Title: Phase Space Quantization of 2+1 Gravity In Chern-Simons Formulation

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Abstract:

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Phase space and Quantization of (2+1)-dimensional gravity in the Chern-Simons formulation

Workshop on quantum gravity in the Americas:
Status and future directions,
October 29-31 2004

References:

1. C. Meusburger, B. J. Schroers: **Poisson structure and symmetry in the Chern-Simons formulation of (2+1)-dimensional gravity**, Class. Quant. Grav. 20 (2003), gr-qc/0301108
2. C. Meusburger, B. J. Schroers: **The quantisation of Poisson structures arising in Chern-Simons theory with gauge group $G \ltimes \mathfrak{g}^*$**, Adv. Theor. Math. Phys. 7 (2003) 1003–1042, hep-th/0310218
3. C. Meusburger, B. J. Schroers: **Mapping class group actions in Chern-Simons theory with gauge group $G \ltimes \mathfrak{g}^*$**, to appear in Nucl. Phys. B, hep-th/0312049

Contents:

1 (2+1) gravity as a Chern-Simons gauge theory

[Achucarro, Townsend, Witten]

spacetime $M \approx \mathbb{R} \times S_{g,n}$
$S_{g,n}$ = genus g surface with n punctures (massive particles with spin)

gauge group $\tilde{P}_3^\uparrow = \tilde{L}_3^\uparrow \ltimes \mathbb{R}^3$, $(u_1, \boldsymbol{a}_1) \cdot (u_2, \boldsymbol{a}_2) = (u_1 u_2, \boldsymbol{a}_1 + \mathrm{Ad}(u_1)\boldsymbol{a}_2)$

$$J^a, P^a \in iso(2,1): \quad [J^a, J^b] = \epsilon^{abc} J_c \quad [J^a, P^b] = \epsilon^{abc} P_c \quad [P^a, P^b] = 0$$
$$\langle J^a, J^b \rangle = 0 \qquad \langle J^a, P^b \rangle = \eta^{ab} \qquad \langle P^a, P^b \rangle = 0$$

gauge connection $A^\mu(x) = \underbrace{e_a^\mu(x)}_{\text{triad}} P^a + \underbrace{\omega_a^\mu(x)}_{\text{spin connection}} J^a$

Chern-Simons action $I_{CS}[A] = \int_M \langle A \wedge dA \rangle + \frac{2}{3}\langle A \wedge A \wedge A \rangle$

equations of motion $\delta A \Rightarrow F(x) = \underbrace{D_\omega e^a(x)}_{\text{torsion}} P_a + \underbrace{F_\omega(x)}_{\text{curvature}} J_a = 0$

gauge transformations $A \to g \cdot A \cdot g^{-1} + g d g^{-1}$

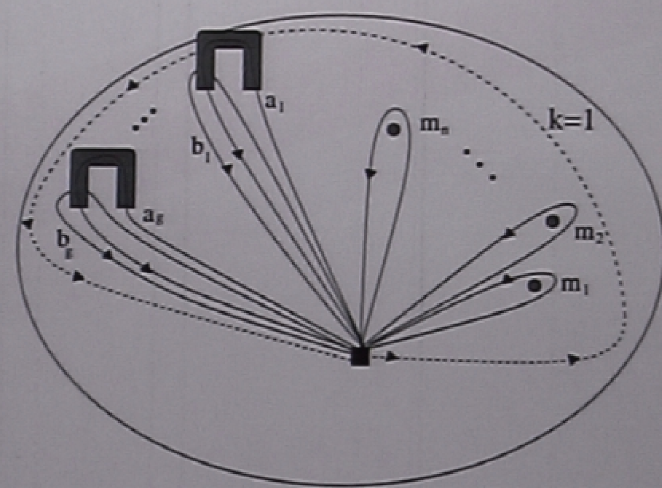
phase space $\mathcal{M}_{S_{g,n}}$ = flat connections on $S_{g,n}$/gauge transformations

- finite dimensional
- Poisson structure from canonical Poisson structure of gauge fields

[Fock, Rosly]: $\mathcal{M}_{S_{g,n}}$ as finite dim. quotient via graph in $S_{g,n}$

- $\mathcal{M}_{S_{g,n}}$ = flat graph connections/graph gauge transformations
- Poisson structure via auxiliary Poisson structure on space of graph connections (need: classical r-matrix)

2 Phase space



graph set of generators of $\pi_1(S_{g,n})$

$$\pi_1(S_{g,n}) = \langle m_1, \ldots, m_n, a_1, b_1, \ldots, a_g, b_g \; ; \; [b_g, a_g^{-1}] \cdots [b_1, a_1^{-1}] m_n \cdots m_1 = 1 \rangle$$

graph connections holonomies

handles: $A_j, B_j \in \tilde{P}_3^\uparrow$

particles: $M_i = (e^{-p_a J^a}, -\mathrm{Ad}(e^{-p_a J^a})\boldsymbol{j}) \in \mathcal{C}_{\mu_i s_i} \;\; \Leftrightarrow \;\; \boldsymbol{p}^2 = \mu_i^2, \boldsymbol{p}\boldsymbol{j} = \mu_i s_i$

$\Rightarrow \boldsymbol{p} \approx$momentum, $\boldsymbol{j} \approx$ angular momentum

graph gauge transformations simultaneous conjugation with $\tilde{P}_3^\uparrow$

phase space

$$\mathcal{M}_{S_{g,n}} = \{(M_1, ..., M_n, A_1, B_1, ..., A_g, B_g) \in \mathcal{C}_{\mu_1 s_1} \times ... \times \mathcal{C}_{\mu_n s_n} \times (\tilde{P}_3^\uparrow)^{2g} | \\ [B_g, A_g^{-1}] \cdots [B_1, A_1^{-1}] M_n \cdots M_1 = 1\} / \text{sim. conjugation with } \tilde{P}_3^\uparrow$$

3 Poisson structure

classical r-matrix $r = P_a \otimes J^a \Rightarrow$ Poisson structure on $(\tilde{P}_3^\uparrow)^{n+2g}$

properties

in terms of functions $F \in \mathcal{C}^\infty((\tilde{L}_3^\uparrow)^{n+2g})$ "on momentum space" and "angular momenta" $\boldsymbol{j}^X$, $X \in \{M_1, \ldots, B_g\}$:

- mixed contributions of quantities associated to different particles and handles
- semidirect-product structure

$$\{F_1, F_2\} = 0$$
$$\{j_a^X, F\} \in \mathcal{C}^\infty((\tilde{L}_3^\uparrow)^{n+2g})$$
$$\{j_a^X, j_b^Y\} = \sum_{Z \in \{M_1, \ldots, B_g\}} F^{abc}_{XYZ}\ j_c^Z, \qquad F^{abc}_{XYZ} \in \mathcal{C}^\infty((\tilde{L}_3^\uparrow)^{n+2g})$$

constraint algebra

- mass-and spin constraints for particles: Casimir functions

$$\boldsymbol{p}_i^2 - \mu_i^2 \approx 0 \qquad \boldsymbol{p}_{M_i} \boldsymbol{j}_{M_i} - \mu_i s_i \approx 0$$

- six first-class constraints

$$K = (e^{-p_a^K J^a}, -\mathrm{Ad}(e^{-p_a^K J^a})\boldsymbol{j}^K) = [B_g, A_g^{-1}] \cdots [B_1, A_1^{-1}] M_n \cdots M_1 \approx 1$$

$$\Rightarrow\ p_a^K \approx 0 \qquad j_a^K \approx 0$$
$$\{j_a^K, j_b^K\} = \epsilon_{abc} j_K^c \qquad \{j_a^K, p_b^K\} = \epsilon_{abc} p_K^c \qquad \{p_a^K, p_b^K\} = 0$$

4 Quantization

- Chern-Simons theory with compact, semisimple gauge group: combinatorial quantization [Alekseev, Grosse, Schomerus]
- generalised to $SL(2,\mathbb{C})$ [Buffenoir, Noui, Roche]

Quantization for $\tilde{P}_3^\uparrow$

1. **decoupling transformation [Alekseev, Malkin]**

   Poisson algebra $\Rightarrow$ direct sum of n particle and g handle algebras

2. **quantization of particle and handle algebra**

   particle algebra $\widehat{\mathcal{P}} = U(so(2,1)) \otimes \mathcal{C}^\infty(\tilde{L}_3^\uparrow)$

   $$(j_a \otimes F_1) \cdot (j_b \otimes F_2) = j_a \cdot_U j_b \otimes F_1 F_2 - i\hbar j_b \otimes F_1 (j_a . F_2)$$
   with $j_a . F(u) = \frac{d}{dt}|_{t=0} F(e^{-tJ^a} u e^{tJ_a})$

   $\Rightarrow$ irreps labelled by μ, s
   $\Rightarrow$ representation spaces isomorphic to representation spaces of $\tilde{P}_3^\uparrow$

   handle algebra $\widehat{\mathcal{H}} = U(so(2,1) \oplus so(2,1)) \otimes \mathcal{C}^\infty(\tilde{L}_3^\uparrow \times \tilde{L}_3^\uparrow)$

   $$(j_a \otimes F_1) \cdot (j_b \otimes F_2) = j_a \cdot_U j_b \otimes F_1 F_2 - i\hbar j_b \otimes F_1 (j_a . F_2)$$
   with $(\alpha j_a^A + \beta j_a^B) . F(u_A, u_B) = \frac{d}{dt}|_{t=0} F(u_A e^{t\alpha J^a}, u_A e^{t\alpha J_a} u_A^{-1} u_B e^{t\beta J_a})$

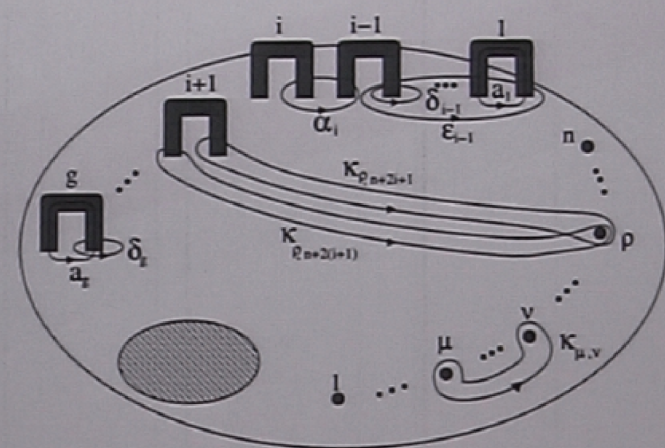
   $\Rightarrow$ single irrep on $L^2(\tilde{L}_3^\uparrow \times \tilde{L}_3^\uparrow)$

3. **quantum theory**

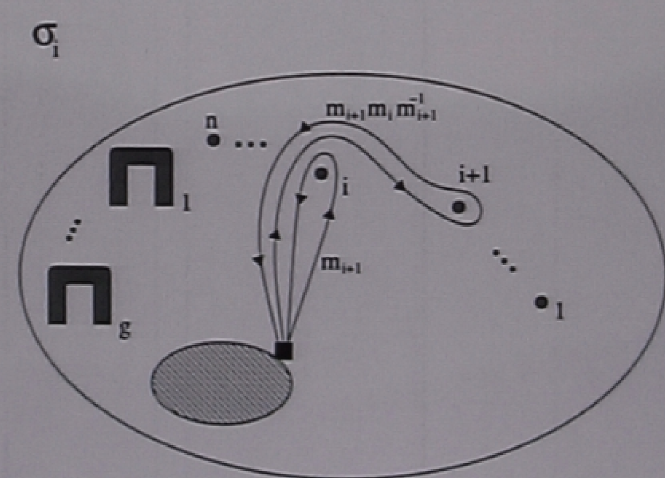
   - representation space of quantum algebra
     $\mathcal{H}_{kin} = \mathcal{H}_{\mathcal{P}_{\mu_1, s_1}} \otimes \ldots \otimes \mathcal{H}_{\mathcal{P}_{\mu_n, s_n}} \otimes \underbrace{\mathcal{H}_{handle} \otimes \ldots \otimes \mathcal{H}_{handle}}_{g\times}$
   - open question: implementation of constraints $\Leftrightarrow$ invariance under action of quantum double $D(\tilde{L}_3^\uparrow)$

Generators of the mapping class group

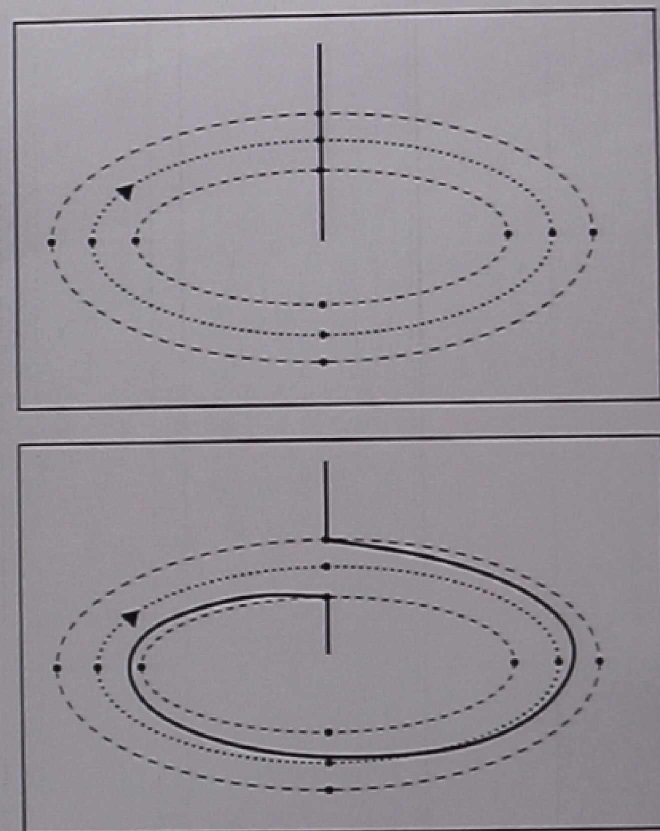
1. Dehn twists around embedded curves on $S_{g,n}$ (particles fixed)



2. Generators of the braid group (exchange particles)



Dehn twist around embedded curve (dotted line)



Generators of the mapping class group

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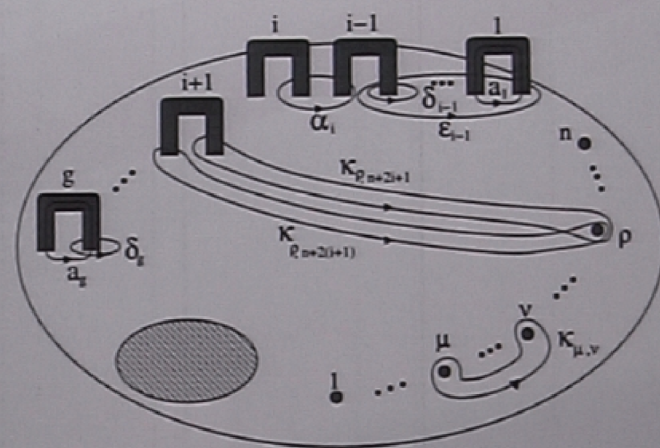

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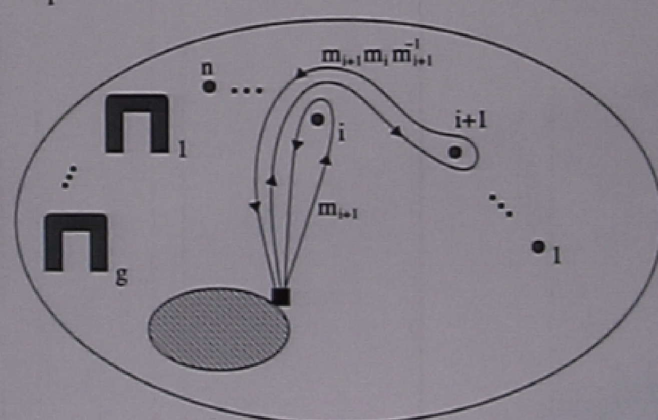
σ_i



5 Symmetries: Action of the mapping class group

Action of $\mathrm{Map}(S_{g,n}\setminus D)$ on fundamental group induces action on auxiliary Poisson algebra

1. **classical action of $\mathrm{Map}(S_{g,n}\setminus D)$**

   - $\mathrm{Map}(S_{g,n}\setminus D)$ acts by Poisson isomorphisms
   - action of Dehn twists around embedded curves γ related to infinitesimally generated group action:
     - parametrize holonomy as $H[\gamma] = (e^{-p_a^\gamma J^a}, -\mathrm{Ad}(e^{-p_a^\gamma J^a})\boldsymbol{j}^\gamma)$
     - consider analogue of spin constraint $c_\gamma = \boldsymbol{p}^\gamma \boldsymbol{j}^\gamma$
     - $\{c_\gamma, \cdot\} \Rightarrow$ one-parameter group of transformations

2. **quantum action of $\mathrm{Map}(S_{g,n}\setminus D)$**

   - $\mathrm{Map}(S_{g,n}\setminus D)$ acts by algebra automorphisms of quantum algebra $\Rightarrow$ action on representation spaces
   - quantum action of Dehn twists: action of ribbon element in representations Π_γ of quantum double $D(\tilde{L}_3^\uparrow)$
   - quantum action of braid group: action of universal R-matrix in representations Π_{ij} of $D(\tilde{L}_3^\uparrow)$

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Action of $\mathrm{Map}(S_{g,n}\setminus D)$ on fundamental group induces action on auxiliary Poisson algebra

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6 Outlook and Conclusions

Description of phase space of (2+1)-dimensional gravity via auxiliary Poisson structure on holonomies of generators of $\pi_1(S_{g,n})$

phase space

- parametrization by finite number of variables with close relation to physical degrees of freedom
- investigation of Poisson structure, decoupling
- study of symmetries (residual gauge symmetries and $\mathrm{Map}(S_{g,n}\setminus D)$)

quantization

- reduced to quantization of two building blocks (particle and handle algebra)
- construction of quantum algebra and irreducible representations
- quantum action of $\mathrm{Map}(S_{g,n}\setminus D)$
- quantum double $D(\tilde{L}_3^\uparrow)$ as quantum symmetry
- general: Chern-Simons theories with gauge groups $G \ltimes \mathfrak{g}^*$

open questions

- construction of physical Hilbert space, implementation of constraint $[B_g, A_g^{-1}]\cdots[B_1, A_1^{-1}]M_n\cdots M_1 \approx 1$
  $\Leftrightarrow$ invariance under action of $D(\tilde{L}_3^\uparrow)$
  $\Rightarrow$ Clebsch-Gordan analysis of tensor product representations of $D(\tilde{L}_3^\uparrow)$
- application to concrete physics problems

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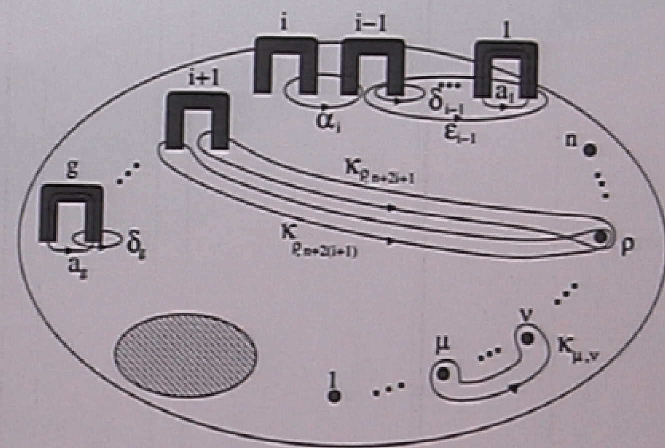

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