Title: Volume Operator and Recoupling Theory

Date: Oct 29, 2004 05:30 PM

URL: http://pirsa.org/04100028

Abstract:

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Recoupling Theory and the Volume Operator in Loop Quantum Gravity

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Quantum Gravity in the Americas Workshop,

work in progress
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Waterloo 29/10/2004

Outline of this Talk

- Motivation
- LQG - Basics
  - Recoupling Schemes
- Volume Operator
  - Definition
  - Simplification of the Matrix Element Expression
- Applications (work in progress)
  - Numerical Studies
  - Triad Operators in LQG
    - Bounded or Unbounded? $\longleftrightarrow$ Comparison to LQC
    - Coherent States
- Conclusions & Outlook

Motivation

$\hat{V}$ is crucial for various Parts of LQG

- Trick: Can treat negative powers of $\det q$ by rewriting as Poisson bracket...

$$H_E = \frac{1}{\kappa\sqrt{\det q}}\left[\, F^j_{ab}\, \epsilon_{jkl}\, E^a_k E^b_l \,\right] \qquad\qquad F^j_{ab} = 2\partial_{[a} A^j_{b]} + \epsilon_{jkl} A^k_a A^l_b$$

  - Identities:

$$\begin{aligned} \frac{\epsilon_{jkl}\; E^a_k E^b_l}{\kappa\sqrt{\det q}} &= \operatorname{sgn}(\det(e))\,\epsilon^{abc} e^j_c \\ e^j_c &\propto \{A^j_c, V\} \propto \operatorname{Tr}\left[\tau_j h_c \{h_c^{-1}, V\}\right] \end{aligned}$$

$$\text{where } V(R) = \int_R d^3x \sqrt{|\det q|}$$

- (Spatial) Diffeomorphism Invariant (Complexifier) Coherent States

$$\begin{aligned} \Psi_{e\in E(\gamma)} &:= \left[\mathrm{e}^{\frac{1}{\hbar}\hat{C}}\; \delta_{h_e} \right]_{G\ni h_e \to h_e^{\mathbb{C}} \in G^{\mathbb{C}}} \\ &= \sum_s \mathrm{e}^{-t\lambda(s)}\; T_s(h_e^{\mathbb{C}})\, \langle\, T_s\;,\;\; \cdot\;\, \rangle_G \end{aligned}$$

LQG - Basics I

- Classical Poisson Structure of ART

$$\left\{ A_a^i(x) \,,\; E_j^b(y) \right\} = \kappa\, \delta_j^i \delta_a^b \delta^{(3)}(x,y) \longrightarrow$$

Electric Fields $E_j(S) = \int_S *E_j$

Holonomies $h_e = \mathcal{P}\, e^{\int_e A}$

- Cylindrical Functions $f_\gamma = f_\gamma\left[\{h_e\}_{e\in E(\gamma)}\right]$ $\quad \mathcal{H} = L_2(f_\gamma, d\mu_\gamma)$

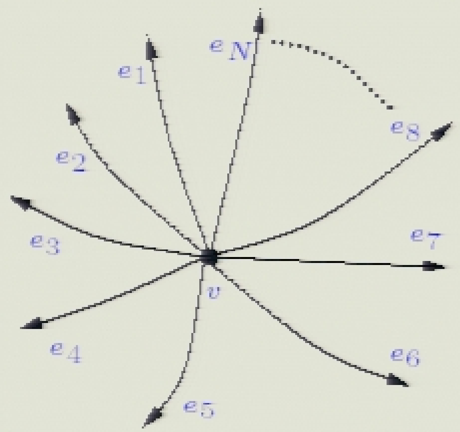
$$\begin{array}{ccc} \hat{E}_j(S) & \rightarrow & X_j \\ \hat{h}_e & \rightarrow & h_e \end{array}$$

- Basis of $\mathcal{H}$ (Peter & Weyl): $\sqrt{2j+1}\left[\pi_j(h_e)\right]_{mn} =: \langle\, h_e \mid j\ m\ ;\ n\,\rangle$ $\longleftarrow$ SNF

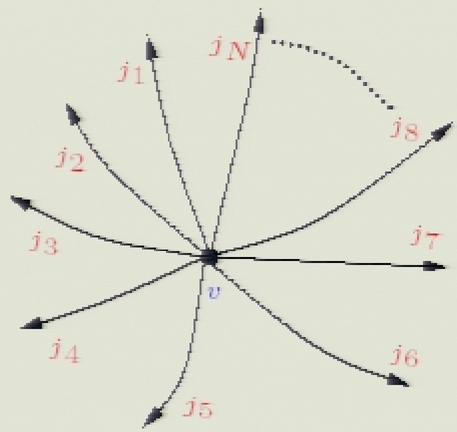
  - Can replace $-\frac{i}{2} X_j \mid j\ m\ ;\ n\,\rangle = J_j \mid j\ m\ ;\ n\,\rangle$

Action of operators only containing X_j can be expressed as action of usual angular momentum operators acting on a spin system!

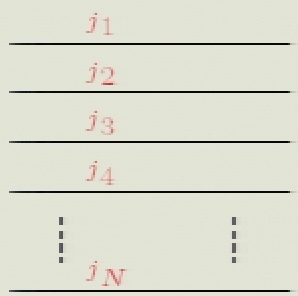
LQG - Basics II



LQG - Basics II

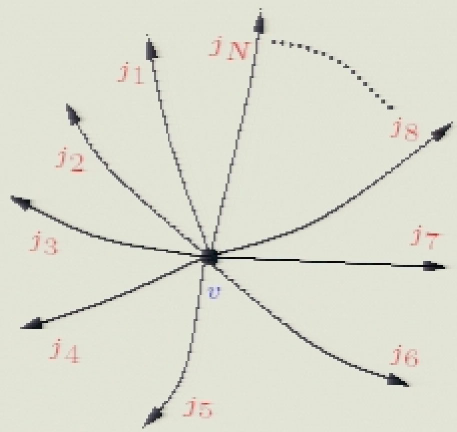


- Tensor Basis

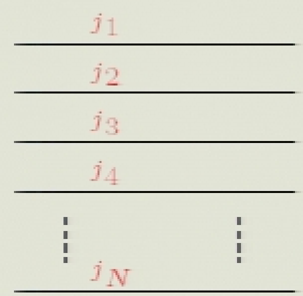


$$T_v = |\; j_1\, m_1\,;\, n_1 > \otimes \ldots \otimes |j_N\, m_N\,;\, n_N \;\; >$$

LQG - Basics II



- Tensor Basis

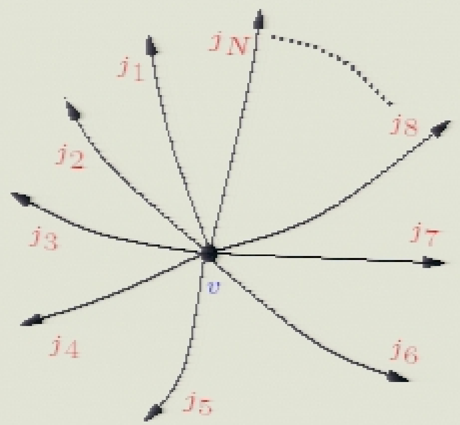


$$T_v = |\, j_1\, m_1\, ;\, n_1 > \otimes \ldots \otimes |j_N\, m_N\, ;\, n_N \;\; >$$

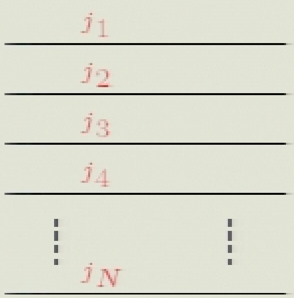
- Decomposition

$$\pi_{j_1} \otimes \pi_{j_2} = \bigoplus_{j_{12}=|j_1-j_2|}^{j_1+j_2} \pi_{j_{12}}$$

LQG - Basics III: Recoupling Basis 1



- Tensor Basis

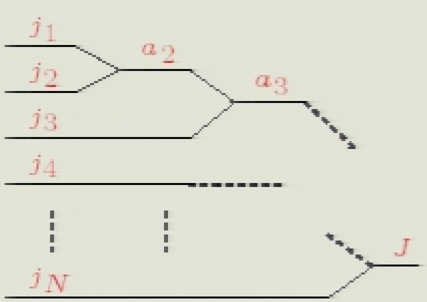


$$T_v = |\ j_1\, m_1\, ;\, n_1 > \otimes \ldots \otimes |j_N\, m_N\, ;\, n_N \ >$$

- Decomposition

$$\pi_{j_1} \otimes \pi_{j_2} = \bigoplus_{j_{12}=|j_1-j_2|}^{j_1+j_2} \pi_{j_{12}}$$

- Recoupling Basis



$$T_v = |\ a_2\ a_3\ \ldots\ a_{N-1}\ J\ M\, ;\, \vec{j}\ \vec{n}\ >$$

LQG - Basics III: Recoupling Basis 2

- $T_v = |\; a_2(j_1\, j_2)\; a_3(a_2\, j_3)\; \ldots\; a_{N-1}(a_{N-2}\, j_{N-1})\; J(a_{N-1}\, j_N)\; M\,;\vec{j}\;\vec{n}\; >$

  diagonalizes

  $$J_1,\; A_2 = J_1 + J_2,\; A_3 = A_2 + J_3,\; \ldots\;,\; J = J_1 + \ldots + J_2$$

  The order of recoupling matters!

- $T_v' = |\; g_2(j_I\, j_J)\; g_3(g_2\, j_1)\; \ldots\; g_{I+2}(g_{I+1}, j_{I+1})\; \ldots\; g_{J+1}(g_J, j_{J+1})\; \ldots\; J(g_{N-1}\, j_N)\; M\,;\vec{j}\;\vec{n}\; >$

  diagonalizes

  $$J_I,\; G_2 = J_I + J_J,\; G_3 = G_2 + J_1,\; \ldots\;,\; J = J_1 + \ldots + J_2$$

- Trafo between the different Bases: $3nj$-Symbols

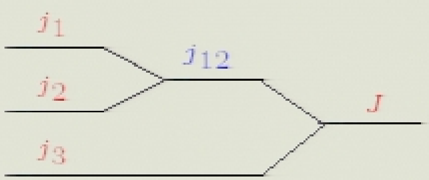
  $$\big|\; \vec{g}(IJ)\; J\; M\;;\;\vec{n}\;\big\rangle = \sum_{\vec{a}(12)} \big\langle\; \vec{a}(12)\; J\; M\;;\;\vec{n}\;\big|\;\vec{g}(IJ)\; J\; M\;;\;\vec{n}\;\big\rangle\;\big|\;\vec{a}(12)\; J\; M\;;\;\vec{n}\;\big\rangle$$

LQG - Basics III: $3nj$-Symbols in Terms of $6j$-Symbols

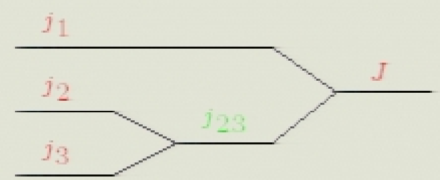
$$
\begin{aligned}
& < \vec{g}(IJ) | \vec{g}'(12) > \;= \\
=\; & \sum_{h_2(j_I, j_1)} < g_2(j_I, j_J), g_3(g_2, j_1) | h_2(j_I, j_1), g_3(h_2, j_J) > \\
\times\; & \sum_{h_3(h_2, j_2)} < g_3(h_2, j_J), g_4(g_3, j_2) | h_3(h_2, j_2), g_4(h_3, j_J) > \\
\vdots\; & \quad \vdots \qquad\qquad\qquad\qquad \vdots \\
\times\; & \sum_{h_{I-1}(h_{I-2}, j_{I-2})} < g_{I-1}(h_{I-2}, j_J), g_I(g_{I-1}, j_{I-2}) | h_{I-1}(h_{I-2}, j_{I_2}), g_I(h_{I-1}, j_J) > \\
& \times < g_I(h_{I-1}, j_J), g_{I+1}(g_I, j_{I-1}) | g'_I(h_{I-1}, j_{I-1}), g_{I+1}(g'_I, j_J) > \\
& \times < g_{I+1}(g'_I, j_J), g_{I+2}(g_{I+1}, j_{I+1}) | g'_{I+1}(g'_I, j_{I+1}), g_{I+2}(g'_{I+1}, j_J) > \\
& \times < g_{I+2}(g'_{I+1}, j_J), g_{I+3}(g_{I+2}, j_{I+2}) | g'_{I+2}(g'_{I+1}, j_{I+2}), g_{I+3}(g'_{I+2}, j_J) > \\
& \quad \vdots \qquad\qquad\qquad\qquad \vdots \\
& \times < g_{J-1}(g'_{J-2}, j_J), g_J(g_{J-1}, j_{J-1}) | g'_{J-1}(g'_{J-2}, j_{J-1}), g_J(g'_{J-1}, j_J) > \\
& \times < h_2(j_I, j_1), h_3(h_2, j_2) | g'_2(j_1, j_2), h_3(g'_2, j_I) > \\
& \times < h_3(g'_2, j_I), h_4(h_3, j_3) | g'_3(g'_2, j_3), h_4(g'_3, j_I) > \\
& \times < h_4(g'_3, j_I), h_5(h_4, j_4) | g'_4(g'_3, j_4), h_5(g'_4, j_I) > \\
& \quad \vdots \qquad\qquad\qquad\qquad \vdots \\
& \times < h_{I-1}(g'_{I-2}, j_I), g'_I(h_{I-1}, j_{I-1}) | g'_{I-1}(g'_{I-2}, j_{I-1}), g'_I(g'_{I-1}, j_I) >
\end{aligned}
$$

LQG - Basics III: $6j$-Symbols

- Definition



$$= \sum_{j_{23}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23} \end{Bmatrix}$$



Volume Operator I: Definition

- Classical Volume Expression:

$$
\begin{aligned}
V(R) &= \int_R d^3x \sqrt{|\det q|} \\
&= \int_R d^3x \sqrt{\left|\frac{1}{3!}\epsilon_{abc}\epsilon^{ijk}E_i^a E_j^b E_k^c\right|}
\end{aligned}
$$

- According Operator in LQG

$$
\hat{V}(R)_\gamma |\vec{a}\ J\ M\ ; \vec{n} > \; = \sqrt{\left|\frac{i}{4}\sum_{I<J<K} \epsilon(I,J,K)\ \hat{q}_{IJK}\right|}\ |\vec{a}\ J\ M\ ;\ \vec{n} >
$$

where $\hat{q}_{IJK} := [(J_{IJ})^2, (J_{JK})^2]$

- Matrix Elements

$$
\begin{aligned}
< \vec{a}|\hat{q}_{IJK}|\vec{a}' > \; = & \sum_{\vec{g}''(12)} \Bigg[\sum_{\vec{g}(IJ)} g_2(IJ)(g_2(IJ)+1) < \vec{g}(IJ)|\vec{g}'' >< \vec{g}(IJ)|\vec{a} > \times \\
& \times \sum_{\vec{g}(JK)} g_2(JK)(g_2(JK)+1) < \vec{g}(JK)|\vec{g}'' >< \vec{g}(JK)|\vec{a}' > \Bigg] \\
& - [\vec{a} \Longleftrightarrow \vec{a}']
\end{aligned}
$$

Volume Operator II: Simplified Expression for the Matrix Element 1

- Basic Structure: $$\sum_{\vec{g}(IJ)} g_2(IJ)\,\big(\,g_2(IJ)+1\,\big) < \vec{g}(IJ)|\vec{g}''(12) >< \vec{g}(IJ)|\vec{a}(12) >$$
- Rewrite as 6j-Symbol $$F(j_{12},j_{12}') = \sum_{j_{23}} (2j_{23}+1)\ j_{23}(j_{23}+1)\ \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_{12}' \\ j_3 & j_4 & j_{23} \end{Bmatrix}$$
- Observation

$$a(a+1) = (-1)^{a+b+c} \begin{Bmatrix} a & b & c \\ 1 & c & b \end{Bmatrix} X(b,c)^{\frac{1}{2}} + \big[(b+1)+c(c+1)\big]$$

Elliot Biedenharn Identity

- Solution

$$\boxed{\begin{aligned} F(j_{12},j_{12}') &:= \sum_{j_{23}} (2j_{23}+1)\ j_{23}(j_{23}+1)\ \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_{12}' \\ j_3 & j_4 & j_{23} \end{Bmatrix} = \\ &= \frac{1}{2}(-1)^{j_1+j_2+j_3+j_4+j_{12}+j_{12}'+1} X(j_1,j_4)^{\frac{1}{2}} \begin{Bmatrix} j_2 & j_1 & j_{12} \\ 1 & j_{12}' & j_1 \end{Bmatrix} \begin{Bmatrix} j_3 & j_4 & j_{12} \\ 1 & j_{12}' & j_4 \end{Bmatrix} \\ &\quad + \frac{j_1(j_1+1)+j_4(j_4+1)}{(2j_{12}+1)} \delta_{j_{12}j_{12}'} \end{aligned}}$$

with $X(b,c) = 2b(2b+1)(2b+2)2c(2c+1)(2c+2)$.

Volume Operator II: Simplified Expression for the Matrix Element 2

$$< \vec{a}|\hat{q}_{IJK}|\vec{a}' > \;=$$

$$= \frac{1}{4}(-1)^{j_K+j_I+a_{I-1}+a_K}(-1)^{a_I-a'_I}(-1)^{\sum_{n=I+1}^{J-1} j_n}(-1)^{-\sum_{p=J+1}^{K-1} j_p} \times$$

$$\times X(j_I,j_J)^{\frac{1}{2}} X(j_J,j_K)^{\frac{1}{2}} \sqrt{(2a_I+1)(2a'_I+1)}\sqrt{(2a_J+1)(2a'_J+1)} \times$$

$$\times \left\{ \begin{matrix} a_{I-1} & j_I & a_I \\ 1 & a'_I & j_I \end{matrix} \right\} \left[\prod_{n=I+1}^{J-1} \sqrt{(2a'_n+1)(2a_n+1)}(-1)^{a'_{n-1}+a_{n-1}+1} \left\{ \begin{matrix} j_n & a'_{n-1} & a'_n \\ 1 & a_n & a_{n-1} \end{matrix} \right\} \right] \times$$

$$\times \left[\prod_{n=J+1}^{K-1} \sqrt{(2a'_n+1)(2a_n+1)}(-1)^{a'_{n-1}+a_{n-1}+1} \left\{ \begin{matrix} j_n & a'_{n-1} & a'_n \\ 1 & a_n & a_{n-1} \end{matrix} \right\} \right] \left\{ \begin{matrix} a_K & j_K & a_{K-1} \\ 1 & a'_{K-1} & j_K \end{matrix} \right\} \times$$

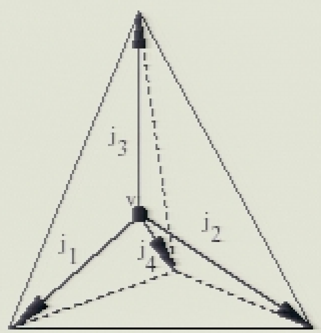
$$\times \Bigg[(-1)^{a'_J+a'_{J-1}} \left\{ \begin{matrix} a_J & j_J & a'_{J-1} \\ 1 & a_{J-1} & j_J \end{matrix} \right\} \left\{ \begin{matrix} a'_{J-1} & j_J & a'_J \\ 1 & a_J & j_J \end{matrix} \right\}$$

$$- (-1)^{a_J+a_{J-1}} \left\{ \begin{matrix} a'_J & j_J & a'_{J-1} \\ 1 & a_{J-1} & j_J \end{matrix} \right\} \left\{ \begin{matrix} a_{J-1} & j_J & a'_J \\ 1 & a_J & j_J \end{matrix} \right\} \Bigg] \times$$

$$\times \prod_{n=2}^{I-1} \delta_{a_n a'_n} \prod_{n=K}^{N} \delta_{a_n a'_n}$$

Application I: Numerical Studies 1

- Gauge Invariant 4-Vertex



- Basis-States

$$T_v = \left| \underbrace{a_2(j_1\, j_2)}_{j_{12}}\; a_3(a_2\, j_3)\; J(a_3\, j_4) = 0\; M = 0\, ;\; \vec{n} \right\rangle$$

- Gauge Invariance

$$J_1 + J_2 + J_3 + J_4 \stackrel{!}{=} 0$$

- Only $\hat{q}_{123}$ matters!

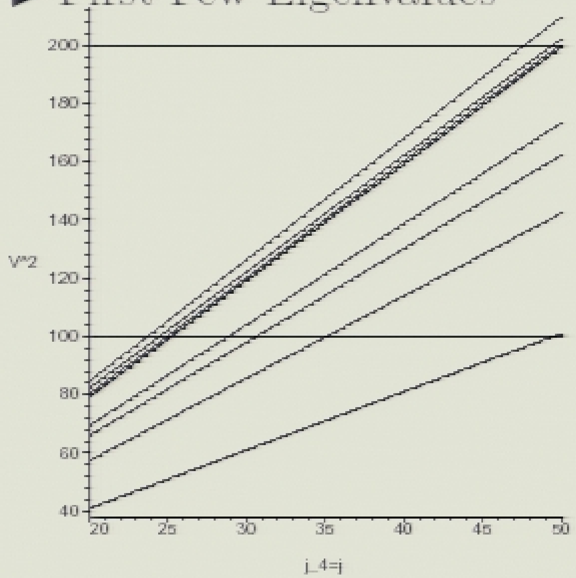
$$\begin{aligned}
< j_{12} | \hat{q}_{123} | j_{12} - 1 > \quad &= \\
&= \frac{1}{\sqrt{(2j_{12}-1)(2j_{12}+1)}} \Big[(j_1 + j_2 + j_{12} + 1)(-j_1 + j_2 + j_{12})(j_1 - j_2 + j_{12})(j_1 + j_2 - j_{12} + 1) \\
&\qquad (j_3 + j_4 + j_{12} + 1)(-j_3 + j_4 + j_{12})(j_3 - j_4 + j_{12})(j_3 + j_4 - j_{12} + 1) \Big]^{\frac{1}{2}} \\
&= \; - \; < j_{12} - 1 | \hat{q}_{123} | j_{12} >
\end{aligned}$$

Application I: Numerical Studies 2

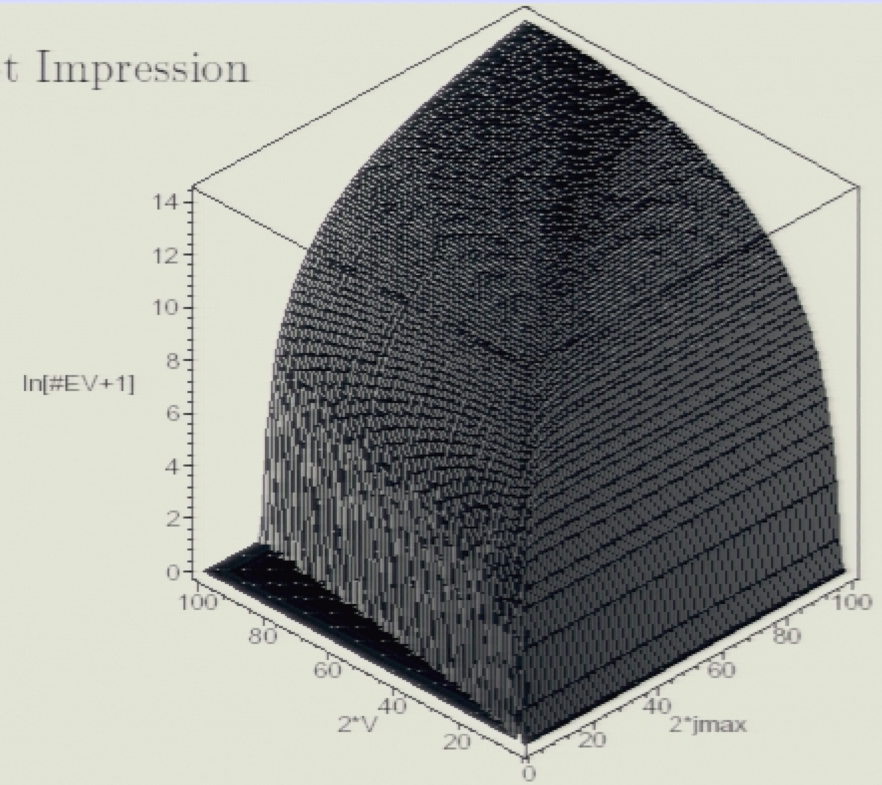
- Matrix Structure

$$A = \begin{pmatrix} 0 & -a_1 & 0 & \cdots & \cdots & 0 \\ a_1 & 0 & -a_2 & & & \vdots \\ 0 & a_2 & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & -a_{n-1} \\ 0 & \cdots & \cdots & \cdots & a_{n-1} & 0 \end{pmatrix}$$

- First Few Eigenvalues

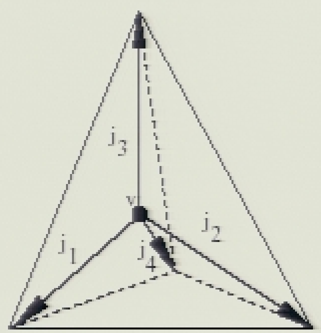


- First Impression



Application I: Numerical Studies 1

- Gauge Invariant 4-Vertex



- Basis-States

$$T_v = \Big|\, \underbrace{a_2(j_1\, j_2)}_{j_{12}}\; a_3(a_2\, j_3)\; J(a_3\, j_4) = 0\; M = 0\, ;\, \vec{n}\Big\rangle$$

- Gauge Invariance

$$J_1 + J_2 + J_3 + J_4 \stackrel{!}{=} 0$$

- Only $\hat{q}_{123}$ matters!

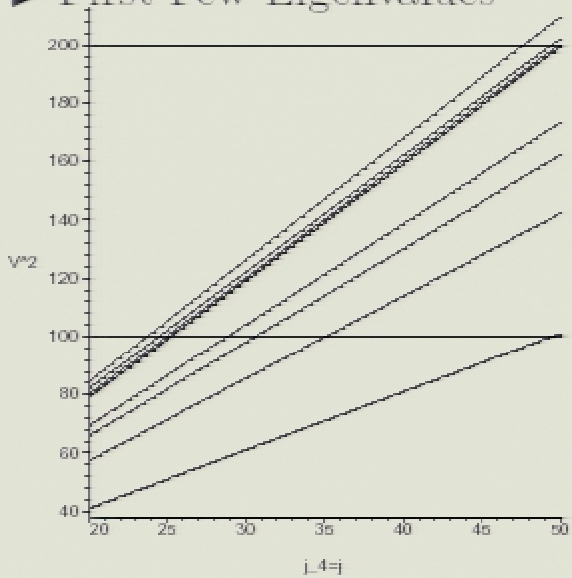
$$\begin{aligned} < j_{12}|\hat{q}_{123}|j_{12}-1> \; &= \\ &= \frac{1}{\sqrt{(2j_{12}-1)(2j_{12}+1)}}\Big[(j_1+j_2+j_{12}+1)(-j_1+j_2+j_{12})(j_1-j_2+j_{12})(j_1+j_2-j_{12}+1) \\ &\qquad (j_3+j_4+j_{12}+1)(-j_3+j_4+j_{12})(j_3-j_4+j_{12})(j_3+j_4-j_{12}+1)\Big]^{\frac{1}{2}} \\ &= \; - \; < j_{12}-1|\hat{q}_{123}|j_{12}> \end{aligned}$$

Application I: Numerical Studies 2

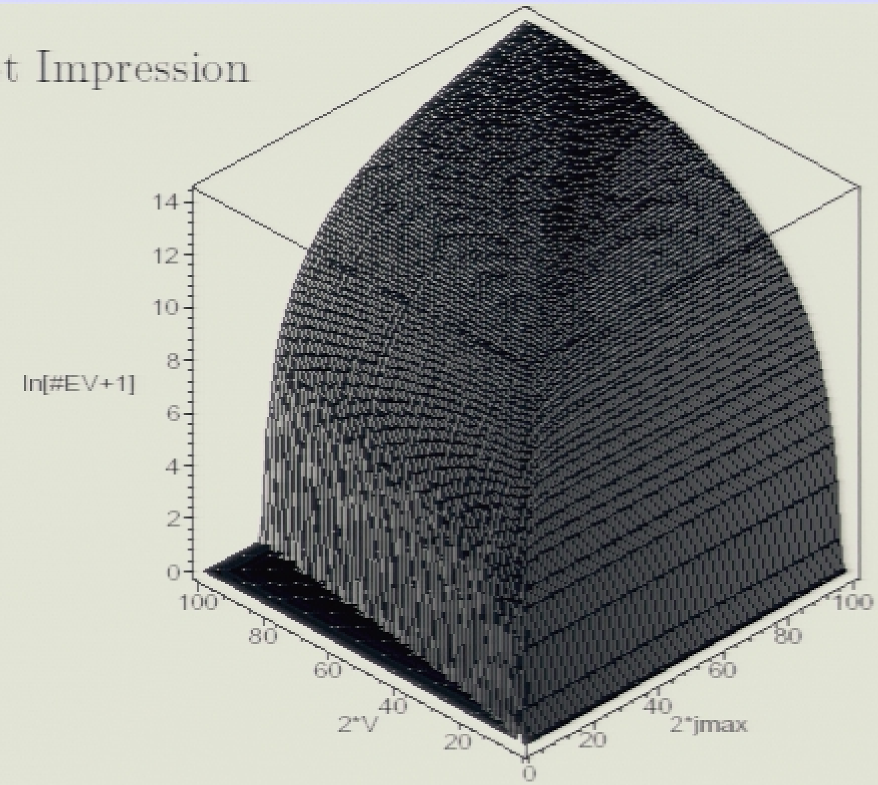
- Matrix Structure

$$A = \begin{pmatrix} 0 & -a_1 & 0 & \cdots & \cdots & 0 \\ a_1 & 0 & -a_2 & & & \vdots \\ 0 & a_2 & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & -a_{n-1} \\ 0 & \cdots & \cdots & \cdots & a_{n-1} & 0 \end{pmatrix}$$

- First Few Eigenvalues



- First Impression



Application II: Triad Operators in LQG 1

- Classical: Euclidean Part of H

$$H_E = \frac{1}{\kappa\sqrt{\det q}}\left[\, F^j_{ab}\,\epsilon_{jkl}\; E^a_k E^b_l \,\right]$$

$$\begin{aligned} E^a_j &= \det(e) e^a_j \\ \frac{\epsilon_{jkl}\; E^a_k E^b_l}{\kappa\sqrt{\det q}} &= \mathrm{sgn}\big(\det(e)\big)\epsilon^{abc} e^j_c \\ e^j_c &\propto \{A^j_c, V\} \propto \mathrm{Tr}\left[\tau_j h_c \{h_c^{-1}, V\}\right] \end{aligned}$$

  - Regularized Expression

$$H^\epsilon_E(N) \sim \sum_{\Delta\in T(\Sigma)} N\big(v(\Delta)\big)\epsilon^{IJK}\,\mathrm{Tr}\left[h_{\alpha_{IJ}}(\Delta) h_{e_K}(\Delta)\left[h^{-1}_{e_K}(\Delta), \hat{V}\right]\right]$$

$\rightarrow$ study operator

$$^{(r)}\hat{e}^k_K \;:\; = \mathrm{Tr}\left[\tau_k\; h_K\; \left[h_K^{-1}, \hat{V}^r\right]\right]$$

- LQC: inverse Scale Factor

$$\widehat{(a^{-1})} \propto \left(\sum_k \mathrm{Tr}\left[\tau_k h_k \{h_k^{-1}, \sqrt{V}\}\right]\right)^2 \quad \rightarrow \text{Bounded Operator in Quantum Theory}$$

Application II: Triad Operators in LQG 2

- What about LQG ??

  - Holonomie Action $[\pi_{\frac{1}{2}}(\cdot)]_{AB} = \frac{1}{2}\ \left|\frac{1}{2}\ A\ ;\ B\ \right\rangle$

$$\frac{1}{2}\ \left|\frac{1}{2}\ A\ ;\ B\ \right\rangle \cdot \left|\ j\ m\ ;\ n\ \right\rangle = \left[\frac{j+2Bn+1}{2\,(j+1)}\right]^{\frac{1}{2}} \left\langle\ j\ m\ ;\ \frac{1}{2}\ A\ \middle|\ j+\frac{1}{2}\ m+A\ \right\rangle \cdot \left|\ j+\frac{1}{2}\ m+A\ :\ n+B\ \right\rangle$$
$$\oplus\ (-2)B\left[\frac{j-2Bn}{2\ j}\right]^{\frac{1}{2}} \left\langle\ j\ m\ ;\ \frac{1}{2}\ A\ \middle|\ j-\frac{1}{2}\ m+A\ \right\rangle \cdot \left|\ j-\frac{1}{2}\ m+A\ :\ n+B\ \right\rangle$$

Application II: Triad Operators in LQG 2

- What about LQG ??

  - Holonomie Action $[\pi_{\frac{1}{2}}(\cdot)]_{AB} = \frac{1}{2}\ \left|\frac{1}{2}\ A\ :\ B\ \right\rangle$

$$\frac{1}{2}\ \left|\frac{1}{2}\ A\ ;\ B\ \right\rangle \cdot \left|\ j\ m\ ;\ n\ \right\rangle = \left[\frac{j+2Bn+1}{2\,(j+1)}\right]^{\frac{1}{2}} \left\langle\ j\ m\ :\ \tfrac{1}{2}\ A\ \right|\ j+\tfrac{1}{2}\ m+A\ \right\rangle \cdot \left|\ j+\tfrac{1}{2}\ m+A\ :\ n+B\ \right\rangle$$

$$\oplus\ (-2)B\left[\frac{j-2Bn}{2\ j}\right]^{\frac{1}{2}} \left\langle\ j\ m\ ;\ \tfrac{1}{2}\ A\ \right|\ j-\tfrac{1}{2}\ m+A\ \right\rangle \cdot \left|\ j-\tfrac{1}{2}\ m+A\ :\ n+B\ \right\rangle$$

  - 3-Vertex-Calculation

$$\left\langle a_2(j_1\ j_2) = j_3\ \ J=1\ M'\ ;\ n_1 n_2 n_3 \right|\ {}^{(r)}\hat{e}^{k}_{K=3}\ \left| a_2(j_1\ j_2) = j_3\ \ J=0\ M\ ;\ n_1 n_2 n_3 \right\rangle\ =$$

$$= C(M,M',k)\frac{\sqrt{j_3(j_3+1)}}{2j_3+1}\Big[\big[(j_1+j_2+j_3+2)(-j_1+j_2+j_3+1)(j_1-j_2+j_3+1)(j_1+j_2-j_3)\big]^{\frac{r}{4}}$$
$$+\big[(j_1+j_2+j_3+1)(-j_1+j_2+j_3)(j_1-j_2+j_3)(j_1+j_2-j_3+1)\big]^{\frac{r}{4}}\Big]$$

$\boxed{\text{Not Bounded!}}$

Application II: Triad Operators in LQG 3

- $U(1)^3$-CS Calculation
  - Study $$\begin{aligned} {}^{(r)}\hat{e}^j_e(v) &= \mathrm{Tr}\Big[\, \tau_j\, \hat{h}_e\, [\, \hat{h}_e^{-1}\,,\, \hat{V}^r_v\,]\,\Big] && \text{for } SU(2) \\ \hat{q}^j_e(v,r) &= \mathbb{i}\, \hat{h}^j_e\, [\, (\hat{h}^j_e)^{-1}\,,\, \hat{V}^r_v\,] && \text{for } U(1)^3 \end{aligned}$$
  - Charge Network States $$T_c = \prod_{\substack{e_I \in E(v) \\ j=1,2,3}} \left[h^j_I(A)\right]^{n^j_I(c)} \quad \text{with} \quad h^j_I(A) = \mathbb{e}^{\mathbb{i}\theta^j_I(A)}$$
  - Coherent States $$\Psi^{(v)}_{m,\gamma} = \prod_{\substack{e\in E(v) \\ j=1,2,3}} \sum_{n^j_e \in \mathbb{Z}} \mathbb{e}^{-\frac{t(e)}{2}[n^j_e]^2} \left[h^j_e(Z(m))h^j_e(A)^{-1}\right]^{n^j_e}$$

- Expectation Value $$\left\langle\, \Psi^{(v)}_{m,\gamma} \,\middle|\, \hat{q}^{j_k}_{e_k}(v,r) \prod_{k=1}^{M} \,\middle|\, \Psi^{(v)}_{m,\gamma} \,\right\rangle$$

  is finite, no matter if peaked at classically singular ($E=0$) configuration)

Conclusions & Outlook

- Simplified Volume Operator in Full Theory

  1) Numerical Studies

  2) Analytical Studies $\rightarrow$ Tools to check LQC and the question of singularity (dis-) appearance in the Full Theory

- Triad - Operator in General not necessarily bounded (3-vertex) on 0-volume Eigenstates $\rightarrow$ Calculation in $U(1)^3$-CS shows finite behavior

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- To Do:
  - Further numerical studies
  - Triad at 4-vertex?
  - Re-Do $U(1)^3$-Calculation for $SU(2)$

  $\rightsquigarrow$ Examine action of $\hat{H}_E$ in detail

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Application II: Triad Operators in LQG 1

- Classical: Euclidean Part of H

$$H_E = \frac{1}{\kappa\sqrt{\det q}}\left[\, F^j_{ab}\, \epsilon_{jkl}\, E^a_k E^b_l \,\right]$$

$$\begin{aligned} E^a_j &= \det(e) e^a_j \\ \frac{\epsilon_{jkl}\ E^a_k E^b_l}{\kappa\sqrt{\det q}} &= \mathrm{sgn}\big(\det(e)\big)\epsilon^{abc} e^j_c \\ e^j_c &\propto \{A^j_c, V\} \propto \mathrm{Tr}\left[\tau_j h_c \{h_c^{-1}, V\}\right] \end{aligned}$$

  - ▶ Regularized Expression

$$H^\epsilon_E(N) \sim \sum_{\Delta \in T(\Sigma)} N\big(v(\Delta)\big)\epsilon^{IJK}\, \mathrm{Tr}\left[h_{\alpha_{IJ}}(\Delta) h_{e_K}(\Delta)\left[h^{-1}_{e_K}(\Delta), \hat{V}\right]\right]$$

$\rightarrow$ study operator

$${}^{(r)}\hat{e}^k_K \ :\ = \mathrm{Tr}\left[\tau_k\ h_K\ \left[h_K^{-1}, \hat{V}^r\right]\right]$$

- LQC: inverse Scale Factor

$$\widehat{(a^{-1})} \propto \left(\sum_k \mathrm{Tr}\left[\tau_k h_k \{h_k^{-1}, \sqrt{V}\}\right]\right)^2 \quad \rightarrow \text{Bounded Operator in Quantum Theory}$$

Volume Operator II: Simplified Expression for the Matrix Element 2

$$
\begin{aligned}
&< \vec{a}|\hat{q}_{IJK}|\vec{a}' > \quad = \\
&= \frac{1}{4}(-1)^{j_K+j_I+a_{I-1}+a_K}(-1)^{a_I-a'_I}(-1)^{\sum_{n=I+1}^{J-1} j_n}(-1)^{-\sum_{p=J+1}^{K-1} j_p} \times \\
&\quad \times X(j_I,j_J)^{\frac{1}{2}} X(j_J,j_K)^{\frac{1}{2}} \sqrt{(2a_I+1)(2a'_I+1)}\sqrt{(2a_J+1)(2a'_J+1)} \times \\
&\quad \times \begin{Bmatrix} a_{I-1} & j_I & a_I \\ 1 & a'_I & j_I \end{Bmatrix} \left[\prod_{n=I+1}^{J-1} \sqrt{(2a'_n+1)(2a_n+1)}(-1)^{a'_{n-1}+a_{n-1}+1} \begin{Bmatrix} j_n & a'_{n-1} & a'_n \\ 1 & a_n & a_{n-1} \end{Bmatrix} \right] \times \\
&\quad \times \left[\prod_{n=J+1}^{K-1} \sqrt{(2a'_n+1)(2a_n+1)}(-1)^{a'_{n-1}+a_{n-1}+1} \begin{Bmatrix} j_n & a'_{n-1} & a'_n \\ 1 & a_n & a_{n-1} \end{Bmatrix} \right] \begin{Bmatrix} a_K & j_K & a_{K-1} \\ 1 & a'_{K-1} & j_K \end{Bmatrix} \times \\
&\quad \times \left[(-1)^{a'_J+a'_{J-1}} \begin{Bmatrix} a_J & j_J & a'_{J-1} \\ 1 & a_{J-1} & j_J \end{Bmatrix} \begin{Bmatrix} a'_{J-1} & j_J & a'_J \\ 1 & a_J & j_J \end{Bmatrix} \right. \\
&\qquad \left. - (-1)^{a_J+a_{J-1}} \begin{Bmatrix} a'_J & j_J & a'_{J-1} \\ 1 & a_{J-1} & j_J \end{Bmatrix} \begin{Bmatrix} a_{J-1} & j_J & a'_J \\ 1 & a_J & j_J \end{Bmatrix} \right] \times \\
&\quad \times \prod_{n=2}^{I-1} \delta_{a_n a'_n} \prod_{n=K}^{N} \delta_{a_n a'_n}
\end{aligned}
$$

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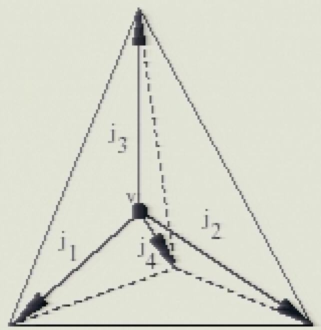
Application I: Numerical Studies 1

- Gauge Invariant 4-Vertex



- Basis-States

$$T_v = \left| \underbrace{a_2(j_1\, j_2)}_{j_{12}}\; a_3(a_2\, j_3)\; J(a_3\, j_4) = 0\; M = 0\, ;\; \vec{n} \right\rangle$$

- Gauge Invariance

$$J_1 + J_2 + J_3 + J_4 \overset{!}{=} 0$$

- Only $\hat{q}_{123}$ matters!

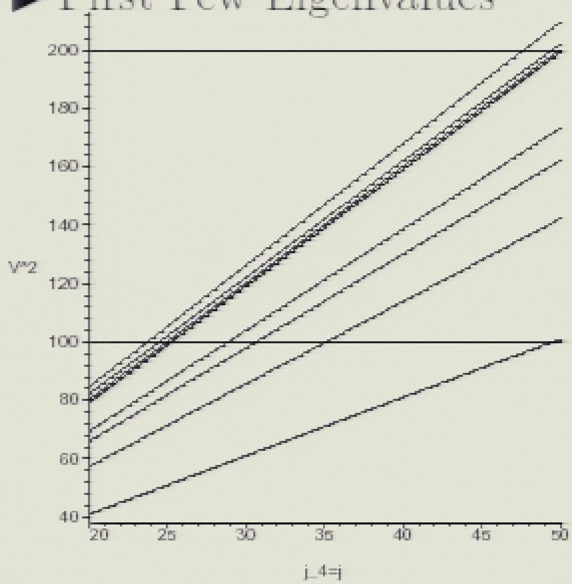
$$
\begin{aligned}
< j_{12}|\hat{q}_{123}|j_{12} - 1 > \; &= \\
&= \frac{1}{\sqrt{(2j_{12}-1)(2j_{12}+1)}}\Big[(j_1 + j_2 + j_{12} + 1)(-j_1 + j_2 + j_{12})(j_1 - j_2 + j_{12})(j_1 + j_2 - j_{12} + 1) \\
&\qquad (j_3 + j_4 + j_{12} + 1)(-j_3 + j_4 + j_{12})(j_3 - j_4 + j_{12})(j_3 + j_4 - j_{12} + 1)\Big]^{\frac{1}{2}} \\
&= \; - \; < j_{12} - 1|\hat{q}_{123}|j_{12} >
\end{aligned}
$$

Application I: Numerical Studies 2

- Matrix Structure

$$A = \begin{pmatrix} 0 & -a_1 & 0 & \cdots & \cdots & 0 \\ a_1 & 0 & -a_2 & & & \vdots \\ 0 & a_2 & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & -a_{n-1} \\ 0 & \cdots & \cdots & \cdots & a_{n-1} & 0 \end{pmatrix}$$

- First Few Eigenvalues



- First Impression