Title: Black Hole Entropy

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Abstract:

BLACK HOLE ENTROPY

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Report on: - joint work with Marcin Domagała
- Krzysztof Meissner's work
- idea's coming from John Baez

The classical phase space: the set of space-times containing an isolated horizon, a location of the horizon and the area a are fixed.

In the 3+1 framework: horizon is represented by a 2-sphere S, the bulk is a 3-manifold Σ bounded by S.
The degrees of freedom:
- a $U(1)$ spin connection defined on the horizon S
- gravitational field data in Σ

such that a consistency condition is satisfied ensuring that the world-surface of S is an isolated horizon.

The quantum Hilbert space: $\mathcal{H}_{\rm kin} \subset \mathcal{H}_S \otimes \mathcal{H}_\Sigma$

$\mathcal{H}_S$: the horizon Hilbert space is understood as the space of the black-hole quantum states. Mathematically, this is the union of the quantum $U(1)$ Chern-Simons theories on a punctured sphere, where all possible sets of punctures are admitted.
$\mathcal{H}_\Sigma$: the bulk Hilbert space is described by Quantum Geometry, consists of excitations of the 3-geometry, which define the quantum area of the horizon.

The quantum consistency condition is equivalent to the quantum Gauss constraint in $\mathcal{H}_S \otimes \mathcal{H}_\Sigma$ at S.

The **quantum constraints** commute with the quantum horizon area operator. All the solutions whose quantum areas fall into any given finite interval $[a-\delta a, a+\delta a]$ can be labeled by a finite number of the quantum black hole states and bulk labels.

IV. $\mathcal{H}_{\rm PHYS}$ AND THE ENTROPY DEFINITION

Solving the vector constraints amounts to the averaging with respect to the S prerving diffeomorphisms. The scalar constraint and the Gauss are already solved on S.
The physical quantum horizon states:
an orthogonal basis $|b_1, ..., b_n$ is labelled by all the sequences,

$$b = (b_1, ...b_n), \;\; b_i \in b_i \; \in \; \mathbb{Z}_k. \tag{17}$$

The full, horizon-bulk physical quantum states:

$$\mathcal{H}_{\rm phys} \; = \bigoplus_{j,m} \mathcal{H}^{b(m),j,m}, \tag{18}$$

where

$$b_i = -2m_i \;\; \mathrm{mod} \;\; k, \;\; m_i \in \{-j_i, -j_i+1, ..., j_i\}, \;\; \sum_i b_i = 0. \tag{19}$$

The horizon area operator $\hat{A}_S$ commutes with all the constraints in this framework, and passes to the physical Hilbert space. *The sequences $j = (j_1, ...j_n)$ are responsible for the area assigned to the 2-surface S of the horizon by the bulk Quantum Geometry, whereas the sequences b represent the intrinsic quantum degrees of freedom of the horizon.*

The ABCK horizon entropy is defined by the number of the quantum horizon states $|b_1, ...b_n > \in \mathcal{H}_{S,\rm phys}$ which correspond to nontrivial subspaces $\mathcal{H}^{b(m),j,m}$ such that

$$a_S^j \; = \; 8\pi\gamma\ell_{\rm Pl}^2 \sum_{i=1}^{n} \sqrt{j_i(j_i+1)} \; \leq \; a. \tag{20}$$

V. A SIMPLER, COMBINATORIC FORMULATION

The entropy S of a quantum horizon of the classical area a according to Quantum Geometry and the Ashtekar-Baez-Corichi-Krasnov framework is

$$S = \ln N(a), \tag{21}$$

where $N(a)$ is 1 plus the number of all the finite sequences $(m_1, ..., m_n)$ of non-zero elements of $\frac{1}{2}\mathbb{Z}$, such that the following equality and inequality are satisfied:

$$\sum_{i=1}^{n} m_i = 0, \tag{22}$$

$$\sum_{i=1}^{n} \sqrt{|m_i|(|m_i|+1)} \leq \frac{a}{8\pi\gamma\ell_{\rm Pl}^2}, \tag{23}$$

where γ is the Barbero-Immirzi parameter of Quantum Geometry.

VI. THE ENTROPY CALCULATIONS: FIRST INEQUALITIES

To find an upper bound for the number $N(a)$ introduced in the previous section (recall that $a = 4\pi\gamma\ell_{\rm Pl}^2 k$, and $k \in \mathbb{N}$), define the area-flux operator whose eigen subspaces are the spaces $\mathcal{H}_\Sigma^{\mathcal{P},j,m}$, but the eigen-values are given by taking the absolute value of each term in the sum defining the flux operator, namely

$$a_S^{+,m} \;:=\; 8\pi\gamma\ell_{\rm Pl}^2 \sum_i |m_i|. \tag{24}$$

Incidently, it may be argued, that this is the way area could be quantized. Consider the set

$$M_k^+ \;:=\; \{(m_1,...,m_n) \;\Big|\; 0 \neq m_i \in \frac{1}{2}\mathbb{Z},\; a_S^{+,m} \leq\; a = 4\pi\gamma\ell_{\rm Pl}^2 k\}\,. \tag{25}$$

Let N_k^+ be the number of elements of M_k^+ plus 1 (the empty sequence). Certainly,

$$N(a) \;\leq\; N_k^+ \tag{26}$$

Next, since k is arbitrarily fixed integer, let it become a variable of the sequence $(N_0^+, N_1^+, ..., N_k^+, ...)$. To establish a recurrence relation satisfied by the sequence $(N_0^+, N_1^+, ..., N_k^+, ...)$, notice that if $(m_1,...,m_n) \in M_{k-1}^+$, then both $(m_1,...,m_n,\frac{1}{2})$, $(m_1,...,m_n,-\frac{1}{2}) \in M_k^+$. In the same way, for arbitrary natural $0 < l \leq k$,

$$(m_1,...,m_n) \in M_{k-l}^+ \;\Rightarrow\; (m_1,...,m_n,\pm\frac{1}{2}l) \in M_k^+. \tag{27}$$

Obviously, if we consider all $0 < l \leq k$, and all the sequences $(m_1,...,m_n) \in M_{k-l}^+$, then the resulting $(m_1,...,m_n,\pm\frac{1}{2}l)$ form the entire set M_k^+. Also, for two different $l \neq l'$,

$$(m_1,...,m_n,\pm\frac{1}{2}l) \;\neq\; (m_1,...,m_n,\pm\frac{1}{2}l'). \tag{28}$$

$$m_1 + ... + m_n = \delta \qquad (38)$$

depends on δ. The average value of δ is $\bar{\delta} = 0$. For large values of k, the number of the paths corresponding to the random walk distance δ should be given by the Gaussian function $\frac{C}{\sqrt{k}} e^{-\frac{\delta^2}{\beta k}} \; N_k^-$. In particular, the value

$$N'^-_k = \frac{C}{\sqrt{k}} N_k^- \qquad (39)$$

corresponds to $\delta = 0$.

Summarizing,

$$\frac{C}{\sqrt{k}} N_k^- \; \leq \; N(a) \leq N_k^+, \qquad (40)$$

where the numbers N_k^- and N_k^+ were calculated in (31, 36). Therefore the entropy is bounded in the following way

$$\frac{\ln 2}{4\pi\gamma\ell_{\rm Pl}^2} a + o(a) \leq S(a) \leq \frac{\ln 3}{4\pi\gamma\ell_{\rm Pl}^2} a. \qquad (41)$$

A necessary condition for the agreement of the entropy $S(a)$ with the Bekenstein-Hawking entropy:

$$\frac{\ln 2}{\pi} \; \leq \; \gamma \; \leq \frac{\ln 3}{\pi}. \qquad (42)$$

VII. EXACT CALCULATION (MEISSNER)

Generalization of the recurrence relation used above to a relation satisfied by the desired number $N(a)$ itself:

$$N(\tilde{a}) = \theta(\tilde{a}-\sqrt{3}/2)\left(2N(\tilde{a}-\sqrt{3}/2)+2N(\tilde{a}-\sqrt{2})+\ldots\right.$$
$$\left.+2N(\tilde{a}-\sqrt{|m_i|(|m_i|+1)}+\ldots+2\left[\sqrt{4\tilde{a}^2+1}-1\right]\right) \quad (43)$$

where [.] stands for the integer part, and $a = \tilde{a}8\pi\gamma\ell_{\rm Pl}^2$. The Laplace transform of $N(\tilde{a})$:

$$P(s) = \int_0^\infty \mathrm{d}\tilde{a}\, N(\tilde{a})\, e^{-s\tilde{a}} \quad (44)$$

$$= \frac{2\sum_{k=1}^\infty e^{-s\sqrt{k(k+2)/4}}}{s\left(1-2\sum_{k=1}^\infty e^{-s\sqrt{k(k+2)/4}}\right)}. \quad (45)$$

The simple real pole is $s_o = 2\pi\gamma_M$, where

$$1-\sum_{0\neq m\in\frac{1}{2}\mathbf{Z}} e^{-2\pi\gamma_{\rm M}\sqrt{|m|(|m|+1)}} = 0. \quad (46)$$

And

$$S(a) = \frac{\gamma_{\rm M}}{\gamma}\frac{a}{4\ell_{\rm Pl}^2} + O(a). \quad (47)$$

Taking into account the condition $\sum_i m_i = 0$ produces the sub-leading term

$$S(a) = \frac{\gamma_{\rm M}}{\gamma}\frac{a}{4\ell_{\rm Pl}^2} - \frac{1}{2}\ln a + O(1). \quad (48)$$

The numerically calculated value:

$$\gamma_{\rm M} = 0.23753295796592\ldots \quad (49)$$

VIII. THE SPIN PROBABILITY DISTRIBUTION

Given any value a of the classical horizon area, and the $N(a)$ quantum states of the horizon labeled by all the finite sequences $(m_1, ..., m_n)$ which contrubute to the entropy, one can fix any arbitrary value

$$0 \neq m \in \frac{1}{2}\mathbb{Z} \tag{50}$$

and consider the subset of states corresponding to the sequences such that

$$m_1 = m. \tag{51}$$

Denote the number of the elements of this subset by $N_{(a)}(m)$. The ratio

$$P_{(a)}(m) \ := \ \frac{N_{(a)}(m)}{N(a)} \tag{52}$$

can be considered as the probability that the first puncture is labeled by $m_1 = m$. The question is, what $P_{(a)}(m)$ is when a is large compared to the minimal area a_m created at the puncture[1],

$$a_m \ = \ 8\pi\gamma\ell_{\rm Pl}^2\sqrt{|m|(|m|+1)}. \tag{53}$$

(Notice, that the answer could make no probabilistic sense, if the $\lim_{a\to\infty} P_{(a)}(m)$ were 0 for all the values of m, for example.) The number $N_{(a)}(m)$ defined above can be also thought of as a number of all the finite sequences $(m_2, ..., m_n)$ such that

$$\sum_{i=2}^{n} m_i \ = \ -m, \quad \sum_{i=2}^{n} \sqrt{|m_i|(|m_i|+1)} \ \leq \ \frac{a - a_m}{8\pi\gamma\ell_{\rm Pl}^2}. \tag{54}$$

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[1]This issue has been raised recently by John Baez.

IX. REANIMATION OF THE QUASI NORMAL MODES LINK

Dreyer's calculations connecting the ABCK entropy from Quantum Geometry, the BH entropy, the quasinormal-modes and the quantum mechanical formula $\Delta E = \omega\hbar$ rely on the 'entropy' derived from the number of states of the type $(\pm j_{\min}, ..., \pm j_{\min}$ only. Our results show, this is not the full entropy.

Remarkably however (Baez, private communication), the link can be found if we use instead of our area operator the flux-area operator defined above by the eigen values

$$a_S^{+,m} := 8\pi\gamma \sum_i |m_i|, \tag{59}$$

($G = \hbar = 1$) which also arguably corresponds to the classical area observable. Recall, that the corresponding entropy was

$$S^+(a) = \ln N^+(a) = \frac{\ln 3}{\pi\gamma}\frac{a}{4\ell_{\rm Pl}^2}. \tag{60}$$

The consistency with BH implies

$$\gamma = \frac{\ln 3}{\pi}. \tag{61}$$

The area spectrum gap is a multiple of

$$\Delta a_S^+ = 8\pi\gamma\frac{1}{2} = 4\pi\gamma = 4\ln 3 \tag{62}$$

Defining after Dreyer the frequency $\omega_{SU(2)}$ by

$$\Delta a_S^+ = 32\pi M\Delta M = 32\pi M\omega_{SU(2)} \tag{63}$$

we find

$$M\omega_{SU(2)} = \frac{\ln 3}{8\pi}, \tag{64}$$

and notice, that $\omega_{SU(2)}$ coincides with the limit of the quasinormal-mode frequences in which the dumping is maximal.

Remarks:

- the calculation uses the SU(2) gauge group of Quantum Geometry

- if one assumes that Quantum Geometry is an SO(3) theory and repeats the calculation of the area, entropy, γ and finally $\omega_{SO(3)}$, the resulting value of $\omega_{SO(3)}$ is

$$\omega_{SO(3)} = \omega_{SU(2)}. \tag{65}$$

- The status of the flux-area operator: there are several remarks related to this point. The flux-area operator used above is given by the flux of the vector field normal to the horizon. The ACBK framework distinguishes this operator in a very special way, specifphic for the horizon only. The normal vector field is defined by the extra structure provided at the horizon, namely the given internal, su(2) valued vector field r. Contracted with the Ashtekar frame field E^i_a, $r^i E^a_i$ defines the vector normal to the horizon. Itself r is NOT a dynamical field. The flux-area operator Baez proposes to use is obtained from the flux integral corresponding to r with the extra absolute value inserted under the integral defining the flux. Classically this an equivalent to the usual definition of the area of the horizon. The quantum operators, on the other hand, are different. If we want to take the QN mode relation seriously, we choose the flux-area definition. Consider now instead of the horizon, a regular 2-surface contained in the bulk. Do we have again two area operatotrs? In this case the internal vector field r normal (via the soldering form) to the surface is not apriori given. We need to construct it from the surface and from the frame. Therefore it involves the dynamical

• on the horizon S:

$$\hat{a}_S \otimes \hat{F}(x)\,\Psi = -2\pi\gamma\,\widehat{\Sigma^r}\otimes 1\,\Psi$$

• the symplectic form:

$$\frac{1}{8\pi G}\left[(\delta',\delta)\mu\Omega_{bulk} + \frac{a_S}{\gamma\pi}\oint_S \delta W\wedge\delta' W\right]$$

$$\Rightarrow\quad k = \frac{a_S}{4\pi\gamma} = \frac{a_S^{t,m}}{4\pi\gamma} = 2\sum_i |m_i|$$

the level of the C-S theory.

$$\mathcal{H} = \bigoplus_{\vec{m}} \mathcal{H}^{\vec{m}}_{bulk}\otimes \mathcal{H}^{C\text{-}S,\,k=\frac{a^{t,m}}{4\pi\gamma}}_{S}$$