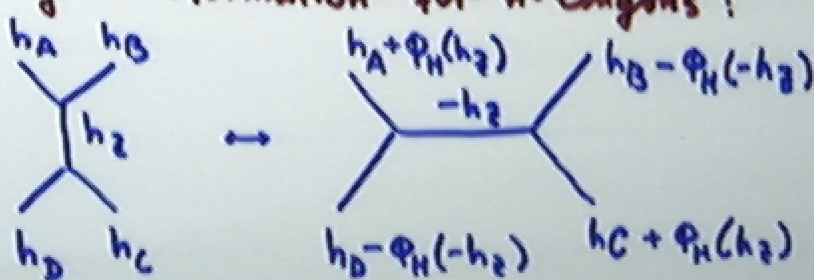
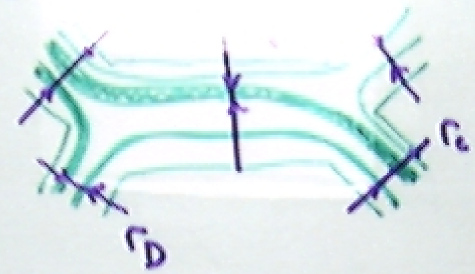


where

$$\varphi_H(x) = (x + |x|)/2 = \lim_{\lambda \to +\infty} \frac{1}{\lambda} \log(1 + e^{\lambda x})$$

h-lengths



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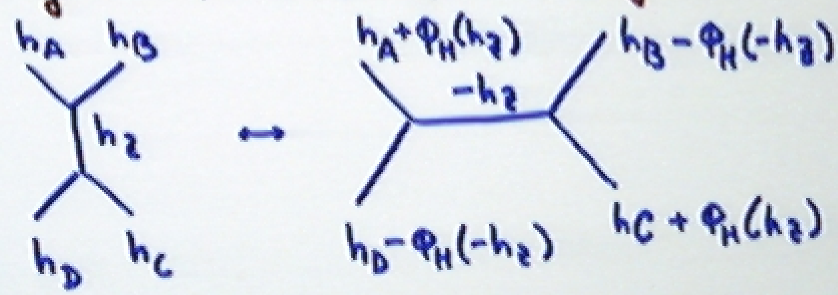

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This picture explains why h-lengths – the shear coordinates

We see that $\varphi_H(x)$ is just a projective limit of $\varphi(x)$ for Teichmüller coordinates

Approximating Laminations

DEF. 5 A sequence $\{n_i\}_\beta \equiv \vec{n}^\beta$ of integers **approximates** $\{h_i\}$ if $\lim_{\beta\to\infty} n_i^\beta / n_j^\beta = h_i / h_j \ \forall i,j$

The **approximating geodesic lamination** $GL_{\vec{m}^\beta}$ is a unique GL passing m_i^β times through i'th edge and having null boundary components

$$\{m_i\} \underset{1-1}{\longleftrightarrow} GL_{\vec{n}_i} \qquad \xi_Z^\beta = \frac{1}{2}(m_A^\beta - m_B^\beta + m_C^\beta - m_D^\beta)$$

DEF. 6 **Graph length** of $GL_{\vec{m}}$ ($g.l.(GL_{\vec{m}})$) is a linear fun. $f(\vec{m}) = \sum_i a_i m_i$, $a_i > 0$

DEF. 7 A **proper length** of a quantum geodesic operator A is

$$\lim_{n\to\infty} \frac{1}{n} \log T_n(A/2) \simeq \frac{1}{2} |\ell_\gamma| \text{ if } A = e^{\ell_\gamma/2} + e^{-\ell_\gamma/2}.$$

MAIN THEOREM Given a set $\vec{h}$ of h-lengths on the fat graph, for any approximating sequence $\vec{n}^\beta$ of n-lengths, the limit

$$\lim_{\beta\to\infty} \frac{p.\ell.(GL_{\vec{m}^\beta})}{g.\ell.(GL_{\vec{m}^\beta})}$$

unique up to "gauge" transform, or reparameterization

exists both in classical and in quantum cases

DEF.5 A ... of integers

approximates ... $\lim_{\beta\to\infty} n_i^\beta/n_j^\beta = h_i/h_j \;\; \forall i,j$

The approximating geodesic lamination $GL_{\vec{m}^\beta}$ is a unique GL passing m_i^β times through i'th edge and having null boundary components

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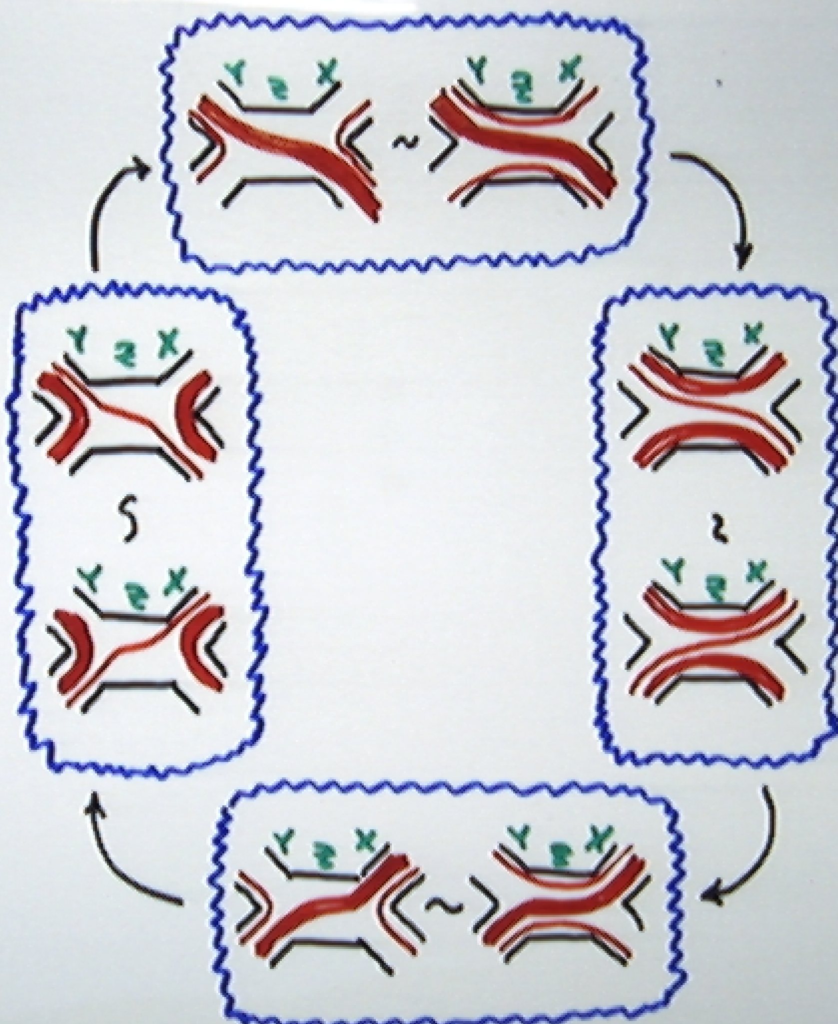
unique up to "gauge" transform., or reparameterization

exists both in classical and in quantum cases. This limit defines the completion of the (projective) set of observables of 3D gravity.

Elements of the proof

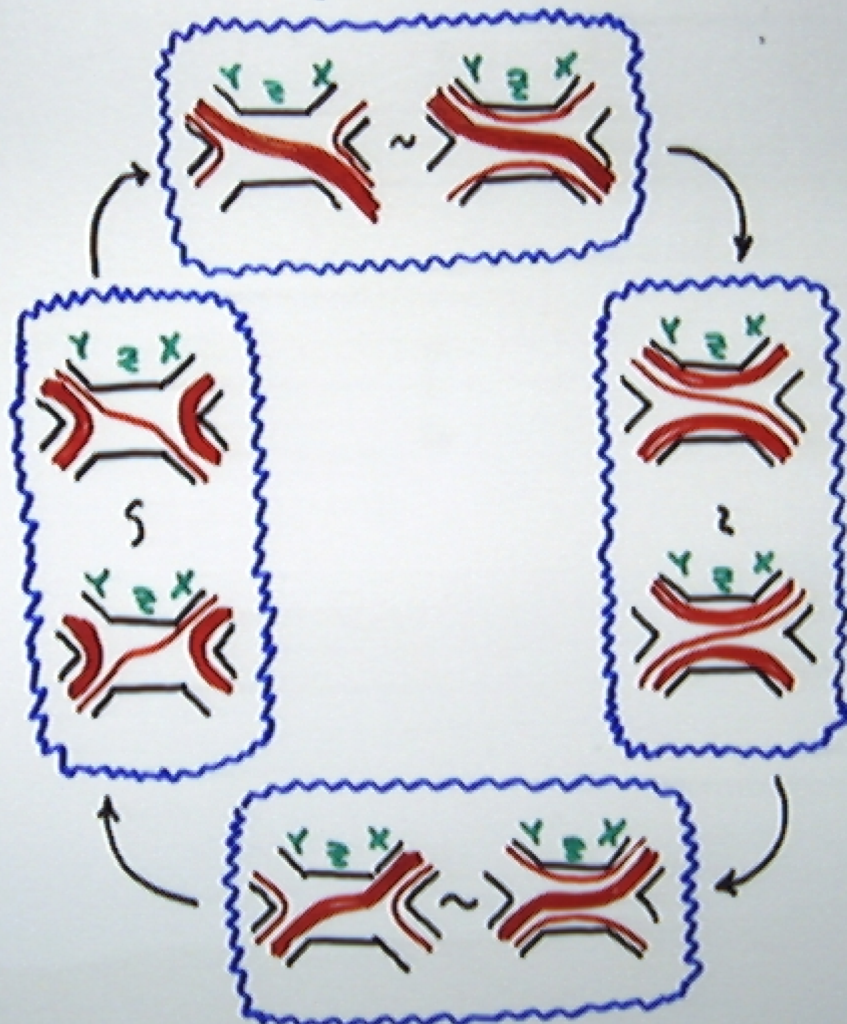
Splitting in sectors



This is circle $P\mathcal{L}_0(F_i')$

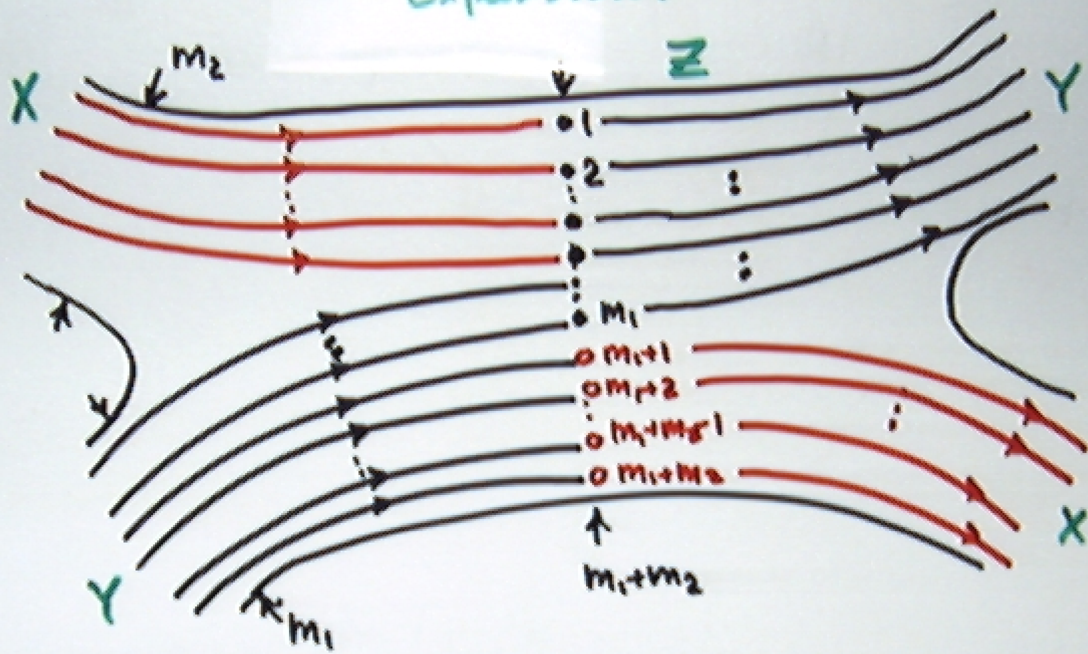
Elements of the proof

Splitting in sectors



This is circle $P\mathcal{L}_0(F_i')$

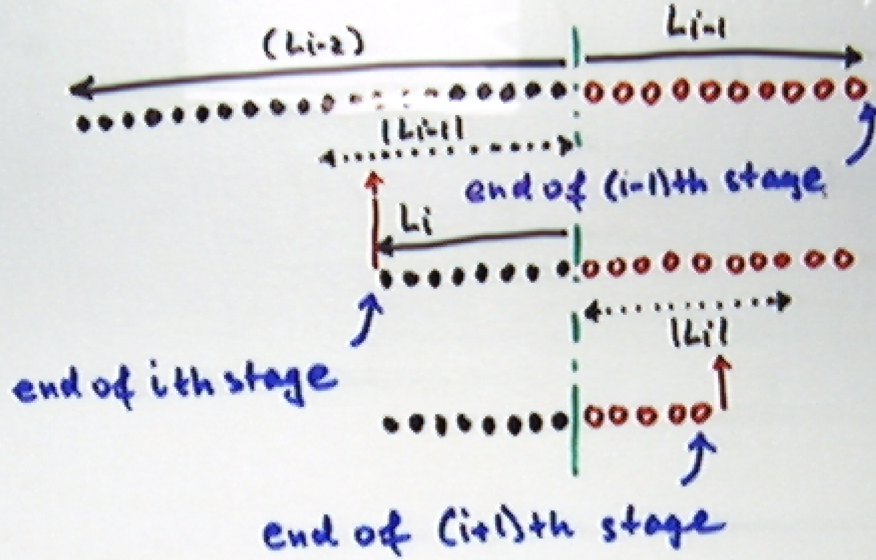
How [illegible] cont. fraction expansion?



$$\frac{m_2}{m_1} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}$$

$1 \quad 2 \quad \cdots \quad m_1-1 \quad m_1 \quad m_1+1 \quad m_1+2 \quad \cdots \quad m_1+m_2-1 \quad m_1+m_2$

starting point



We get every time closer to the starting thread

$$L_1 = (G_X)^{a_1} G_Y$$

<u>Notation</u> $L_i = \ldots\ G_X G_Y$

Then: $\tilde{L}_i \equiv \ldots\ G_Y G_X$

<u>Lemma</u>

$$L_{2i} = (L_{2i-1})^{a_{2i}-1} \tilde{L}_{2i-2} L_{2i-1} \qquad i \geq 1$$

$$L_{2i+1} = (\tilde{L}_{2i})^{a_{2i+1}-1} L_{2i-1} L_{2i}$$

$$L_0 = G_X ; \quad L_1 = (G_X)^{a_1} G_Y$$

Quantum operators $\mathcal{L}_i = :\mathrm{tr}\, L_i:$

Since L_i and $\tilde{L}_i$ are in the same homotopy class,

$$\mathcal{L}_i = \tilde{\mathcal{L}}_i$$

② Proof for ... points

α. Unzipping procedure

$$D_Y^{-1}: (m_1, m_2, m_1+m_2) \to (m_1-m_2, m_2, m_1)$$
$$D_X^{-1}: (m_1, m_2, m_1+m_2) \to (m_1, m_2-m_1, m_2)$$

For a c.f. expansion, we have the sequence of quantum Dehn twist operators

$$D_{(X \text{ or } Y)}^{-a_n} D_{(Y \text{ or } X)}^{-a_{n-1}} \cdots D_Y^{-a_3} D_X^{-a_2} D_Y^{-a_1}$$

β. Asymptotic regime

$X_{(n)}, Y_{(n)}, Z_{(n)}$ become "large" in the sense that for functions for which e.v.'s $\langle f | X_{(0)}, Y_{(0)}, Z_{(0)} | f \rangle$ were bounded, e.v.'s $\langle f | X_{(n)}, Y_{(n)}, Z_{(n)} | f \rangle$ become infinitely large as $n \to \infty$.

If $D_X^{-j} \{ X^{(0)}, Y^{(0)} \} = \{ X^{(j)}, Y^{(j)} \}$ and $D_Y^{-a_i} \{ X^{(j)}, Y^{(j)} \} = \{ X^{(i,j)}, Y^{(i,j)} \}$, then

Lemma
$$\begin{cases} p.l.(\gamma_{(i,j)}) = a_i \, p.l.(\gamma_{X(j)}) + p.l.(\gamma_{(0)}) \\ g.l.(\gamma_{(i,j)}) = a_i \, g.l.(\gamma_{X(j)}) + g.l.(\gamma_{(0)}) \end{cases}$$

That is, their ratio is well-defined.

As L_i and L_{i+1} have a single intersection

$$[\mathcal{L}_{2k}, \mathcal{L}_{2k+1}]_q = \xi \, {:}\, \mathrm{tr}\, L_{2k} L_{2k+1} {:}$$

$$[\mathcal{L}_{2k+1}, \mathcal{L}_{2k}]_q = \xi \, {:}\, \mathrm{tr}\, L_{2k-1} (\tilde{L}_{2k})^{a_{2k+1}-1} {:}$$

① Proof of continuity at rat. points.

$$I_m := {:}\, \mathrm{tr} (L_{2i-1})^{m-1} \check{L}_{2i-2} L_{2i-1} {:}$$

$$\mathcal{L}_{2i-1} := e^{\ell x/2} + e^{-\ell x/2} \quad ; \ \ell_x \text{ positively definite}$$

$$I_0 = e^{\ell_Y/2} + e^{-\ell_Y/2} \ ; \ I_{-1} = e^{\ell_Z/2} + e^{-\ell_Z/2}$$

Basic quantum relations

$$I_{m-1} \mathcal{L}_{2i-1} = q^{1/2} I_m + q^{-1/2} I_{m-2}$$

$$\mathcal{L}_{2i-1} I_{m-1} = q^{-1/2} I_m + q^{1/2} I_{m-2}$$

We then have

$$I_m = q^{-m/2} I_0 (e^{m\ell x/2} + e^{-m\ell x/2}) - q^{-m/2-1/2} I_{-1} \cdot (e^{(m-1)\ell x/2} + e^{-(m-1)\ell x/2}) \quad (*)$$

We want to present these op's in the form

$e^{mH_1 + H_0}$ with H_1, H_0 Hermitian ; the p.l. is then just H_1, but **we need** H_0 to ensure the proper comm. relations.

As $m \to \infty$:

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As $m \to \infty$:

$$q^{-m/2} I_0 (e^{m\ell_x/2} + \dots) \sim q^{m/2} (e^{m\ell_x/2} + \dots) I_0$$

Hence $\quad I_0 \sim e^{-2\pi i \hbar \partial/\partial|\ell_x|}$ in this limit

② Proof for irrational points

α. Unzipping procedure

$D_Y^{-1}: (m_1, m_2, m_1+m_2) \to (m_1-m_2, m_2, m_1)$

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For a c.f. expansion, we have the sequence of quantum Dehn twist operators

$$D_{(X\,or\,Y)}^{-a_n} D_{(Y\,or\,X)}^{-a_{n-1}} \cdots D_Y^{-a_3} D_X^{-a_2} D_Y^{-a_1}$$

β. Asymptotic regime

$X_{(n)}, Y_{(n)}, Z_{(n)}$ become "large" in the sense that for functions for which e.v.'s $\langle f| X_{(0)}, Y_{(0)}, Z_{(0)} |f\rangle$ were bounded, e.v.'s $\langle f| X_{(n)}, Y_{(n)}, Z_{(n)} |f\rangle$ become infinitely large as $n \to \infty$.

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For a c.f. expansion, we have the sequence of quantum Dehn twist operators

$$D^{-a_n}_{(X \text{ or } Y)} D^{-a_{n-1}}_{(Y \text{ or } X)} \cdots D_Y^{-a_3} D_X^{-a_2} D_Y^{-a_1}$$

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$X_{(n)}, Y_{(n)}, Z_{(n)}$ become "large" in the sense that for functions for which e.v.'s $\langle f | X_{(0)}, Y_{(0)}, Z_{(0)} | f \rangle$ were bounded, e.v.'s $\langle f | X_{(n)}, Y_{(n)}, Z_{(n)} | f \rangle$ become infinitely large as $n \to \infty$.

If $D_X^{-j} \{ X^{(0)}, Y^{(0)} \} = \{ X^{(j)}, Y^{(j)} \}$ and $D_Y^{-a_i} \{ X^{(j)}, Y^{(j)} \} = \{ X^{(i;j)}, Y^{(i;j)} \}$, then

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a. Unel...

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perspectives

1. Attaining description of 2+1 gravity and comparing with quantum Liouville program

2. Relations to Ponzano - Regge calculus (L. Freidel, K. Krasnov et al.)

3. Another (unusual) approach to number theory — "quantum" cont. fractions and quantum operators associated with periodic c.f. For instance, for $\frac{1}{1+\frac{1}{1+\frac{1}{\ddots}}}$ we have recurrence relations of form $G_{i-2} + G_{i-1}G_i + G_{i+1} = 0$ classical, or $[G_{i-1}, G_i]_q = (q-q^{-1})G_{i-2}$ quantum.

4. (Remark) In principle, instead of graph lengths, we can take a normalization point at classical Teichmüller space, e.g., the point $X=Y=Z=0$ in the punctured case. Then, the normalization procedure resembles "gauge" fixing in field theory.

Problems & perspectives

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