Title: Status of the Master Constraint Programme

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Abstract:

Status of the Master Constraint Programme

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QG in the Americas

B.D. and T. Thiemann:

"Testing the MCP for LQG"
I - V

to appear

T. Thiemann, gr-qc/030580

Outline

- Introduction & Motivation
- Master Constraint Programme
- Uniqueness
- Observables
- Examples
- Outlook and open issues

Introduction & Motivation

Canonical Quantization of Constrained Systems:

- Find phys. Hilbert space $\boxed{\mathcal{H}_{phys}}$, i.e. a space of states $|\psi\rangle$ s.t.
  - $\hat{C}_\ell |\psi\rangle = 0$
  - inner product on $\mathcal{H}_{phys}$

(usually) given

- Kinematical Hilbert space $\boxed{\mathcal{H}_{kin}}$
- $\hat{C}_\ell$'s on $\mathcal{H}_{kin}$

Problems:

- $\mathcal{H}_{phys} \not\subset \mathcal{H}_{kin}$ in general
- non-commuting constraints (spectral analysis not possible)

GR:

- cannot define $\hat{C}(N)$ directly on $\mathcal{H}_{Diff}$, but concrete implementation uses $\mathcal{H}_{Diff}$
- makes semi-classical analysis

The Master Constraint

Simplify constraint algebra!

finite dim. system

$$M = \sum_{i,j} C_i K_{ij} C_j$$

strictly pos. matrix

constraints

field theory

$$M = \int d^n x \; C(x) \, (K \cdot C)(x)$$

∞ many constraints

strictly pos. op

gravity

$$M = \int d^3 x \; \frac{C(x)\, C(x)}{\sqrt{\det q}}$$

3-diffeo-invariant

$$\boxed{M = 0 \iff \begin{array}{ll} C_i = 0 & \forall i \\ C(x) = 0 & \forall x \end{array}}$$

We are left with ONE constraint!

MCP: Physical Hilbert Space

$\mathcal{H}_{kin}$ (separable), $\hat{M}$ (pos., s.a.)

$\downarrow$

direct integral decomposition
('generalized eigenspace' decomp)

$$\mathcal{H}_{kin} \simeq \int_{spec(\hat{M})}^{\oplus} \mathcal{H}_{kin}^{\oplus}(\lambda)\; d\mu(\lambda)$$

carries induced inner product   spectral measure

$\downarrow$

$$\boxed{\mathcal{H}_{phys} := \mathcal{H}_{kin}^{\oplus}(0)}$$

Recipe for $\mathcal{H}_{phys}$

- Find cyclic ON system $\{\Omega_j\}_{j\in\mathcal{J}}$

$$\mathcal{H}_{kin} = \bigoplus_j \overline{\mathrm{span}\{\hat{M}^k \Omega_j \mid k \in \mathbb{N}\}}$$

- calculate spectral measures

$$\mu_j(\lambda) = \langle \Omega_j, \Theta(\lambda - \hat{M})\,\Omega_j \rangle$$

- calculate spectral measure $\mu(\lambda)$

$$\mu(\lambda) = \sum_j \alpha_j\, \mu_j(\lambda) \qquad \sum_j \alpha_j = 1$$

- find Radon-Nikodym derivatives

$$\rho_j(\lambda) = \frac{d\mu_j(\lambda)}{d\mu(\lambda)}$$

$\mathcal{H}^{\oplus}_{kin}(\lambda)$ has ON basis $\{e_j(\lambda) \mid \rho_j(\lambda) > 0\}$

$\Rightarrow$ determines $\dim\left(\mathcal{H}^{\oplus}_{kin}(\lambda)\right) =$ multiplicity of λ

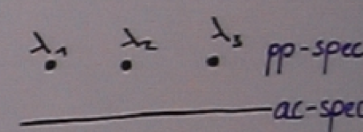
- result does not depend on choice of $\{\Omega_j\}$ (in the following sense)

Uniqueness

$$\mathcal{H}_{KIN} \simeq \int^{\oplus} \mathcal{H}^{\oplus}_{KIN}(\lambda)\, d\mu(\lambda)$$

- measure theoretic formula

$\rightarrow$ (given $\hat{M}$) uniqueness (of $\dim \mathcal{H}^{\oplus}_{kin}(\lambda)$)
μ - a.e.



- for mixed spectrum
  points λ_i have <u>finite</u> measure

  $\Rightarrow$ contributions to $\mathcal{H}^{\oplus}_{kin}(\lambda_i)$ from ac-spec are suppressed

  - decompose $\mathcal{H}_{kin} = \mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sing}$

- dependence of $\mathcal{H}_{phys}$ on $\hat{M}$ (that is K)?

Example: Abelian Constraints

$\mathcal{H}_{kin} = \mathcal{L}^2(\mathbb{R}^2, d^2x)$, $\hat{C}_1 = \hat{p}_1$, $\hat{C}_2 = \hat{p}_2$

$$\hat{M} = \hat{p}_1^2 + \hat{p}_2^2 = -\hbar^2 \Delta$$

sol.: harmonic functions

'eigenfunctions' $|k_1 k_2\rangle = \exp(i\vec{k}\cdot\vec{x})$

- change to $\mathcal{L}^2(\mathbb{R}^2, d^2p)$

$$\Omega_j = N_j \, \rho^{|j|} \exp(ij\varphi) \exp(-\tfrac{\rho^2}{2}) \quad , \quad j \in \mathbb{Z}$$

spherical coord. in $\mathbb{R}^2$

$$\rho_j(\lambda) = \frac{\lambda^{|j|}/|j|!}{2\exp(\frac{\lambda}{2}) - 1} \xrightarrow{\lambda \to 0} \begin{cases} 0 & |j| \geq 1 \\ 1 & j = 0 \end{cases}$$

$$\Rightarrow \quad \mathcal{H}^{\oplus}_{kin}(0) = \mathcal{H}_{phys} \simeq \mathbb{C}$$

$$\mathcal{H}^{\oplus}_{kin}(\lambda > 0) \simeq \mathcal{L}^2(S^1, d\varphi)$$

Observables

<u>classical:</u> How to obtain them?

- framework available (BD., to appear)
- gives in general <u>weak</u> Dirac obs.

<u>quantum:</u> for 'strong' Dirac observables, i.e. $[\mathcal{O}, M] = 0$

$$(A|\psi\rangle)_{\chi_{kin}} \simeq \sum_j \int a_j(\lambda) \left(A(\lambda)\,|e_j(\lambda)\rangle\right)_{\chi(\lambda)} d\mu(\lambda)$$

can be calculated explicitly

- representation for 'weak' Dirac observables?

Examples

- finite dim. examples
- Maxwell theory, Linearized gravity: squaring of constraints $\longrightarrow$ worsened uv-behaviour $\longrightarrow$ choose K such that $\hat{M}$ is densely defined

  K provides regularization of $\hat{M}$

- Gauss constraints in Einstein-YM

$$M = \int d^3x \, \frac{G_J G^J}{\sqrt{\det(q)}}$$

background independent theories regulate themselves

result: gauge inv. states and metric-degenerate states

Conclusions & Outlook

- method is widely applicable, in particular to open algebras
- provides construction of physical inner product and representation of 'strong observables'
- representation of 'weak' observables?
- dependence on $\hat{M}$?

$\boxed{\text{LQG:}}$

- $\hat{M}$ can be defined on $\mathcal{H}_{Diff}$ (T. Thiemann, to appear)
- need separable Hilbert spaces
- semiclass. limit (is easier now)
  - 3-diff.-invariant semiclass. states
- MCP gives construction principle for $\mathcal{H}_{phys}$