Title: Quantum Liouville Theory and Black Hole Entropy

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Abstract:

# Quantum Liouville Theory and Black Hole Entropy

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## Content

### 1. (2+1)-dim black hole entropy

- BTZ black hole entropy
- Asymptotic conformal algebra and Cardy formula

### 2. Boundary Liouville field theory

- Boundary Liouville field theory and spectrum
- Vertex operators as representation of quantum algebra
- Decoupling states and Hilbert space structure
- Counting of BTZ black hole entropy

### 3. Generalization to higher dimensions

## BTZ black hole entropy

(Banãdos-Teitelboim-Zanelli)

(2+1)-dim axially symmetric stationary solution with negative $\underline{\Lambda = -1/l^2}$

Asymptotically <u>Anti-de Sitter</u> space-time

Entropy $S = \frac{2\pi r_+}{4G}$

$r_+$ — radius of the horizon

**What quantum states is the entropy counting?**

– related to near horizon geometry of higher-dimensional black holes

– as a model for realistic black hole physics

Brown & Henneaux:

Asymptotic symmetry group: **2-d conformal group** with generators $L_0$, $\bar{L}_0$, central charge $c = 3l/2G$

Strominger: use **Cardy formula**

Asymptotic density of states

$$\ln \rho(\Delta, \bar{\Delta}) \sim 2\pi \sqrt{\frac{c\Delta}{6}} + 2\pi \sqrt{\frac{c\bar{\Delta}}{6}}$$

$\Delta, \bar{\Delta}$ — eigenvalues of $L_0$, $\bar{L}_0$

$$M = (\Delta + \bar{\Delta})/l, \;\; J = \Delta - \bar{\Delta},$$

obtain the correct BTZ black hole entropy

**Problems:**

1. What states are we counting?

2. Cardy formula assumes that the lowest eigenvalues of $L_0$ and $\bar{L}_0$ are zero — does the condition hold?

# Boundary Liouville field theory

Achúcarro & Townsend, Witten:

(2+1)-dim general relativity $\Longleftrightarrow$ Chern-Simons theory

Coussaert, Henneaux & van Driel:

Dynamics reduces to boundary Liouville theory

$$S_{\mathrm{L}} = \frac{1}{4\pi}\int \sqrt{\bar{g}}\times \left(\frac{1}{2}\,\bar{g}^{ab}\,\partial_a\Phi\,\partial_b\Phi + \frac{\mu}{2\gamma^2}\,e^{\gamma\Phi} + \frac{1}{\gamma}\Phi R(\bar{g})\right)$$

Classical central charge

$$c = \frac{12}{\gamma^2} = \frac{3l}{2G}$$

## Spectrum

zero mode

$$\Delta = \frac{1}{2\gamma^2} + \frac{p_0^2}{2}$$

## Normalizable states

– Normalizable zero-modes wave functions

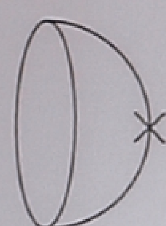
$$p_0 \text{ real}, \quad \Delta > 1/(2\gamma^2)$$

## Nonnormalizable states

– Local operator insertions $e^{\alpha\Phi}$

– Nonnormalizable zero-modes wave functions

$$p_0 \text{ imaginary}, \quad \Delta < 1/(2\gamma^2)$$



Not enough normalizable states (lowest $L_0 > 0$)

Study **nonnormalizable** states for state counting

**The general locally AdS metric**

$$ds^2 = 4\frac{G}{l}(Ld\omega^2 + \bar{L}d\bar{\omega}^2) + (e^{2\rho} + 16\frac{G^2}{l^2}L\bar{L}e^{-2\rho})d\omega d\bar{\omega} + l^2 d\rho^2$$

$L$, $\bar{L}$ — stress tensor in boundary Liouville theory

**Compare constant $L_c$ and $\bar{L}_c$ solutions with the spectrum of Liouville field theory**

| AdS solutions | Boundary Liouville theory |
|---|---|
| black hole | $\Delta > 1/(2\gamma^2)$ normalizable |
| extremal black hole | $\Delta = 1/(2\gamma^2)$ |
| conical singularity | $\Delta < 1/(2\gamma^2)$ nonnormalizable |

## Classical solution

Periodic boundary condition

Spin-$j$ $sl(2)$ representations:

$$e^{-j\gamma\phi} = \left(\frac{16}{\mu}\right)^{-j} \sum_{m=-j}^{j} \psi^j_m(x^+)\, \bar{\psi}^j_m(x^-)$$

$$2j \in N$$

## Canonical quantization

(Gervais, etc.)

classical $sl(2) \Rightarrow$ quantum algebra

Chiral operators $\psi^{j,\bar{j}}_{m,\bar{m}}(x+)$, $\bar{\psi}^{j,\bar{j}}_{m,\bar{m}}(x^-)$ obey the **exchange algebra** and **fusion rule** of $U_q(sl_2) \odot U_{\tilde{q}}(sl_2)$

$$q = e^{i\pi\tilde{\gamma}^2/2}, \quad \tilde{q} = e^{i\pi 2/\tilde{\gamma}^2}$$

## Representations of quantum algebra

$q = e^{i2\pi/p}, \ (p \in Z)$ <u>root of unity</u>

$j = 0, ..., (p'-1)/2$
$m = -j, ..., j$

$$p' = \begin{cases} p & p \text{ odd} \\ p/2 & p \text{ even} \end{cases}$$

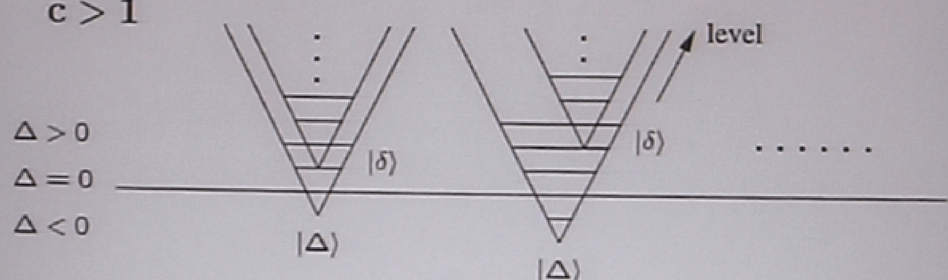
## Kac formula and reducible representations

$$\Delta_{\textsf{Kac}}(J, \hat{J}, c) = \frac{c-1}{24} + \frac{1}{4}[2\hat{J}\alpha_+ + 2J\alpha_-]^2$$

$$\alpha_+ = i\sqrt{2}/\tilde{\gamma}, \ \alpha_- = i\tilde{\gamma}/\sqrt{2}$$

$$J = j + 1/2, \ \hat{J} = \hat{j} + 1/2$$

$c > 1$



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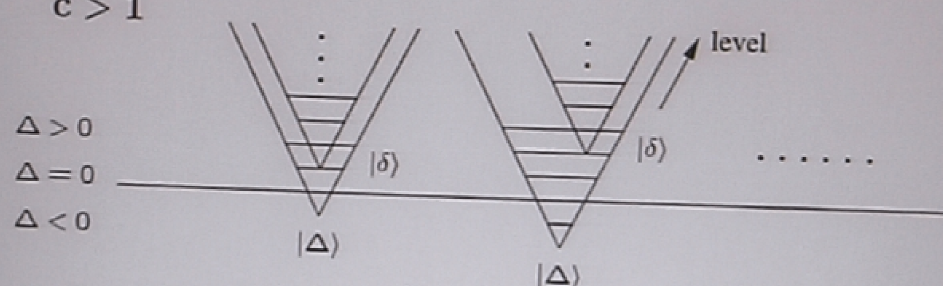
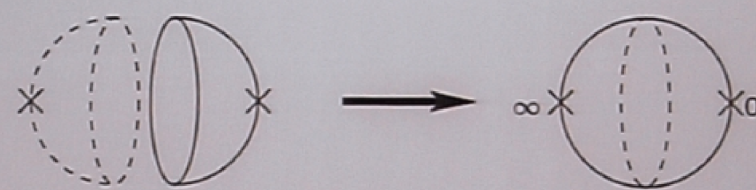


## Decoupling states




- **Finite positive-definite norm**

— Nonexistence of $SL(2,C)$-invariant vacuum

— Nonstandard Ward identity for two-point correlation functions

- **Decoupling**

— Standard Ward identity for three- or higher- point correlation functions

## BTZ black hole entropy

Hilbert space structure

$$\mathcal{H} = \mathbf{H} \otimes \mathbf{H} \otimes \ldots \otimes \mathbf{H}$$

$K$ unitary representations

Asymptotic density of states

$$\ln \rho(\Delta) = 2\pi \sqrt{\frac{K\Delta}{6}}$$

For $4/\tilde{\gamma}^2 = 2N + 1$,

$$K = c = 3l/2G$$

## Generalization to higher dimensions

Dimensional reduction near horizon of arbitrary black hole: Liouville theory governs the dynamics at the horizon

• Spherically symmetric case (Solodukhin) – Liouville theory in vicinity of horizon

• Observation: Boundary LFT interpretation of Hawking radiation, reflection amplitude

Boundary operators — horizon states

Bulk operators — propogating state

Work in progress: general stationary black hole space-time