Title: Proofs and Pictures: The Role of Visualization in Mathematical and Scientific Reasoning

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Abstract: Do you have to see it to believe it? James Robert Brown, Professor of Philosophy at the University of Toronto, will discuss the highly interesting but controversial topic of the legitimate role of visual thinking in mathematics and science. Examples of picture proofs and thought experiments will be given. An explanation of how they work will be sketched. <kw> Proof, pictures, James Brown, axioms, sketches, experiment, Barwise, Godel, isomorphc homomorphic, intuitions, continuum hypothesis, refutation </kw>

# Proofs & Pictures

Common view of pictures and diagrams in math

Joseph-Louis Lagrange: "No figures will be found in this work. The methods that I set forth require neither constructions nor geometrical or mechanical arguments, but only algebraic operations, subject to a regular and uniform procedure." (*Mécanique Analytic*,1788, *Pref.*)

Pictures and diagrams are:

- useful psychological aids
- often provide insight
- heuristically suggestive
- fun

- but they do ***not*** provide evidence
- only a proof can do that

- pictures can be dangerously misleading

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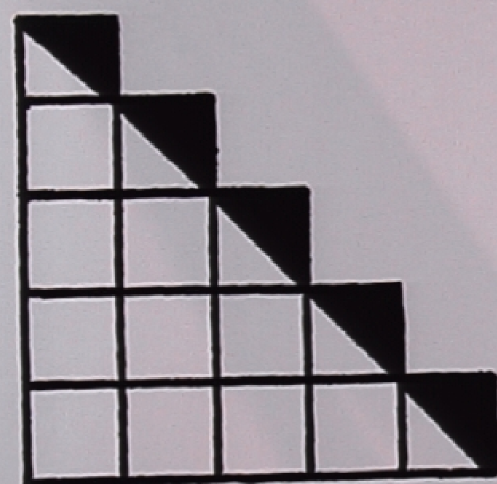
**Thm.**

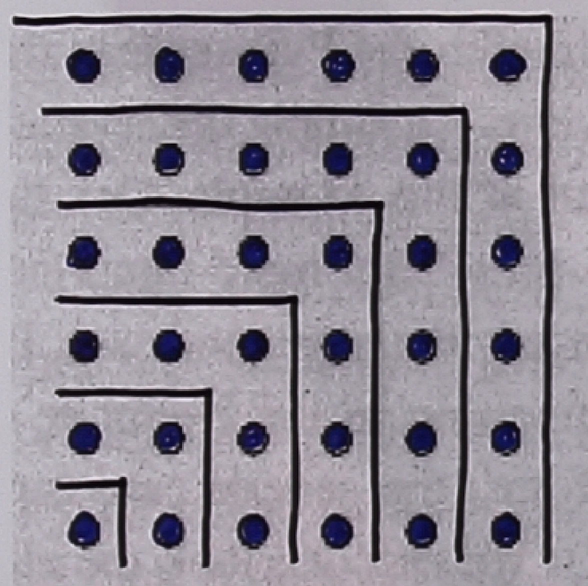
$$1+2+3+\cdots+n = \frac{n^2}{2} + \frac{n}{2}$$

**Proof**



**Thm** $1+3+5+\dots+(2n-1) = n^2$

**Proof**

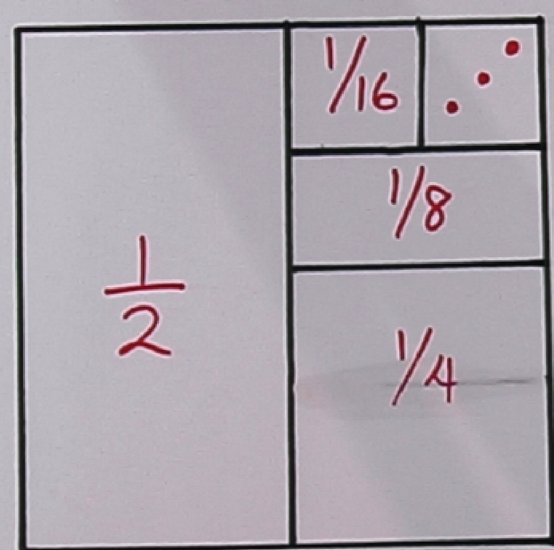


**Thm** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

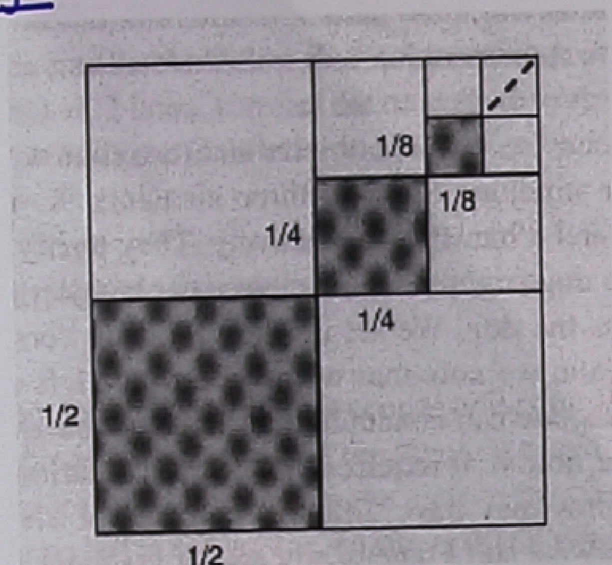
$$= 1$$

**Proof**



**Thm** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{1}{3}$

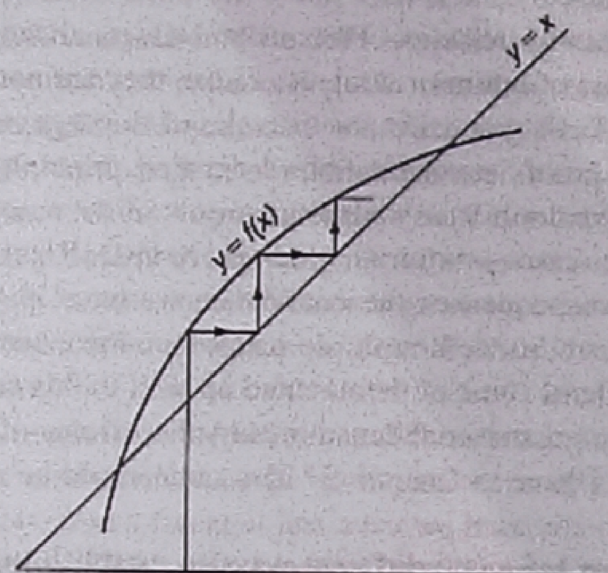
**Proof**



One of the best pictorial arguments is a proof of the 'fixed point theorem' in one dimension: *Let $f(x)$ be continuous and increasing in $0 \leq x \leq 1$, with values satisfying $0 \leq f(x) \leq 1$, and let $f_2(x) = f[f(x)]$, $f_n(x) = f[f_{n-1}(x)]$. Then under iteration of f every point is either a fixed point, or else converges to a fixed point.*

For the professional the only proof needed is [Figure 11.6].

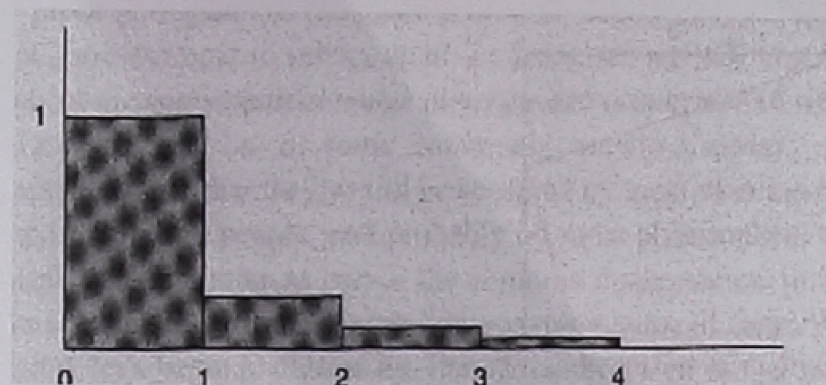
(Littlewood 1953/1986: 55)



So far, so good.
So why worry?

How much paint to cover the shaded area?

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

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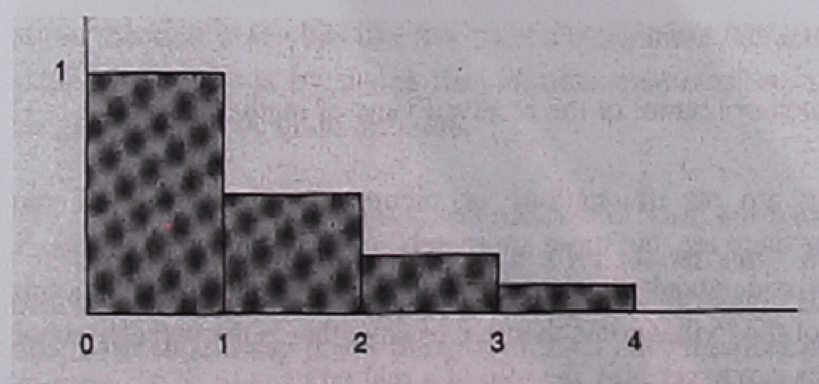

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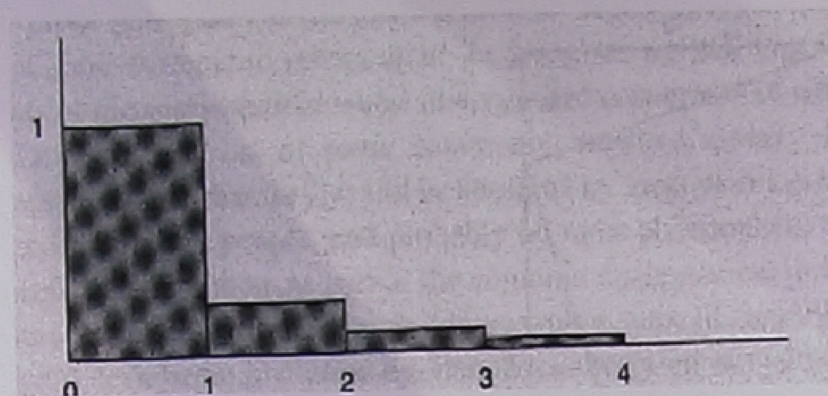


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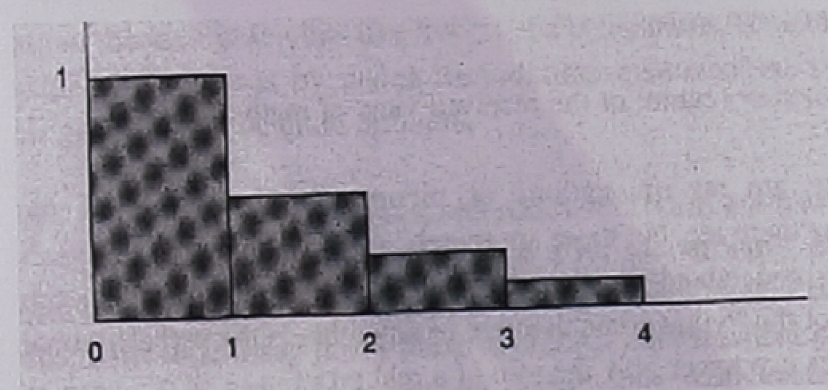

Pictures can mislead.
But so can ordinary proofs.

- Proofs are usually sketches.
- $2^{nd}$-order logic is used.
- Axioms are not proven.
- Naive set theory used.
- In computer proof hardware and software problems.

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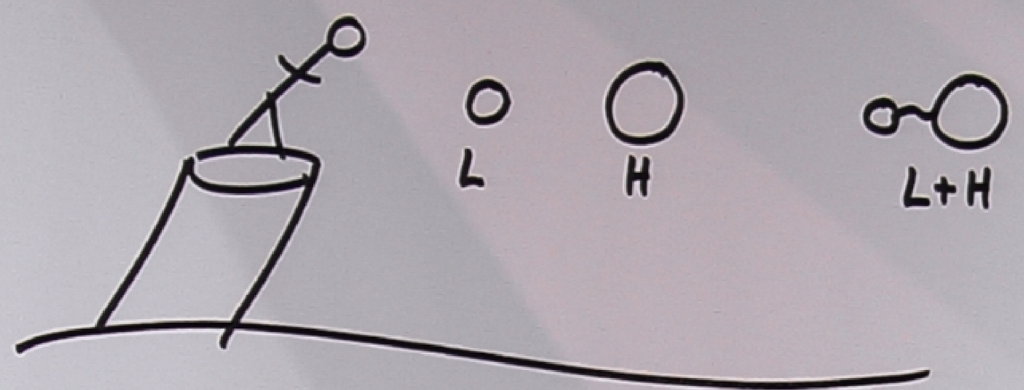
# Thought Experiments




Aristotle : $H > L$

$\therefore$ $L+H > H$

But $H > L+H$

Absurd

$\therefore$ $H = L = H+L$

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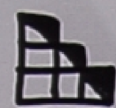
Possibly related problems

- How do thought experiments work?
- How do math pictures work?

Why do some math pictures work as proofs?

- eg. 

- Barwise: picture is isomorphic (or homomorphic) to its object.

- The picture suggests an inductive proof (which is the <u>real</u> proof).

- Gödel-type intuition. The picture is a telescope for the mind's eye looking into Plato's heaven.

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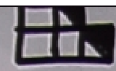
- eg. 

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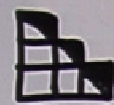
- eg. 
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# Gödel

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- They often require real ingenuity (which a picture might provide).

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# Insight vs Evidence

- Common view: pictures provide insight, not evidence.

- On the contrary. The  example <u>does</u> provide evidence.

  But greater insight comes from the verbal/symbolic proof by induction.

## How to Refute the Continuum Hypothesis

CH: $2^{\aleph_0} = \aleph_1$

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Gödel claims "that a question not decidable now has meaning and may be decided in the future.... New mathematical intuitions leading to a decision of such problems as Cantor's continuum hypothesis are perfectly possible..."

The following "refutation" is from Freiling (1986).

It has been ignored. Probably because of the style of argument.

We assume ZFC (Zermelo-Frankel set theory with the Axiom of Choice).

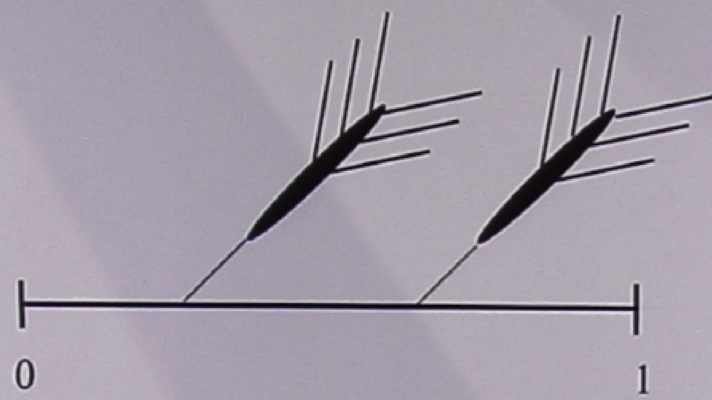
Assume the continuum hypothesis (CH) is true.

Points can be well-ordered so that for each $q$, the set $\{p \in [0, 1]: p \prec q\}$ is countable.

Let $S_q =$ the set of elements that are earlier than the point $q$ in the well ordering.

The fact that $S_p$ is countable stems from the way cardinal numbers are defined and the fact that $[0,1]$ has cardinality $\aleph_1$.

Imagine two people simultaneously throw darts at the at the interval [0,1]. Thus the darts are independent and two random numbers are selected.




Suppose the throwers hits points $p$ and $q$.

Each thrower argues as follows: the set of real numbers, say $S_q$, preceding the number I picked out is a subset of [0,1].

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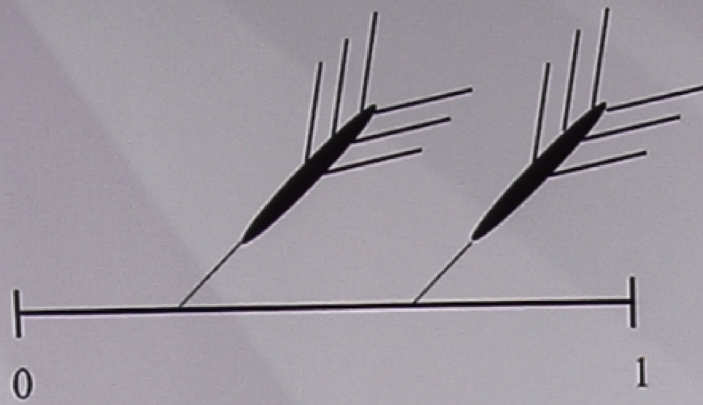


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Note that $S_q$ is a *countable* subset of points on the line.

The measure of any countable set is 0. So the probability of the other dart picking out a point in $S_q$ is 0.

The other thrower argues the same way.

The consequence is that there will be a zero probability event every time there is a pair of darts thrown. This is absurd.

Consequently, we should abandon the initial assumption, CH, since it leads to this absurdity.

Thus, CH is refuted. The number of points on the line is greater than $\aleph_1$.

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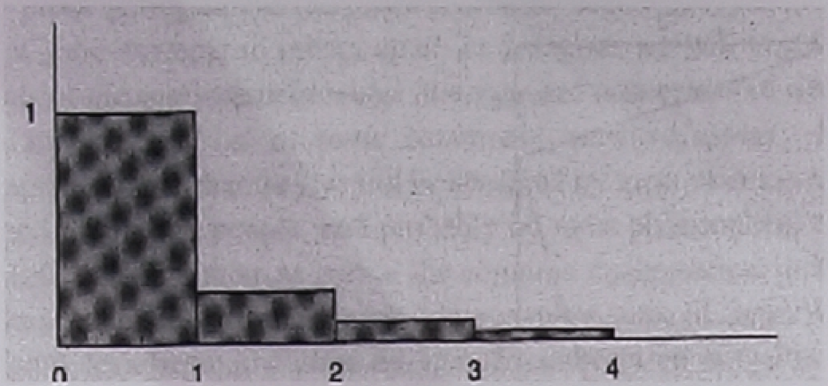


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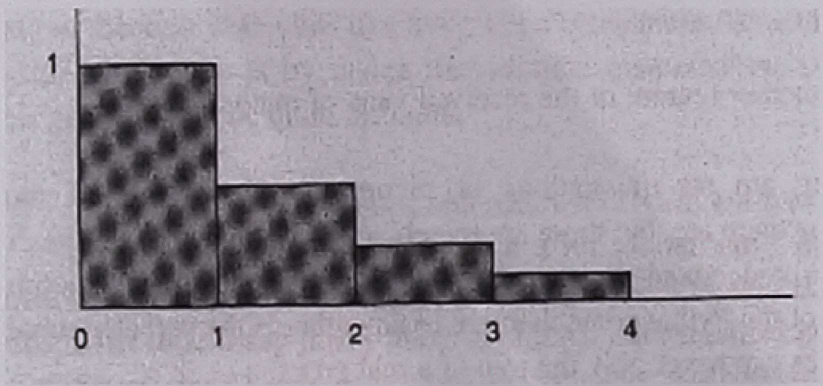



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