Title: Contextuality for Preparations, Transformations, and Unsharp Measurements

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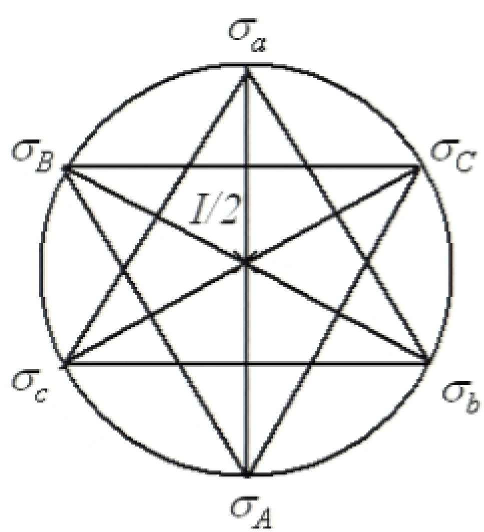
Abstract:

Contextuality for preparations, transformations and unsharp measurements

Rob Spekkens



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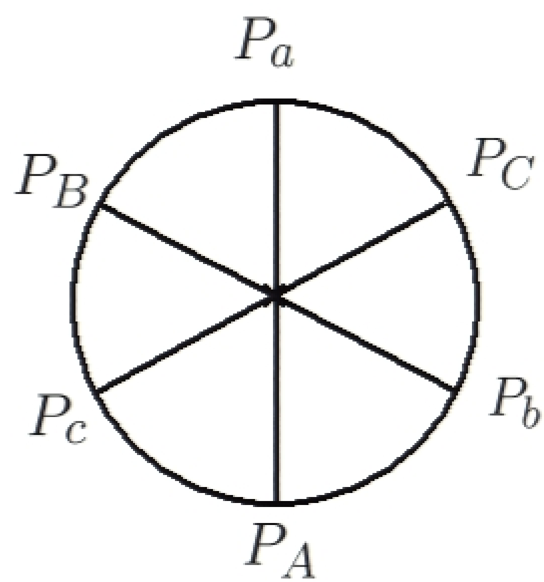
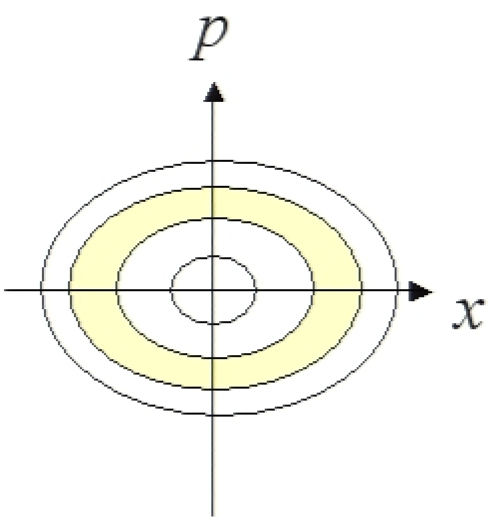


See quant-ph/0406166

Piqudos seminar Oct. 27, 2004

The idea of a hidden variable model of quantum mechanics

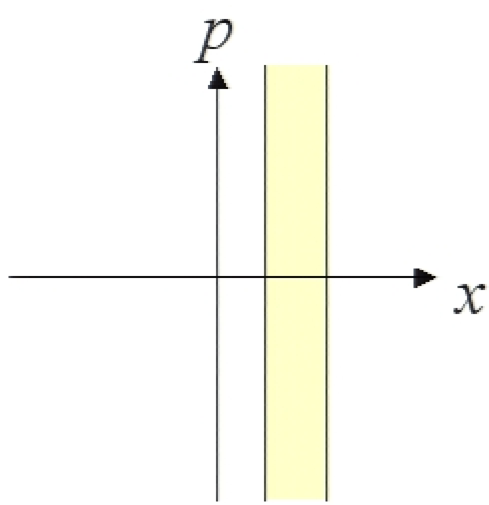
In a classical theory, properties are associated with regions of the state space



$$E(x,p) = \frac{p^2}{2m} + \frac{kx^2}{2}$$

$$E_1 \leq E \leq E_2$$

Consider $$g_A(x,p) = a_1 \text{ if } x < x_1,$$
$$= a_2 \text{ if } x_1 \le x \le x_2,$$
$$= a_3 \text{ if } x > x_2.$$



Consider

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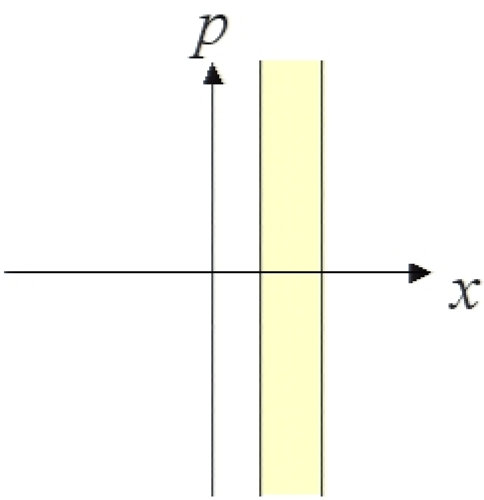


Equivalently, $g_A(x,p) = \sum_k a_k \chi_k(x,p)$

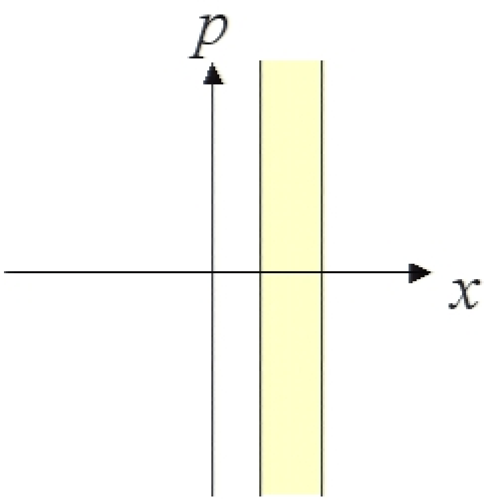
where

$$\chi_1(x,p) = 1 \text{ if } x < x_1, \quad = 0 \text{ otherwise},$$

$$\chi_2(x,p) = 1 \text{ if } x_1 \le x \le x_2, \quad = 0 \text{ otherwise},$$

$$\chi_3(x,p) = 1 \text{ if } x > x_2, \quad = 0 \text{ otherwise},$$

Consider $$\begin{aligned} g_A(x,p) &= a_1 \text{ if } x < x_1, \\ &= a_2 \text{ if } x_1 \le x \le x_2, \\ &= a_3 \text{ if } x > x_2. \end{aligned}$$



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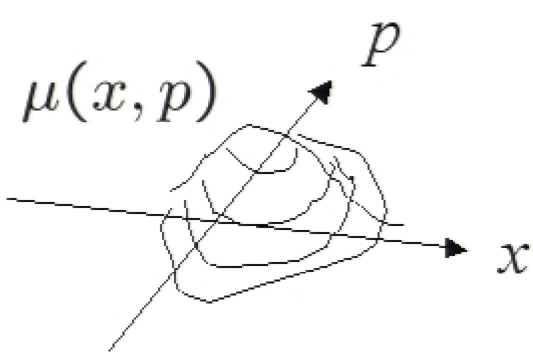

Can still have probabilistic outcomes if x,p is unknown

$$\text{Prob}(k) = \int dx\ dp\ \mu(x,p)\ \chi_k(x,p)$$

In quantum theory, we have

$$A = \sum_k a_k P_k$$

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The idea of a deterministic hidden variable theory is that

$$|\psi\rangle \leftrightarrow \mu(\lambda)$$

$$A \leftrightarrow g_A(\lambda)$$

$$\{P_k\} \leftrightarrow \{\chi_k(\lambda)\}$$

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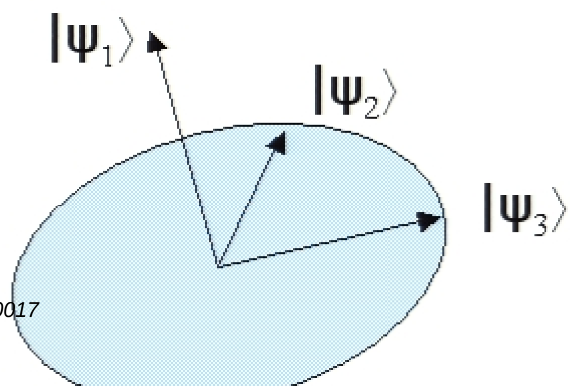
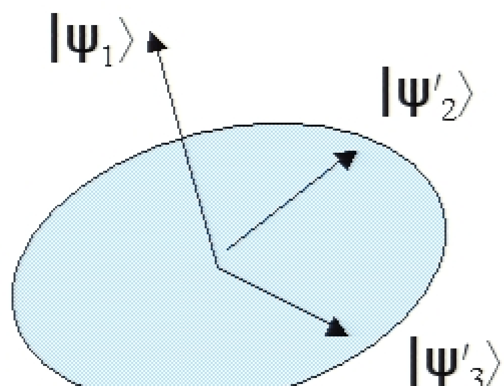
There are many ways of measuring $\{P_k\}$

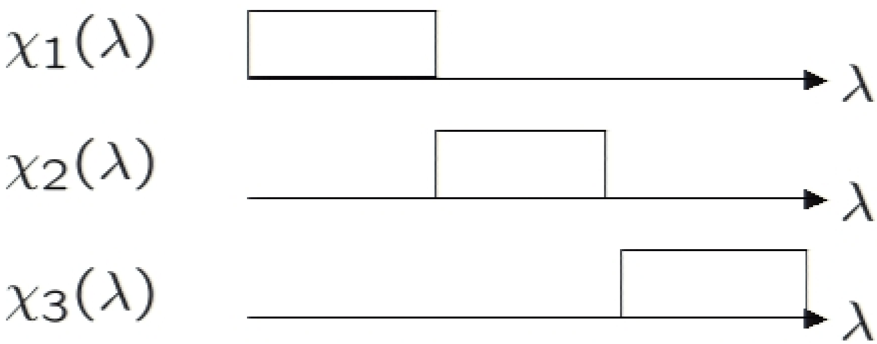
Example: $\{|\psi_1\rangle\langle\psi_1|, I - |\psi_1\rangle\langle\psi_1|\}$



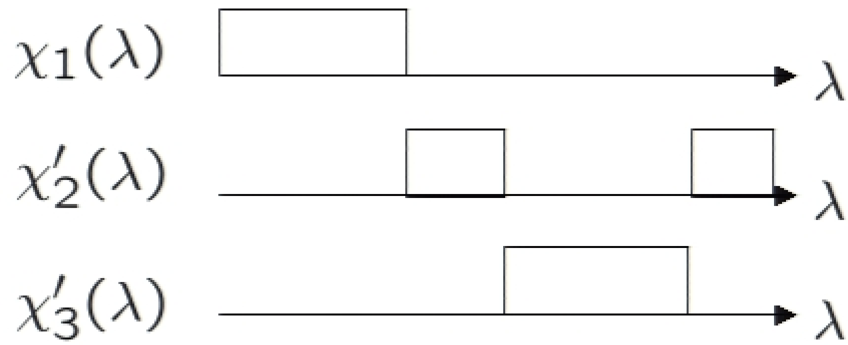


Naively, one might hope to represent this as follows:

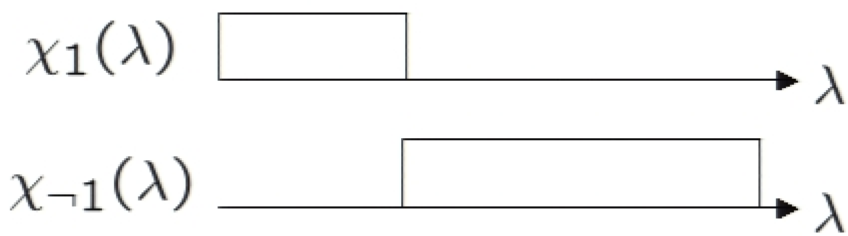
$\{|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|, |\psi_3\rangle\langle\psi_3|\}$



$\{|\psi_1\rangle\langle\psi_1|, |\psi_2'\rangle\langle\psi_2'|, |\psi_3'\rangle\langle\psi_3'|\}$



$\{|\psi_1\rangle\langle\psi_1|, I - |\psi_1\rangle\langle\psi_1|\}$



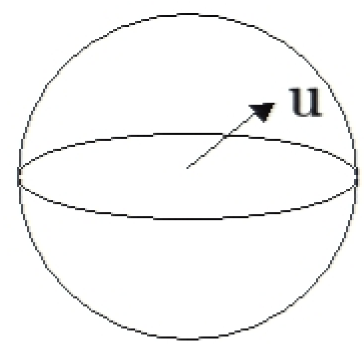
The traditional notion of noncontextuality:
Every P is associated with the same $\chi(\lambda)$ regardless of how it is measured

Achieving noncontextuality for pure preparations and sharp measurements in 2d

$$|+\mathbf{n}\rangle \leftrightarrow \qquad \mu_{\mathbf{n}}(\mathbf{u}) = \begin{cases} \frac{1}{\pi}\mathbf{n}\cdot\mathbf{u} & \text{for } \mathbf{n}\cdot\mathbf{u} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$|+\mathbf{m}\rangle \leftrightarrow \qquad \chi_{\mathbf{m}+}(\mathbf{u}) = \begin{cases} 1 & \text{for } \mathbf{m}\cdot\mathbf{u} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\int \mu_{\mathbf{n}}(\mathbf{u})\chi_{\mathbf{m}+}(\mathbf{u})d\mathbf{u} = \frac{1}{2}(1+\mathbf{m}\cdot\mathbf{n}) = |\langle +\mathbf{m}|+\mathbf{n}\rangle|^2$$



It was shown by Bell (1966) and Kochen and Specker (1967) that a noncontextual hidden variable model of quantum theory for Hilbert spaces of dimensionality 3 or greater is impossible. That is, quantum theory is contextual

This is called the
Bell-Kochen-Specker theorem

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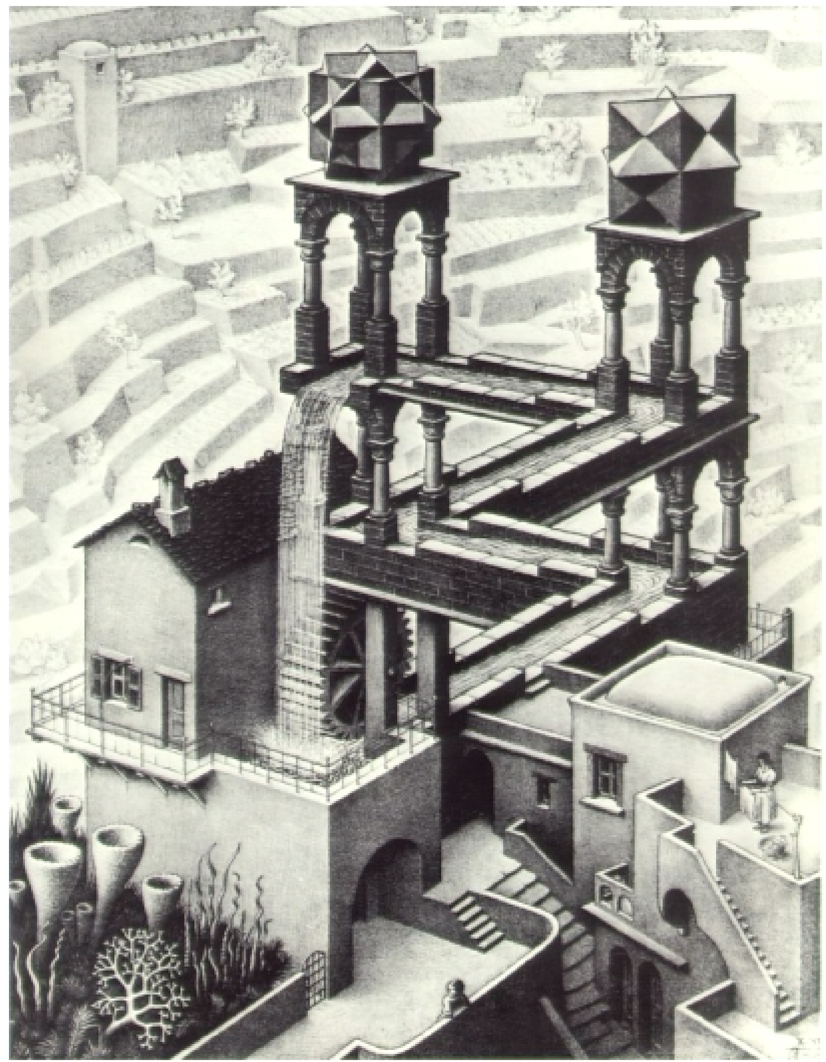

The traditional definition of contextuality does not apply to:

(1) arbitrary operational theories
(2) preparations, transformations, or unsharp measurements
(3) indeterministic ontological models

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(1) arbitrary operational theories
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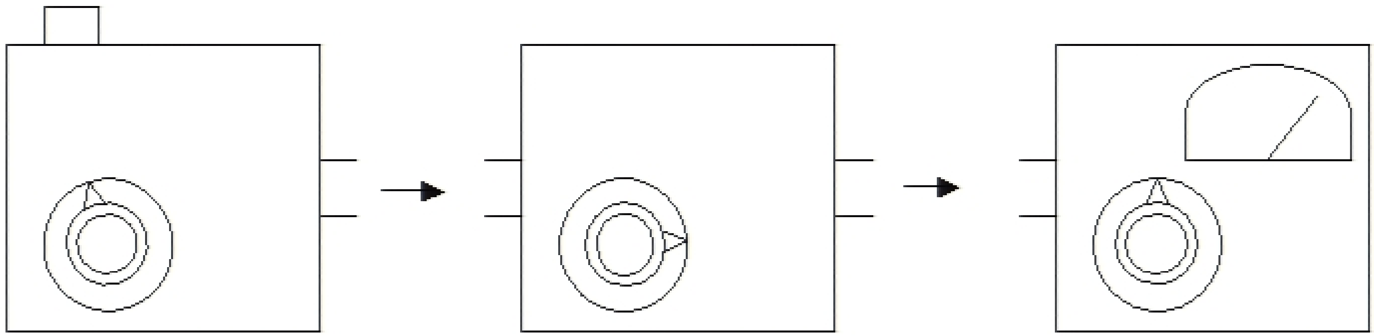
(3) indeterministic ontological models

Proposed new definition:
A noncontextual ontological model of an operational theory is one wherein if two experimental procedures are operationally equivalent, then they have equivalent representations in the ontological model.

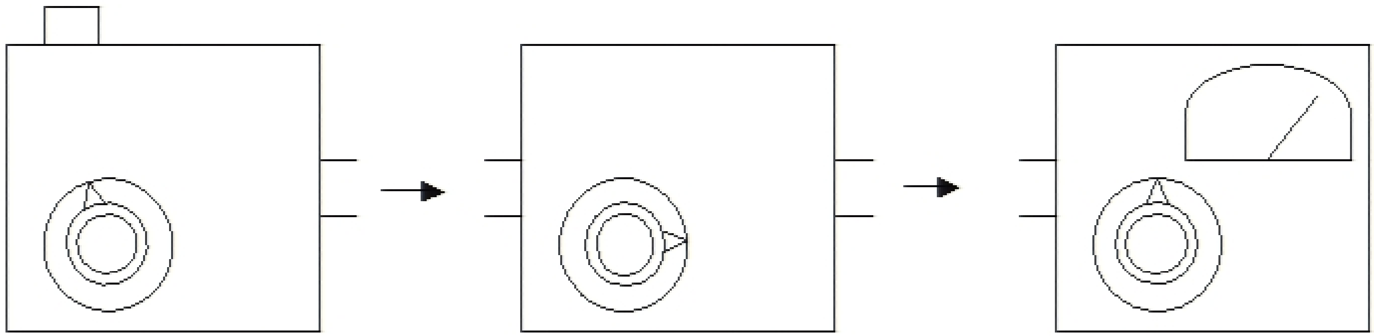
Operational theories



Preparation P Transformation T Measurement M

These are defined as lists of instructions

Operational theories



Preparation	Transformation	Measurement
P	T	M

These are defined as lists of instructions

An operational theory specifies

$p(k|\mathrm{P}, \mathrm{T}, \mathrm{M}) \equiv$ The probability of outcome k of M given P and T.

Defining operational equivalence of procedures

For preparations

$\mathsf{P} \simeq \mathsf{P}'$ if

$p(k|\mathsf{P},\mathsf{M}) = p(k|\mathsf{P}',\mathsf{M})$ for all M.

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For measurements

$\mathsf{M} \simeq \mathsf{M}'$ if

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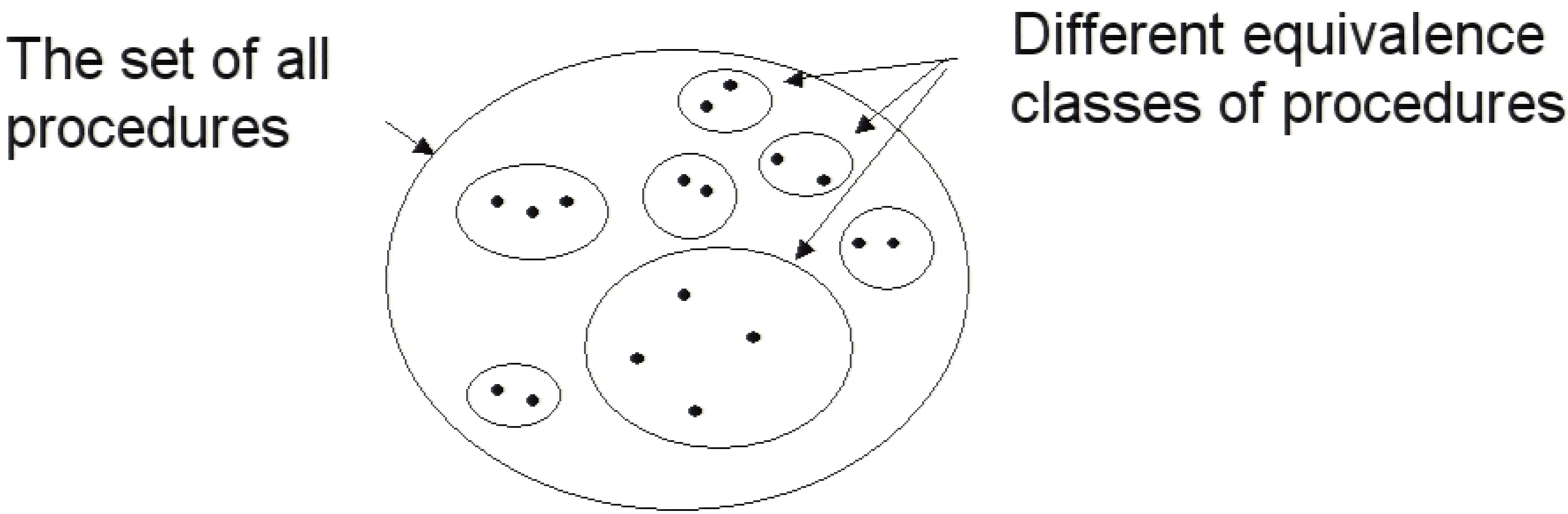
For transformations

$\mathsf{T} \simeq \mathsf{T}'$ if

$p(k|\mathsf{P},\mathsf{T},\mathsf{M}) = p(k|\mathsf{P},\mathsf{T}',\mathsf{M})$ for all P,M.

Defining the context of a procedure



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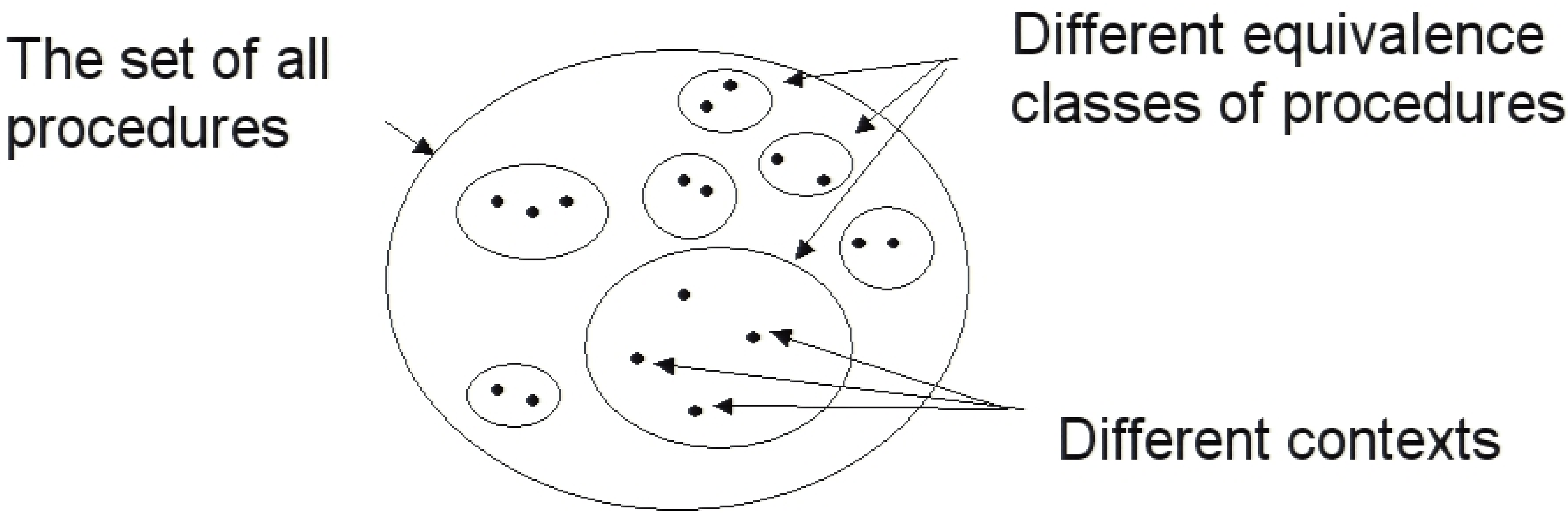

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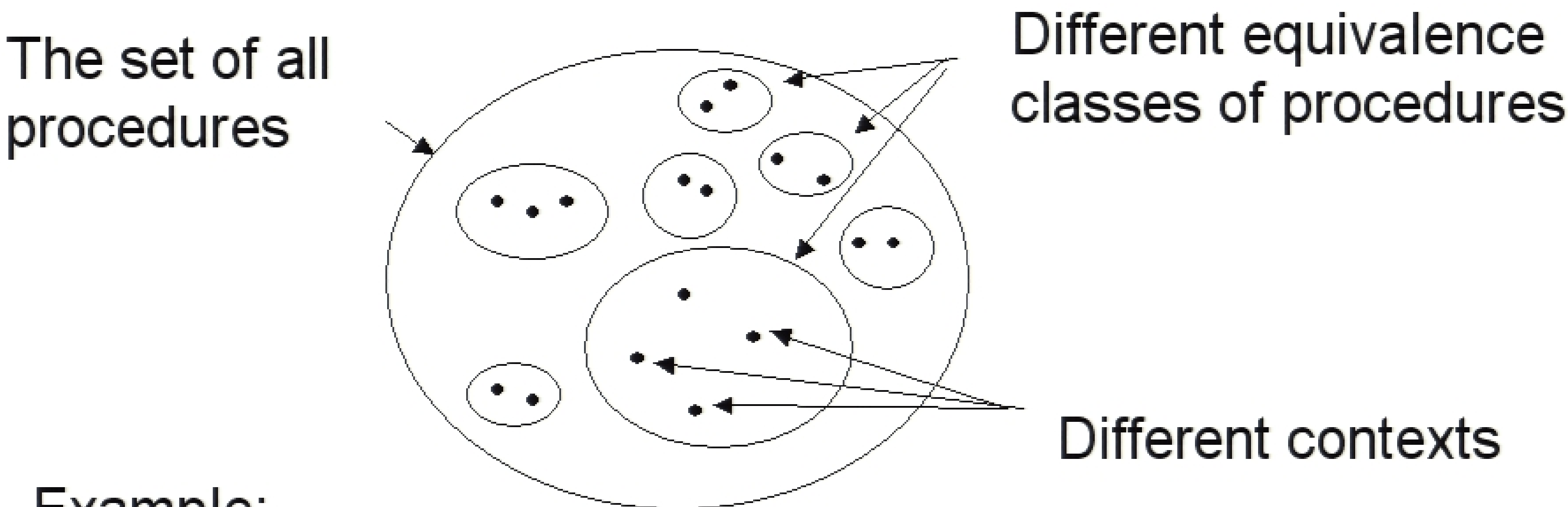
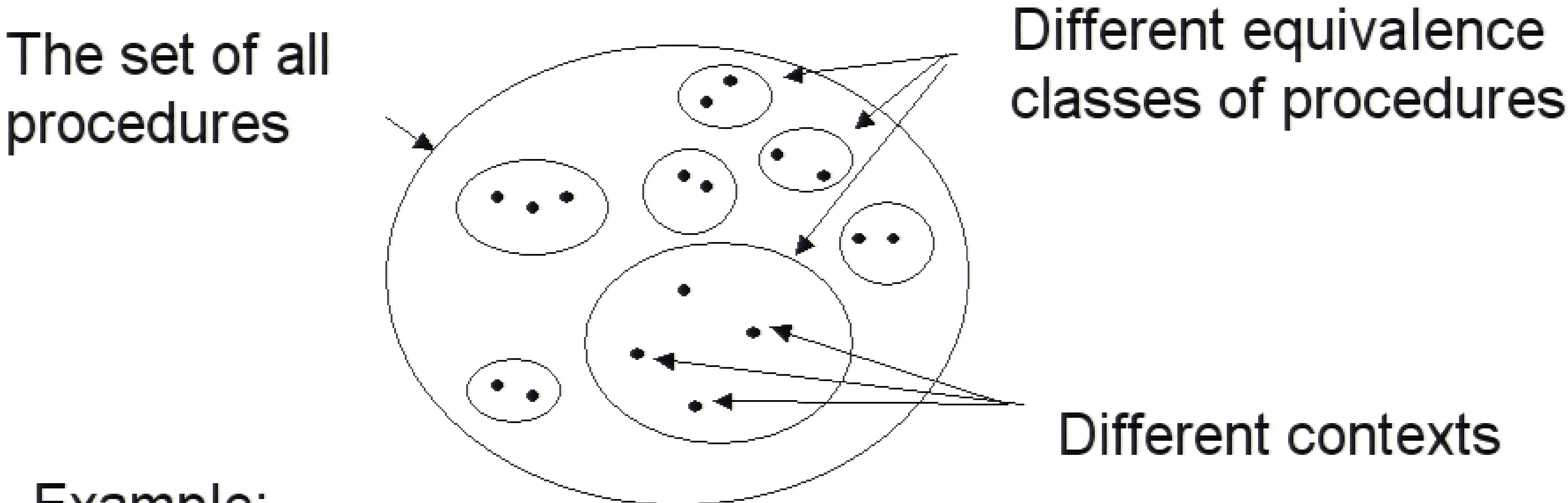


Example:

Defining the context of a procedure



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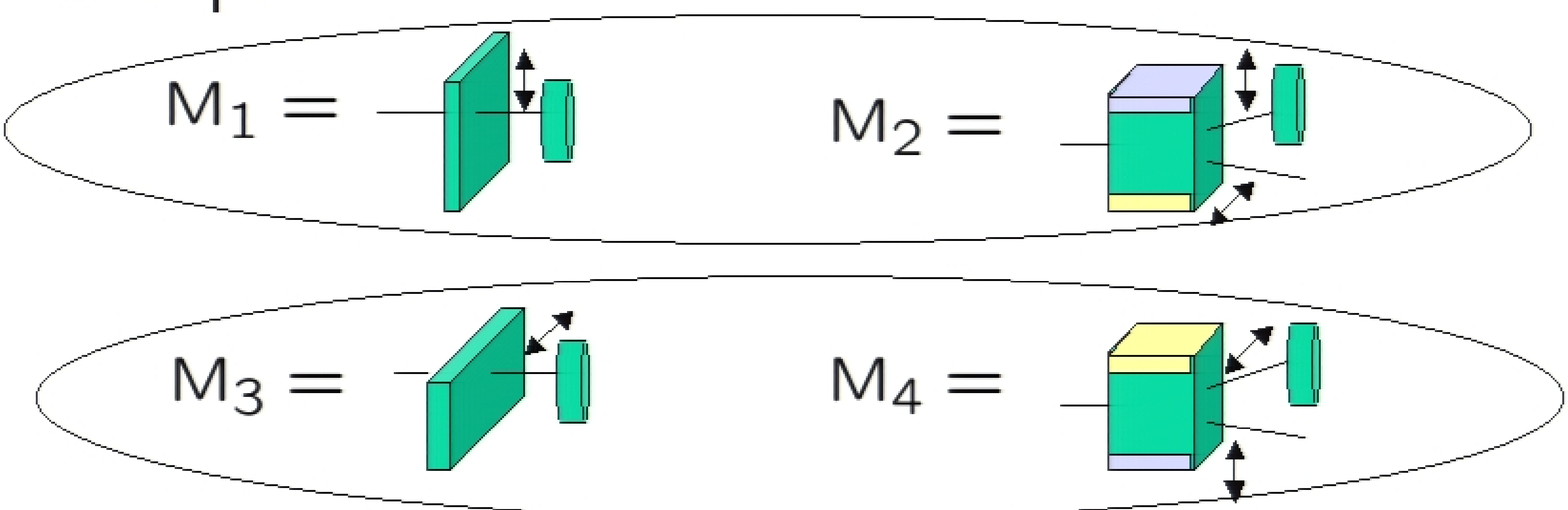

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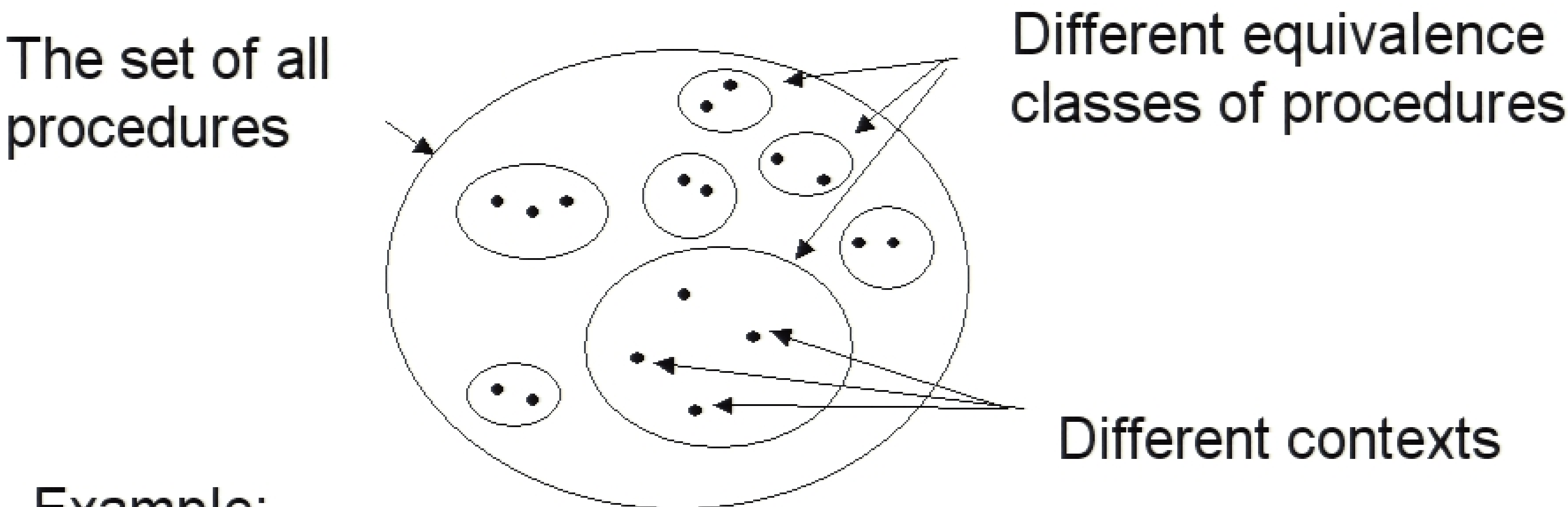

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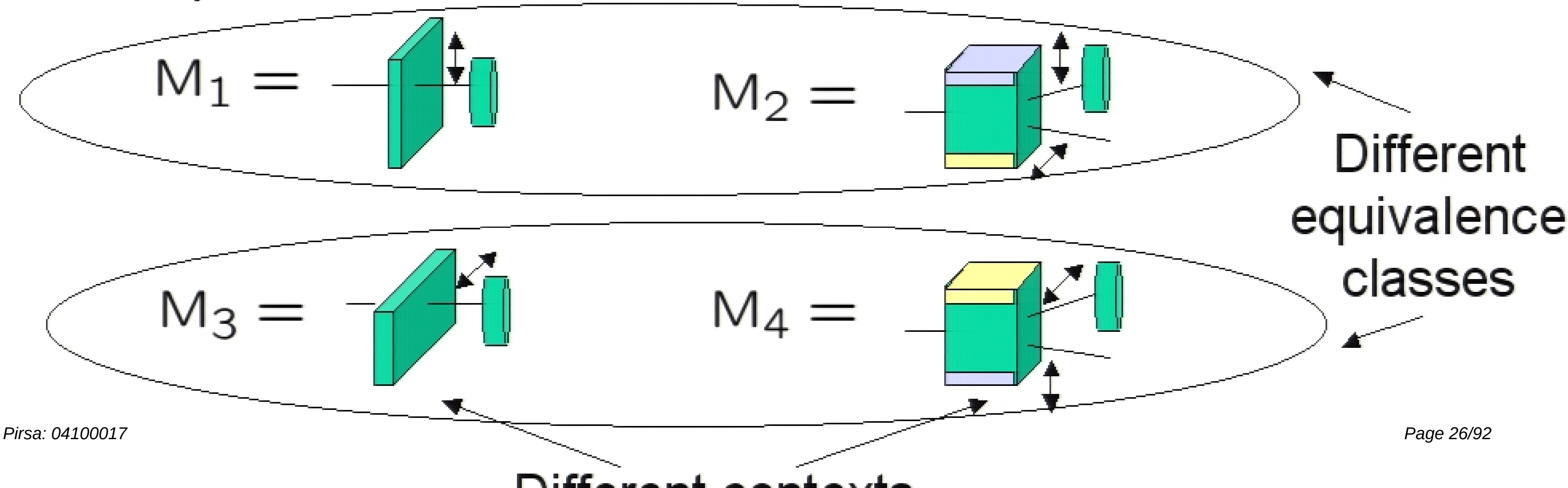
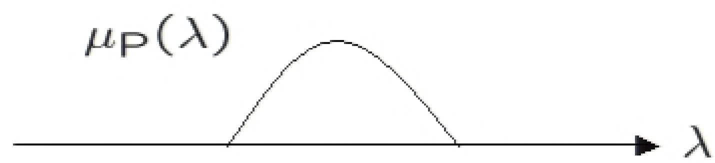


An ontological model of an operational theory

Preparations

$\mu_{\mathsf{P}} : \Omega \rightarrow [0,1]$

$\int \mu_{\mathsf{P}}(\lambda) d\lambda = 1$



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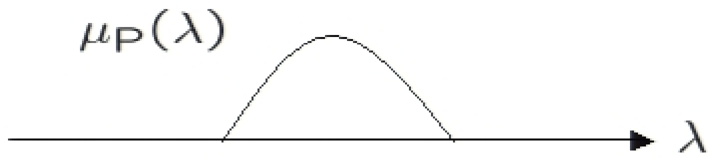
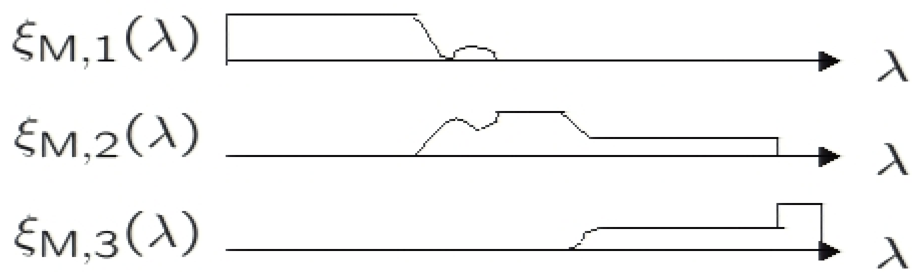


Measurements

$\xi_{\mathsf{M},k} : \Omega \to [0,1]$

$\sum_k \xi_{\mathsf{M},k}(\lambda) = 1$ for all λ



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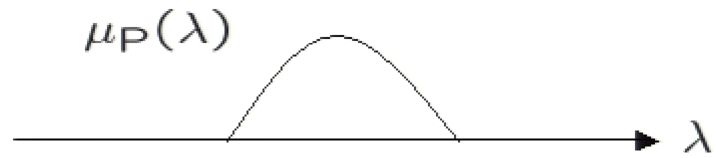

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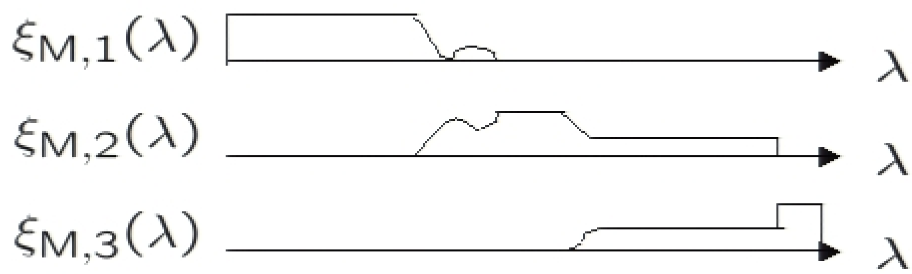
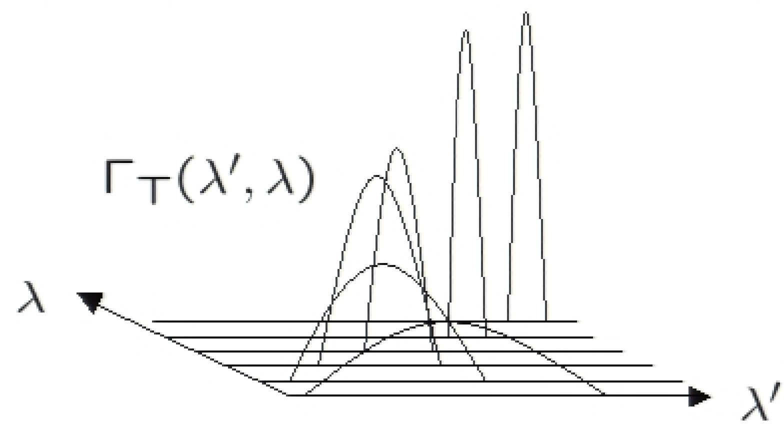


Transformations

$\Gamma_{\mathsf{T}} : \Omega \times \Omega \rightarrow [0,1]$

$\int \Gamma_{\mathsf{T}}(\lambda', \lambda) d\lambda' = 1$ for all λ



An ontological model of an operational theory

Preparations

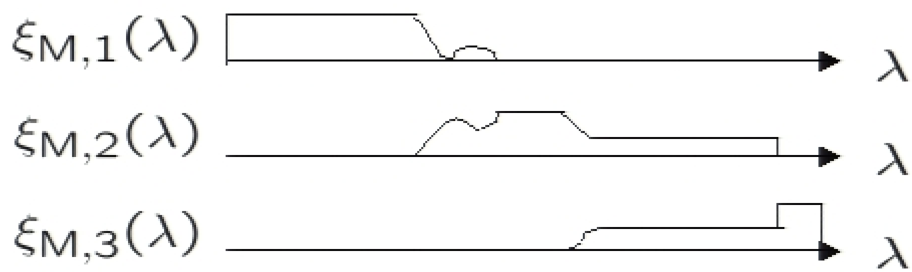
$\mu_{\mathsf{P}} : \Omega \rightarrow [0,1]$

$\int \mu_{\mathsf{P}}(\lambda) d\lambda = 1$

Measurements

$\xi_{\mathsf{M},k} : \Omega \rightarrow [0,1]$

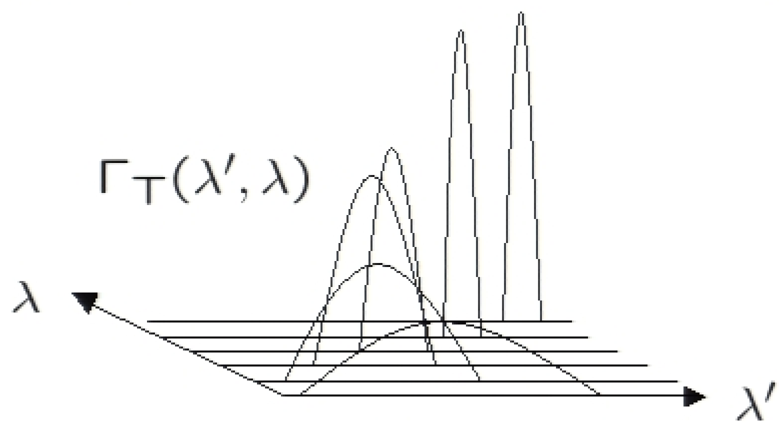
$\sum_k \xi_{\mathsf{M},k}(\lambda) = 1$ for all λ



Transformations

$\Gamma_{\mathsf{T}} : \Omega \times \Omega \rightarrow [0,1]$

$\int \Gamma_{\mathsf{T}}(\lambda', \lambda) d\lambda' = 1$ for all λ



$$p(k|\mathsf{P},\mathsf{T},\mathsf{M}) = \int d\lambda' d\lambda \, \xi_{\mathsf{M},k}(\lambda') \, \Gamma_{\mathsf{T}}(\lambda', \lambda) \, \mu_{\mathsf{P}}(\lambda)$$

Defining noncontextuality

Preparation Noncontextuality

$$\mu_{\mathsf{P}}(\lambda) = \mu_{e(\mathsf{P})}(\lambda)$$

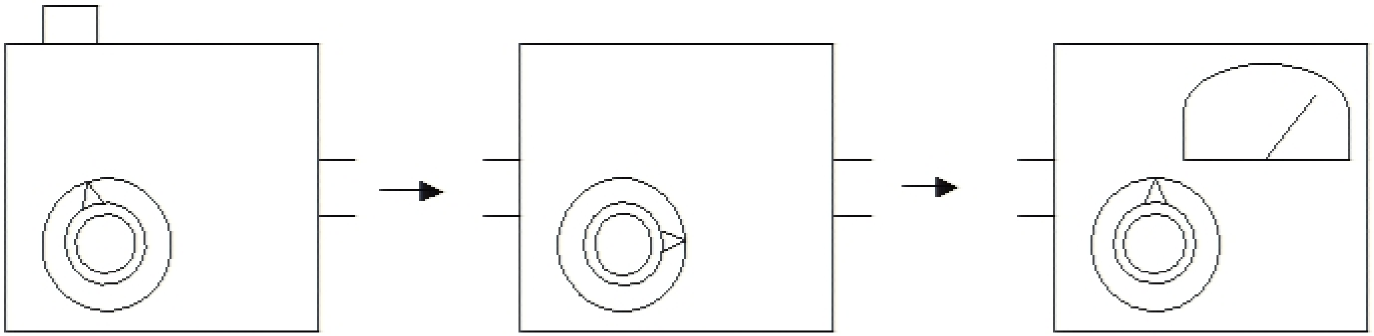
Measurement Noncontextuality

$$\xi_{\mathsf{M},k}(\lambda) = \xi_{e(\mathsf{M}),k}(\lambda)$$

Transformation Noncontextuality

$$\Gamma_{\mathsf{T}}(\lambda', \lambda) = \Gamma_{e(\mathsf{T})}(\lambda', \lambda)$$

Operational Quantum Mechanics



Preparation	Transformation	Measurement
P	T	M
Vectors	Unitary maps	Projector-valued measures (PVMs)
$\|\psi\rangle$	U	$\{P_k\}$
$\|\psi\rangle \in \mathbb{C}_d$	$U \in \mathcal{L}(\mathbb{C}_d)$	$P_k \in \mathcal{L}(\mathbb{C}_d)$
	$U^\dagger U = UU^\dagger = I$	$\sum_{k=1}^{d} P_k = I$
		$P_k^2 = P_k$

$$\text{Prob(k)} \quad = \langle\psi|U^\dagger P_k U|\psi\rangle$$

More general preparations

Probability p, prepare $|\psi\rangle$
Probability q, prepare $|\chi\rangle$
Measure $\{P_k\}$

$$
\begin{aligned}
\text{Prob}(k) &= p\langle\psi|P_k|\psi\rangle + q\langle\chi|P_k|\chi\rangle \\
&= p\,\text{Tr}(|\psi\rangle\langle\psi|P_k) + q\,\text{Tr}(|\chi\rangle\langle\chi|P_k) \\
&= \text{Tr}(\rho P_k)
\end{aligned}
$$

$$\rho = p|\psi\rangle\langle\psi| + q|\chi\rangle\langle\chi|$$

A density operator

$$\rho \in \mathcal{L}(\mathbb{C}_d)$$

$$\langle\psi|\rho|\psi\rangle \geq 0, \forall\psi$$

$$\text{Tr}(\rho) = 1$$

$\rho = |\psi\rangle\langle\psi| \quad \leftrightarrow$ Pure preparation

$\rho \neq |\psi\rangle\langle\psi| \quad \leftrightarrow$ Mixed preparation

More general transformations

Prepare ρ
Probability p, transform with U
Probability q, transform with V
measure $\{\mathrm{E_k}\}$

$$\text{Prob}(k) = p\,\text{Tr}(U\rho U^\dagger E_k) + q\,\text{Tr}(V\rho V^\dagger E_k)$$
$$= \text{Tr}(\mathcal{T}(\rho)E_k)$$
$$\mathcal{T}(\cdot) = pU(\cdot)U^\dagger + qV(\cdot)V^\dagger$$

A completely positive map (CP map)

$$\mathcal{T} : \mathcal{L}(\mathbb{C}_d) \to \mathcal{L}(\mathbb{C}_d)$$
$$\mathcal{T}(\rho) = \sum_\mu A_\mu \rho A_\mu^\dagger$$
$$\sum_\mu A_\mu^\dagger A_\mu = I$$

$\mathcal{T}(\rho) = U\rho U^\dagger \quad \leftrightarrow$ Reversible transformation
$\mathcal{T}(\rho) \neq U\rho U^\dagger \quad \leftrightarrow$ Irreversible transformation

More general measurements

Prepare ρ
Probability p, measure the PVM $\{P_k\}$
Probability q, measure the PVM $\{Q_k\}$

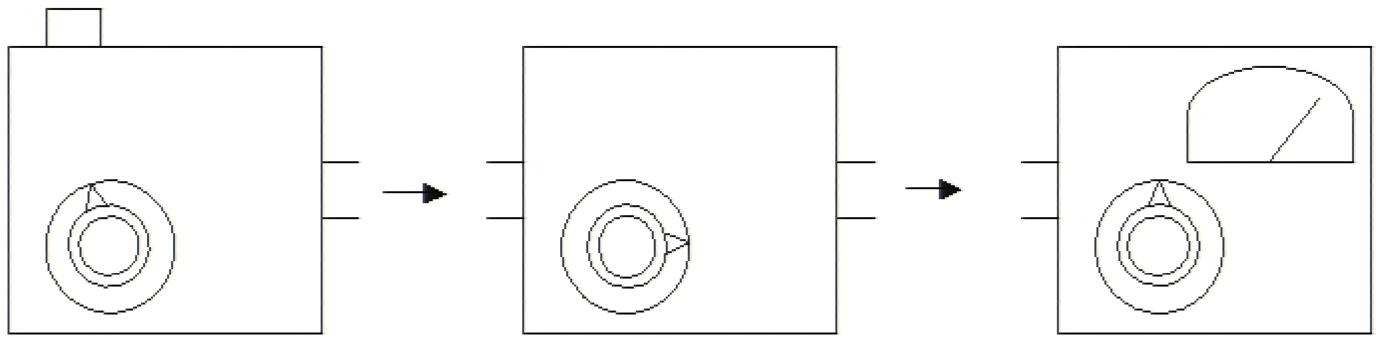
$$\text{Prob}(k) = p\,\text{Tr}(\rho P_k) + q\,\text{Tr}(\rho Q_k)$$
$$= \text{Tr}(\rho E_k)$$
$$E_k = pP_k + qQ_k$$

A Positive operator valued measure (POVM)

$$E_k \in \mathcal{L}(\mathbb{C}_d)$$
$$\langle\psi|E_k|\psi\rangle \geq 0, \forall\psi$$
$$\sum_{k=1}^{d} E_k = I$$

$\{E_k\} = \{P_k\}$ $\leftrightarrow$ Sharp measurement
$\{E_k\} \neq \{P_k\}$ $\leftrightarrow$ Unsharp measurement

Operational Quantum Mechanics

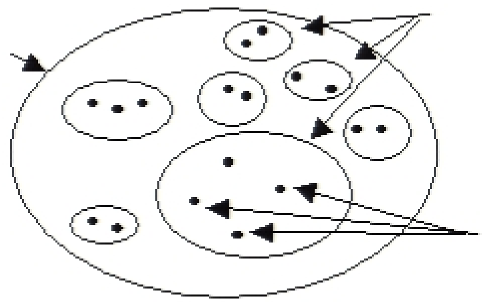


Preparation	Transformation	Measurement
P	T	M
Density operators	Completely positive maps (CP maps)	Positive operator-valued measures (POVMs)
ρ	$\mathcal{T}$	$\{E_k\}$
$\rho \in \mathcal{L}(\mathbb{C}_d)$	$\mathcal{T} : \mathcal{L}(\mathbb{C}_d) \to \mathcal{L}(\mathbb{C}_d)$	$E_k \in \mathcal{L}(\mathbb{C}_d)$
$\langle\psi\|\rho\|\psi\rangle \geq 0, \forall\psi$	$\mathcal{T}(\rho) = \sum_\mu A_\mu \rho A_\mu^\dagger$	$\sum_{k=1}^d E_k = I$
$\mathrm{Tr}(\rho) = 1$	$\sum_\mu A_\mu^\dagger A_\mu = I$	$\langle\psi\|E_k\|\psi\rangle \geq 0, \forall\psi$

$$\text{Prob(k)} = \mathrm{Tr}[\mathcal{T}(\rho) E_k]$$

Contexts for preparations in QM

The set of all preparation procedures



Different density operators ρ

Different contexts

Preparation Noncontextuality

$$\mu_{\mathrm{P}}(\lambda) = \mu_\rho(\lambda)$$

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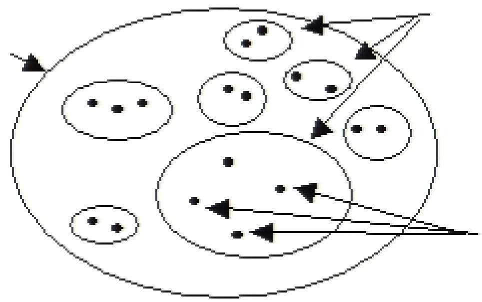

Different density operators ρ

Different contexts

Examples of contexts for mixed preparations:

Different convex decompositions of ρ

Many $\{p_j, |\psi_j\rangle\}$ such that
$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$

a.k.a. the ambiguity of mixtures

Preparation Noncontextuality

$$\mu_{\mathrm{P}}(\lambda) = \mu_\rho(\lambda)$$

Contexts for preparations in QM

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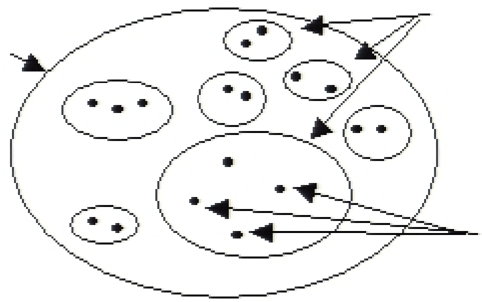

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Preparation Noncontextuality

$$\mu_{\mathsf{P}}(\lambda) = \mu_\rho(\lambda)$$

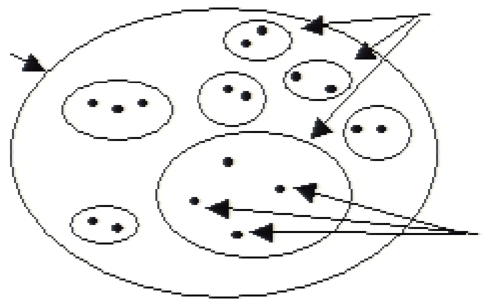
Different purifications of ρ

Many $|\Psi\rangle_{AB}$ such that

$\rho = \mathrm{Tr}_B(|\Psi\rangle_{AB}\langle\Psi|)$

Contexts for measurements in QM

The set of all measurement procedures



Different POVMs $\{E_k\}$

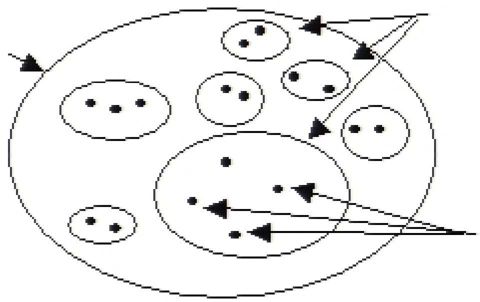
Different contexts

Measurement Noncontextuality

$$\xi_{\mathrm{M},j}(\lambda) = \xi_{\{E_k\},j}(\lambda)$$

Contexts for measurements in QM

The set of all measurement procedures



Different POVMs $\{E_k\}$

Different contexts

Examples of contexts for unsharp measurements:

Different convex decompositions of $\{E_k\}$

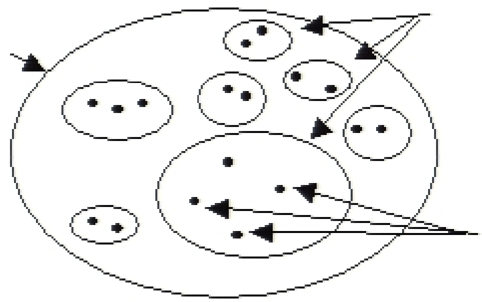
Many $\{p_j, \{E_k^j\}\}$ such that

$E_k = \sum_j p_j E_k^j$

Measurement Noncontextuality

$$\xi_{\mathrm{M},j}(\lambda) = \xi_{\{E_k\},j}(\lambda)$$

Contexts for measurements in QM



Examples of contexts for unsharp measurements:

Different convex decompositions of $\{E_k\}$

Many $\{p_j, \{E_k^j\}\}$ such that

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Different fine-grainings of $\{E_k\}$

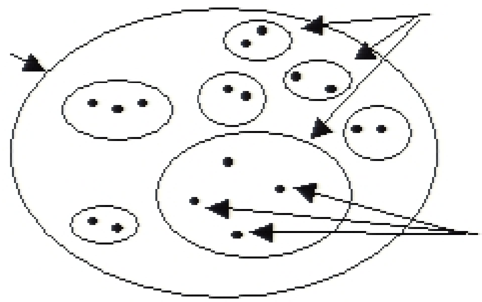
Many $\{E_{k,s}\}$ such that

$E_k = \sum_s E_{k,s}$

Measurement Noncontextuality

$$\xi_{\mathrm{M},j}(\lambda) = \xi_{\{E_k\},j}(\lambda)$$

Contexts for measurements in QM



Examples of contexts for unsharp measurements:

Different convex decompositions of $\{E_k\}$

Many $\{p_j, \{E_k^j\}\}$ such that

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Different fine-grainings of $\{E_k\}$

Many $\{E_{k,s}\}$ such that

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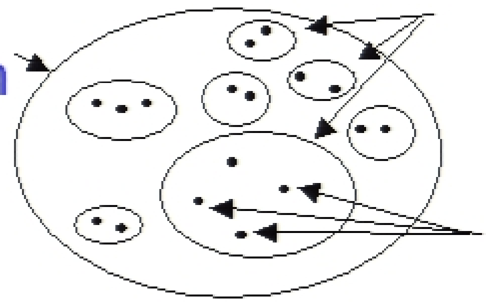
Different Neumark extensions of $\{E_k\}$

Measurement Noncontextuality

$$\xi_{\mathrm{M},j}(\lambda) = \xi_{\{E_k\},j}(\lambda)$$

Contexts for transformations in QM

The set of all transformation procedures

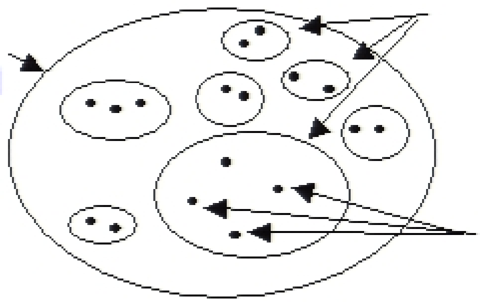


Transformation Noncontextuality

$$\Gamma_{\mathrm{T}}(\lambda', \lambda) = \Gamma_{\mathcal{T}}(\lambda', \lambda)$$

Contexts for transformations in QM

The set of all transformation procedures



Different CP maps $\mathcal{T}$

Different contexts

Examples of contexts for irreversible transformations:

Different convex decompositions of $\mathcal{T}$

Many $\{p_j, U_j\}$ such that

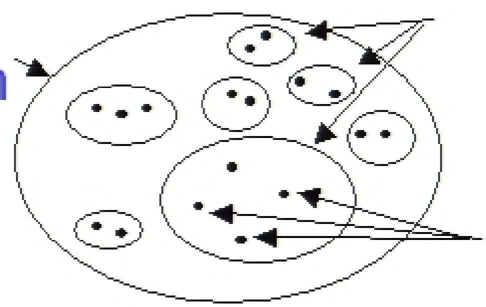
$$\mathcal{T}(\cdot) = \sum_j p_j U_j (\cdot) U_j^\dagger$$

Transformation Noncontextuality

$$\Gamma_{\mathrm{T}}(\lambda', \lambda) = \Gamma_{\mathcal{T}}(\lambda', \lambda)$$

Contexts for transformations in QM

The set of all transformation procedures



Different CP maps $\mathcal{T}$

Different contexts

Examples of contexts for irreversible transformations:

Different convex decompositions of $\mathcal{T}$

Many $\{p_j, U_j\}$ such that

$$\mathcal{T}(\cdot) = \sum_j p_j U_j (\cdot) U_j^{\dagger}$$

Different unitary extensions of $\mathcal{T}$

Transformation Noncontextuality

$$\Gamma_{T}(\lambda', \lambda) = \Gamma_{\mathcal{T}}(\lambda', \lambda)$$

Proof of preparation contextuality

(a preparation noncontextual ontological model is impossible)

Important features of ontological models

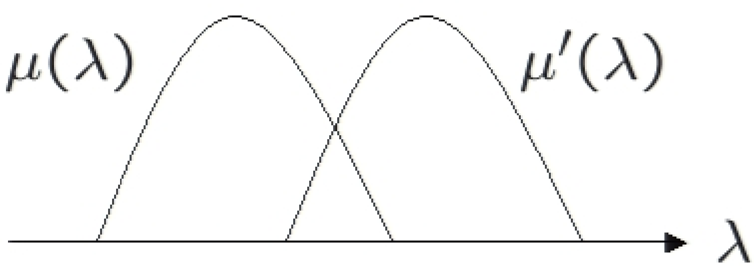
Let $\text{P} \leftrightarrow \mu(\lambda)$

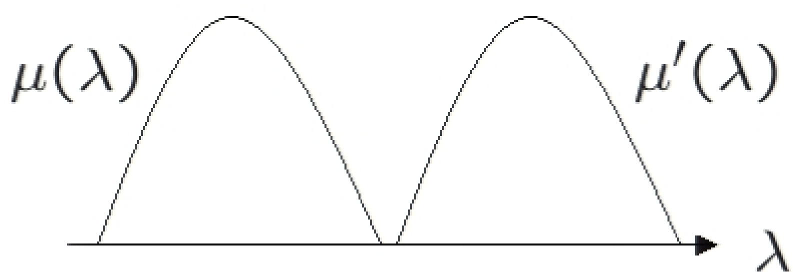
$\text{P}' \leftrightarrow \mu'(\lambda)$

Representing distinguishability:

If P and P′ are distinguishable with certainty

then $\mu(\lambda)\,\mu'(\lambda) = 0$





Representing convex combination:

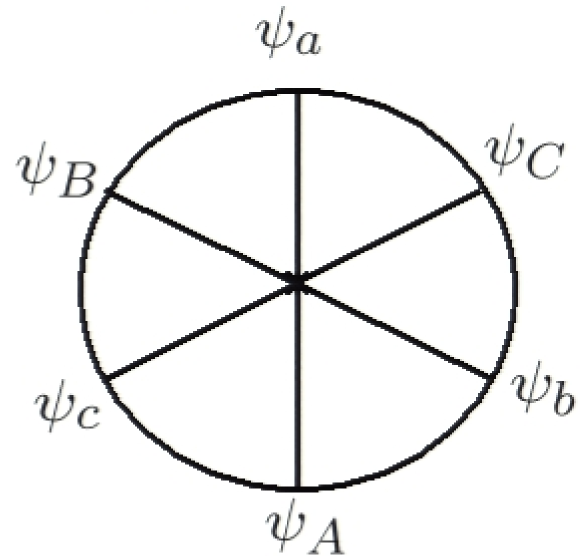
If $\text{P}'' = \text{P}$ with prob. p and P′ with prob. $1-p$

Then $\mu''(\lambda) = p\,\mu(\lambda) + (1-p)\,\mu'(\lambda)$

Proof of preparation contextuality in 2d

$$
\begin{aligned}
\mathsf{P}_a &\leftrightarrow \psi_a = (1,0) \\
\mathsf{P}_A &\leftrightarrow \psi_A = (0,1) \\
\mathsf{P}_b &\leftrightarrow \psi_b = (1/2, \sqrt{3}/2) \\
\mathsf{P}_B &\leftrightarrow \psi_B = (\sqrt{3}/2, -1/2) \\
\mathsf{P}_c &\leftrightarrow \psi_c = (1/2, -\sqrt{3}/2) \\
\mathsf{P}_C &\leftrightarrow \psi_C = (\sqrt{3}/2, 1/2)
\end{aligned}
$$

$$
\psi_a \perp \psi_A \\
\psi_b \perp \psi_B \\
\psi_c \perp \psi_C
$$



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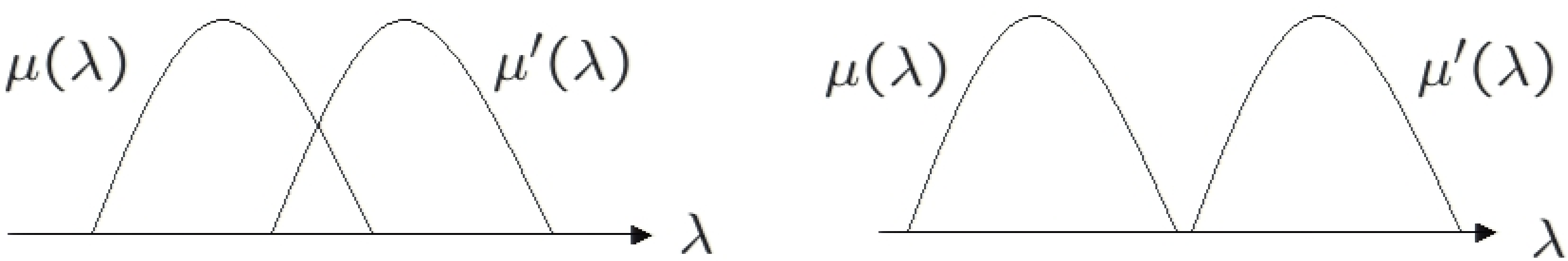

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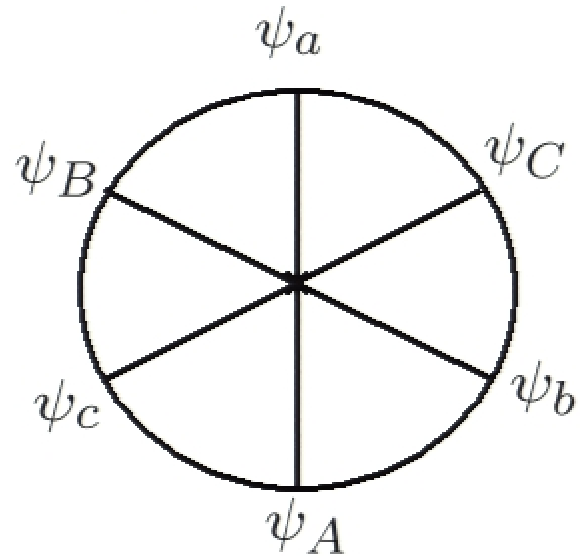
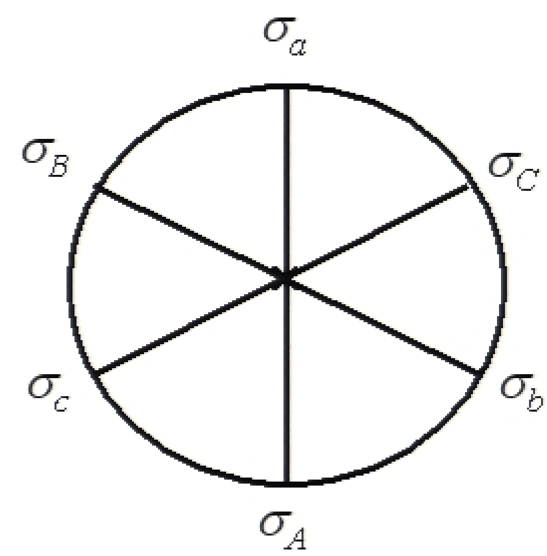
\psi_c &\perp \psi_C
\end{aligned}
$$



Proof of preparation contextuality in 2d

$$
\begin{aligned}
P_a &\leftrightarrow & \sigma_a &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
P_A &\leftrightarrow & \sigma_A &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
P_b &\leftrightarrow & \sigma_b &= \begin{pmatrix} \frac{1}{4} & \frac{1}{4}\sqrt{3} \\ \frac{1}{4}\sqrt{3} & \frac{3}{4} \end{pmatrix} \\
P_B &\leftrightarrow & \sigma_B &= \begin{pmatrix} \frac{3}{4} & -\frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{1}{4} \end{pmatrix} \\
P_c &\leftrightarrow & \sigma_c &= \begin{pmatrix} \frac{1}{4} & -\frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{3}{4} \end{pmatrix} \\
P_C &\leftrightarrow & \sigma_C &= \begin{pmatrix} \frac{3}{4} & \frac{1}{4}\sqrt{3} \\ \frac{1}{4}\sqrt{3} & \frac{1}{4} \end{pmatrix}
\end{aligned}
$$

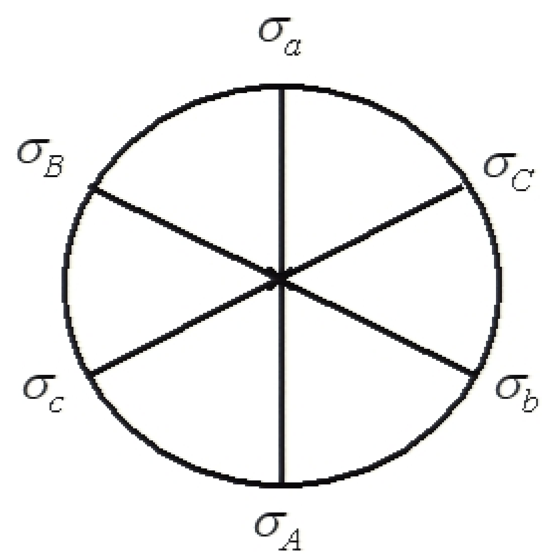
$$
\begin{aligned}
\sigma_a\sigma_A &= 0 \\
\sigma_b\sigma_B &= 0 \\
\sigma_c\sigma_C &= 0
\end{aligned}
$$



Proof of preparation contextuality in 2d

$$\begin{array}{lll}
\mathsf{P}_a & \leftrightarrow & \sigma_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
\mathsf{P}_A & \leftrightarrow & \sigma_A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
\mathsf{P}_b & \leftrightarrow & \sigma_b = \begin{pmatrix} \frac{1}{4} & \frac{1}{4}\sqrt{3} \\ \frac{1}{4}\sqrt{3} & \frac{3}{4} \end{pmatrix} \\
\mathsf{P}_B & \leftrightarrow & \sigma_B = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{1}{4} \end{pmatrix} \\
\mathsf{P}_c & \leftrightarrow & \sigma_c = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{3}{4} \end{pmatrix} \\
\mathsf{P}_C & \leftrightarrow & \sigma_C = \begin{pmatrix} \frac{3}{4} & \frac{1}{4}\sqrt{3} \\ \frac{1}{4}\sqrt{3} & \frac{1}{4} \end{pmatrix}
\end{array}$$

$$\begin{aligned}
\sigma_a \sigma_A &= 0 \\
\sigma_b \sigma_B &= 0 \\
\sigma_c \sigma_C &= 0
\end{aligned}$$



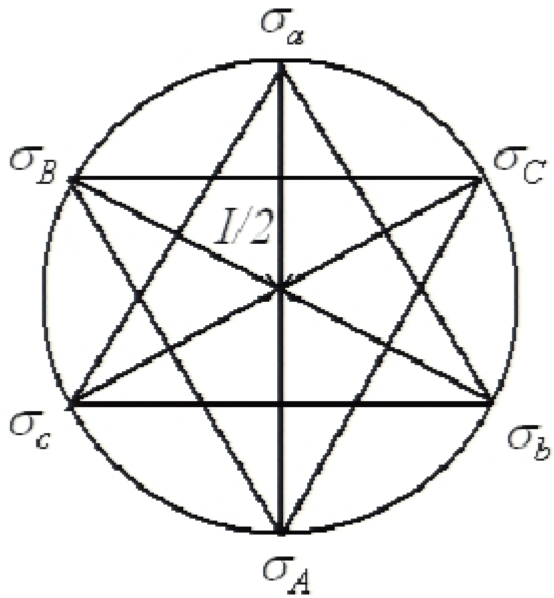
P_a and P_A are distinguishable with certainty
P_b and P_B are distinguishable with certainty
P_c and P_C are distinguishable with certainty

$$\longrightarrow \quad \begin{aligned}
\mu_a(\lambda)\,\mu_A(\lambda) &= 0 \\
\mu_b(\lambda)\,\mu_B(\lambda) &= 0 \\
\mu_c(\lambda)\,\mu_C(\lambda) &= 0
\end{aligned}$$

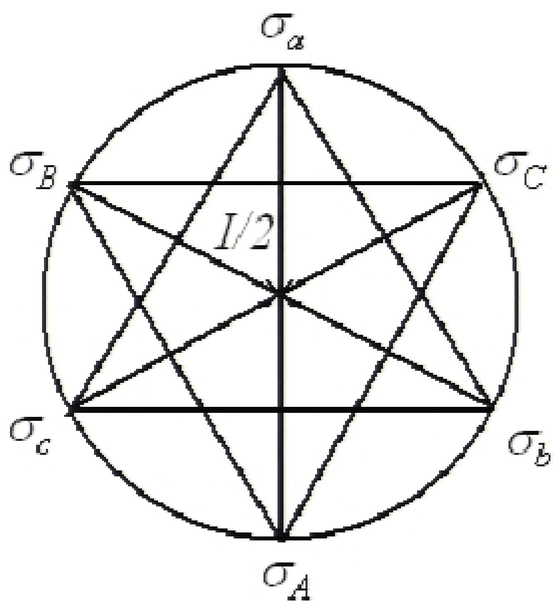
$$
\begin{aligned}
\mathsf{P}_{aA} &\equiv \mathsf{P}_a \text{ and } \mathsf{P}_A \text{ with prob. } 1/2 \text{ each} \\
\mathsf{P}_{bB} &\equiv \mathsf{P}_b \text{ and } \mathsf{P}_B \text{ with prob. } 1/2 \text{ each} \\
\mathsf{P}_{cC} &\equiv \mathsf{P}_c \text{ and } \mathsf{P}_C \text{ with prob. } 1/2 \text{ each} \\
\mathsf{P}_{abc} &\equiv \mathsf{P}_a,\ \mathsf{P}_b \text{ and } \mathsf{P}_c \text{ with prob. } 1/3 \text{ each} \\
\mathsf{P}_{ABC} &\equiv \mathsf{P}_A,\ \mathsf{P}_B \text{ and } \mathsf{P}_C \text{ with prob. } 1/3 \text{ each}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{P}_{aA} &\equiv \mathsf{P}_a \text{ and } \mathsf{P}_A \text{ with prob. } 1/2 \text{ each} \\
\mathsf{P}_{bB} &\equiv \mathsf{P}_b \text{ and } \mathsf{P}_B \text{ with prob. } 1/2 \text{ each} \\
\mathsf{P}_{cC} &\equiv \mathsf{P}_c \text{ and } \mathsf{P}_C \text{ with prob. } 1/2 \text{ each} \\
\mathsf{P}_{abc} &\equiv \mathsf{P}_a, \mathsf{P}_b \text{ and } \mathsf{P}_c \text{ with prob. } 1/3 \text{ each} \\
\mathsf{P}_{ABC} &\equiv \mathsf{P}_A, \mathsf{P}_B \text{ and } \mathsf{P}_C \text{ with prob. } 1/3 \text{ each}
\end{aligned}
$$

$$
\longrightarrow \quad
\begin{aligned}
\mu_{aA}(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\
\mu_{bB}(\lambda) &= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\
\mu_{cC}(\lambda) &= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\
\mu_{abc}(\lambda) &= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\
\mu_{ABC}(\lambda) &= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda)
\end{aligned}
$$

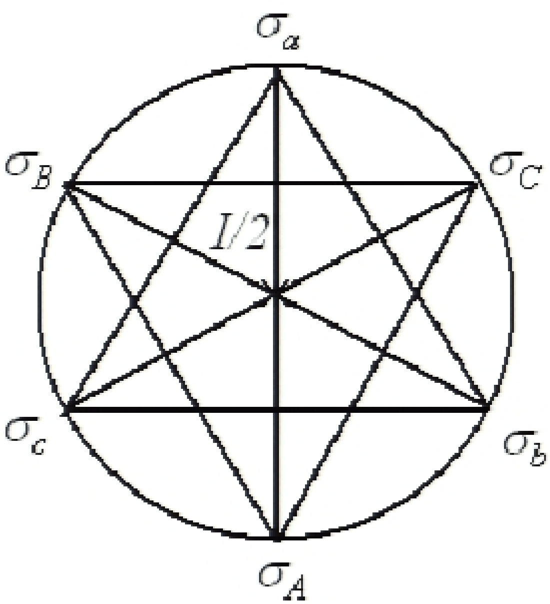


$$
\begin{aligned}
I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\
&= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
&= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\
&= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\
&= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.
\end{aligned}
$$



$$\begin{aligned} I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\ &= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\ &= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\ &= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\ &= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C. \end{aligned}$$

$$\mathsf{P}_{aA} \simeq \mathsf{P}_{bB} \simeq \mathsf{P}_{cC}$$
$$\simeq \mathsf{P}_{abc} \simeq \mathsf{P}_{ABC}$$



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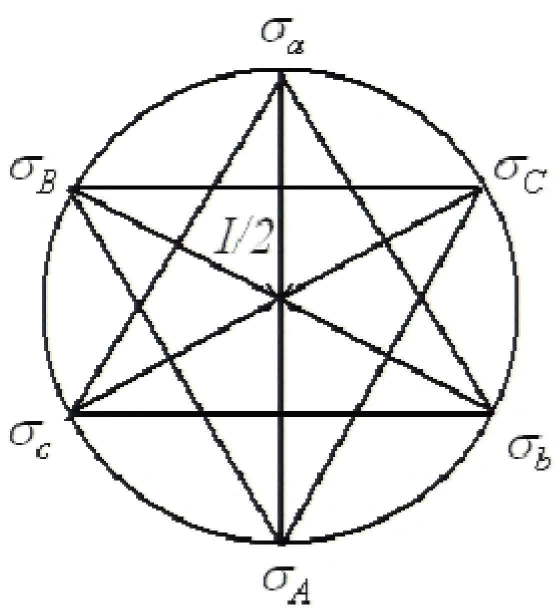
&= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\
&= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\
&= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{P}_{aA} \simeq \mathsf{P}_{bB} \simeq \mathsf{P}_{cC} \\
&\simeq \mathsf{P}_{abc} \simeq \mathsf{P}_{ABC}
\end{aligned}
$$

By preparation noncontextuality

$$
\begin{aligned}
&\mu_{aA}(\lambda) = \mu_{bB}(\lambda) = \mu_{cC}(\lambda) \\
&= \mu_{abc}(\lambda) = \mu_{ABC}(\lambda) \\
&\equiv \nu(\lambda)
\end{aligned}
$$



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\begin{aligned}
I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\
&= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
&= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\
&= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\
&= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{P}_{aA} \simeq \mathsf{P}_{bB} \simeq \mathsf{P}_{cC} \\
&\simeq \mathsf{P}_{abc} \simeq \mathsf{P}_{ABC}
\end{aligned}
$$

By preparation noncontextuality

$$
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&\mu_{aA}(\lambda) = \mu_{bB}(\lambda) = \mu_{cC}(\lambda) \\
&= \mu_{abc}(\lambda) = \mu_{ABC}(\lambda) \\
&\equiv \nu(\lambda)
\end{aligned}
$$

$$
\begin{aligned}
\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\
&= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\
&= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\
&= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\
&= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda).
\end{aligned}
$$

Our task is to find $\mu_a(\lambda)$, $\mu_A(\lambda)$, $\mu_b(\lambda)$, $\mu_B(\lambda)$, $\mu_c(\lambda)$, $\mu_C(\lambda)$, and $\nu(\lambda)$ such that

$$
\begin{aligned}
\mu_a(\lambda)\ \mu_A(\lambda) &= 0 \\
\mu_b(\lambda)\ \mu_B(\lambda) &= 0 \\
\mu_c(\lambda)\ \mu_C(\lambda) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\
&= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\
&= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\
&= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\
&= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda).
\end{aligned}
$$

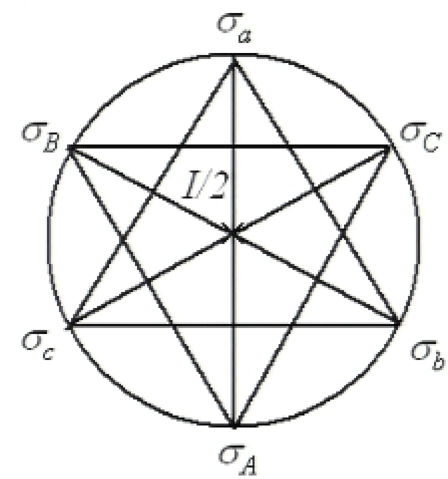
Our task is to find $\mu_a(\lambda)$, $\mu_A(\lambda)$, $\mu_b(\lambda)$, $\mu_B(\lambda)$, $\mu_c(\lambda)$, $\mu_C(\lambda)$, and $\nu(\lambda)$ such that

$$\begin{aligned}\mu_a(\lambda)\,\mu_A(\lambda) &= 0\\ \mu_b(\lambda)\,\mu_B(\lambda) &= 0\\ \mu_c(\lambda)\,\mu_C(\lambda) &= 0\end{aligned}$$

$$\begin{aligned}\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda)+\frac{1}{2}\mu_A(\lambda)\\ &= \frac{1}{2}\mu_b(\lambda)+\frac{1}{2}\mu_B(\lambda)\\ &= \frac{1}{2}\mu_c(\lambda)+\frac{1}{2}\mu_C(\lambda)\\ &= \frac{1}{3}\mu_a(\lambda)+\frac{1}{3}\mu_b(\lambda)+\frac{1}{3}\mu_c(\lambda)\\ &= \frac{1}{3}\mu_A(\lambda)+\frac{1}{3}\mu_B(\lambda)+\frac{1}{3}\mu_C(\lambda).\end{aligned}$$

i.e., paralleling the quantum structure:

$$\begin{aligned}\sigma_a\sigma_A &= 0\\ \sigma_b\sigma_B &= 0\\ \sigma_c\sigma_C &= 0\end{aligned}$$



$$\begin{aligned}I/2 &= \frac{1}{2}\sigma_a+\frac{1}{2}\sigma_A\\ &= \frac{1}{2}\sigma_b+\frac{1}{2}\sigma_B\\ &= \frac{1}{2}\sigma_c+\frac{1}{2}\sigma_C\\ &= \frac{1}{3}\sigma_a+\frac{1}{3}\sigma_b+\frac{1}{3}\sigma_c\\ &= \frac{1}{3}\sigma_A+\frac{1}{3}\sigma_B+\frac{1}{3}\sigma_C.\end{aligned}$$

Our task is to find $\mu_a(\lambda)$, $\mu_A(\lambda)$, $\mu_b(\lambda)$, $\mu_B(\lambda)$, $\mu_c(\lambda)$, $\mu_C(\lambda)$, and $\nu(\lambda)$ such that

$$
\begin{aligned}
\mu_a(\lambda)\,\mu_A(\lambda) &= 0 \\
\mu_b(\lambda)\,\mu_B(\lambda) &= 0 \\
\mu_c(\lambda)\,\mu_C(\lambda) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\
&= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\
&= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\
&= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\
&= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda)
\end{aligned}
$$

Our task is to find
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$\mu_B(\lambda)$, $\mu_c(\lambda)$, $\mu_C(\lambda)$,
and $\nu(\lambda)$ such that

$$\begin{aligned}\mu_a(\lambda)\,\mu_A(\lambda) &= 0\\ \mu_b(\lambda)\,\mu_B(\lambda) &= 0\\ \mu_c(\lambda)\,\mu_C(\lambda) &= 0\end{aligned}$$

$$\begin{aligned}\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda)\\ &= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda)\\ &= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda)\\ &= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda)\\ &= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda)\end{aligned}$$

Suppose

$$\begin{aligned}\mu_a(\lambda) &= 0\\ \mu_b(\lambda) &= 0\\ \mu_c(\lambda) &= 0\end{aligned}$$

Then we obtain
the all-zero solution

Our task is to find $\mu_a(\lambda)$, $\mu_A(\lambda)$, $\mu_b(\lambda)$, $\mu_B(\lambda)$, $\mu_c(\lambda)$, $\mu_C(\lambda)$, and $\nu(\lambda)$ such that

$$
\begin{aligned}
\mu_a(\lambda)\,\mu_A(\lambda) &= 0 \\
\mu_b(\lambda)\,\mu_B(\lambda) &= 0 \\
\mu_c(\lambda)\,\mu_C(\lambda) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\
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&= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\
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\end{aligned}
$$

Suppose

$$
\begin{aligned}
\mu_a(\lambda) &= 0 \\
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\end{aligned}
$$

Then we obtain the all-zero solution

Suppose

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\begin{aligned}
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$$

Then

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\begin{aligned}
\nu(\lambda) &= \frac{1}{3}\mu_c(\lambda) \\
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$$

Thus $\mu_c(\lambda) = 0$

Again yielding the all-zero solution

Our task is to find $\mu_a(\lambda)$, $\mu_A(\lambda)$, $\mu_b(\lambda)$, $\mu_B(\lambda)$, $\mu_c(\lambda)$, $\mu_C(\lambda)$, and $\nu(\lambda)$ such that

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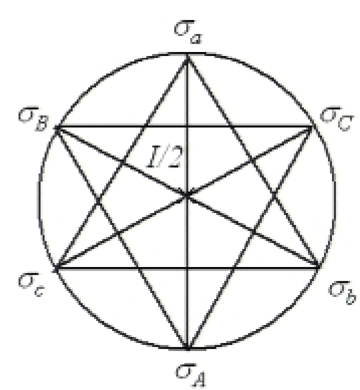
By symmetry, all other cases are similar



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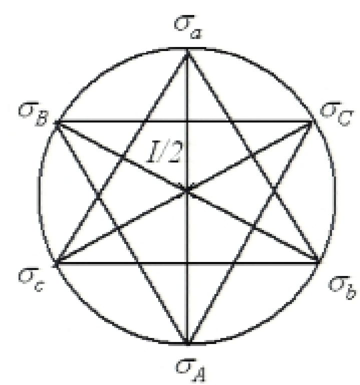

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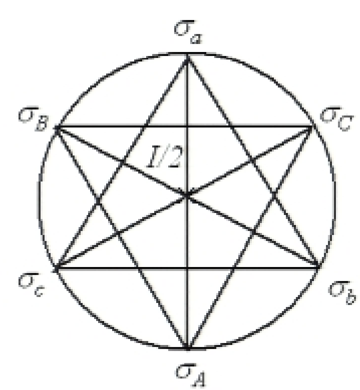


For all λ, we have the all-zero solution

CONTRADICTION

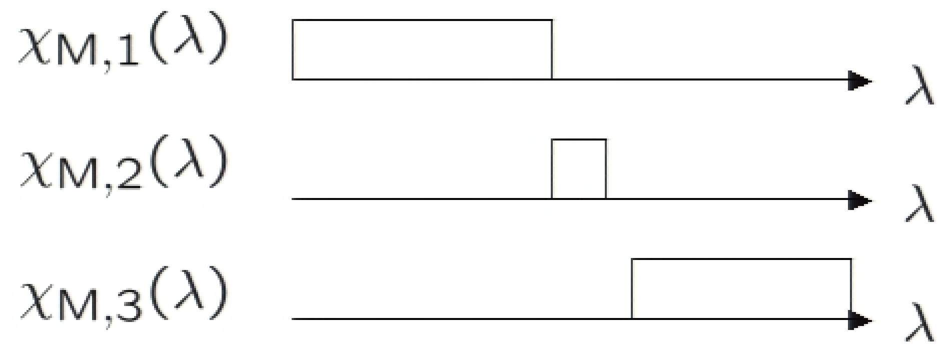
Proof of contextuality for unsharp measurements

(an unsharp-measurement noncontextual ontological model is impossible)

The assumption of outcome determinism for a measurement

$$\chi_{M,k} : \Omega \rightarrow 0 \text{ or } 1$$

$$\sum_k \chi_{M,k}(\lambda) = 1 \text{ for all } \lambda$$



In our language,
traditional notion of noncontextuality
= noncontextuality for sharp measurements
+ outcome determinism for sharp measurements

The traditional notion of noncontextuality concerns whether outcomes depend on the context
The generalized notion of noncontextuality concerns whether probabilities of outcomes depend on the context

Noncontextuality and determinism are separate issues

However,

Outcome determinism does not seem to be a natural assumption for unsharp measurements

Also, preparation noncontextuality implies outcome determinism for sharp measurements

Thus, no-go theorems for the traditional notion of noncontextuality are still no-go theorems for *universal noncontextuality*

Proof of contextuality for unsharp measurements in 2d

$$\mathsf{M}_a \leftrightarrow \{P_a, P_A\}$$
$$\mathsf{M}_b \leftrightarrow \{P_b, P_B\}$$
$$\mathsf{M}_c \leftrightarrow \{P_c, P_C\}$$

P_x projects onto ψ_x

$$P_a + P_A = I$$
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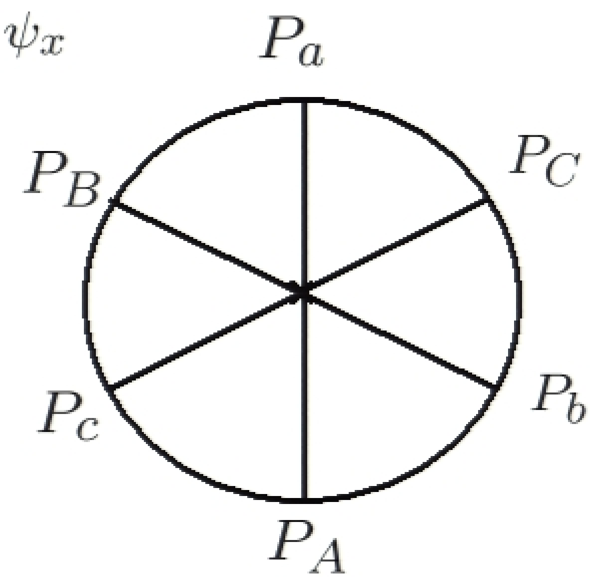

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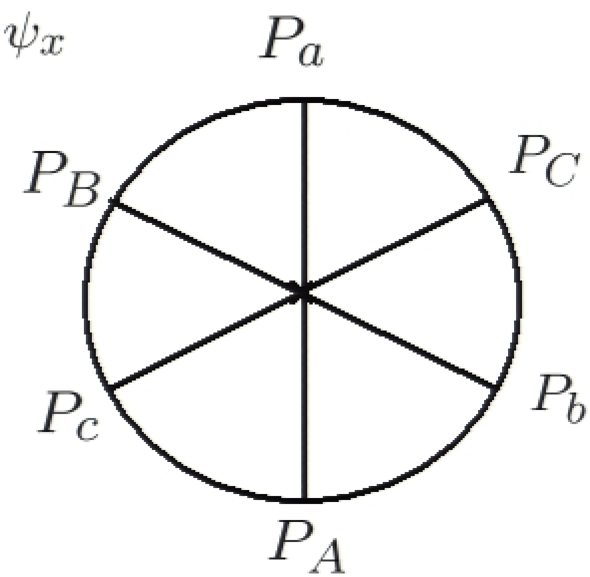

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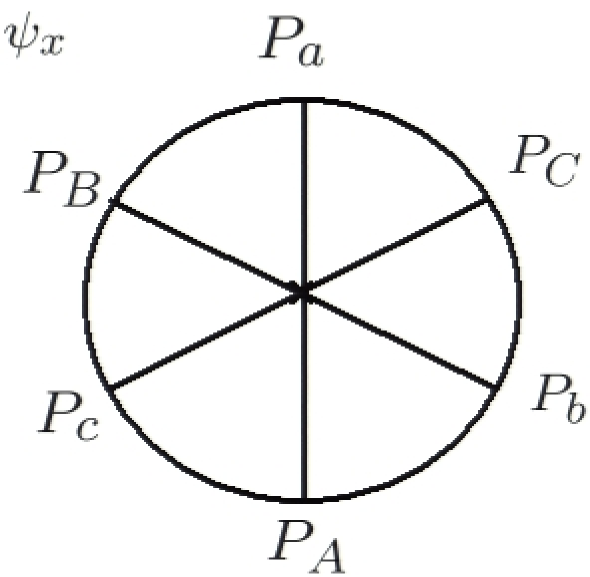

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M $\equiv$ implement one of M_a, M_b and M_c with prob. 1/3 each, register only whether first or second outcome ocurred

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By the assumption of measurement noncontextuality

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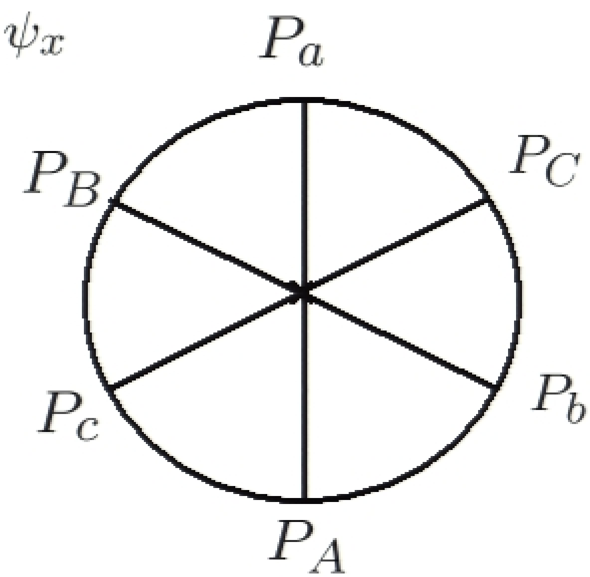

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But $\{0,1\}, \{\tfrac{1}{3}, \tfrac{2}{3}\}, \{1,0\}, \{\tfrac{2}{3}, \tfrac{1}{3}\} \neq \{\tfrac{1}{2}, \tfrac{1}{2}\}$

CONTRADICTION

Proof of transformation contextuality

(a transformation noncontextual ontological model is impossible)

It's similar…

Summary

We have generalized the notion of contextuality to:

(1) arbitrary operational theories
(2) preparations, transformations, and unsharp measurements
(3) indeterministic ontological models

We have provided proofs of contextuality for preparations, transformations, and unsharp measurements in quantum theory for a 2D Hilbert space

Relevance of these results to the hidden variable program

Perhaps surprisingly, these results do not provide support for the idea that the quantum state is a complete description of reality (just as Bell's theorem does not)

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Whether it is complete or not, quantum theory is still nonlocal

Whether it is complete or not, quantum theory is still (preparation) contextual

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- It complicates the explanation of the reproducibility of sharp measurements (particularly if one assumes that hidden variables of the apparatus affect the outcome)

- There is a tension between
  the dependence of representation on certain details of the experimental procedure
  and
  the independence of outcome statistics on those details of the experimental procedure

Relevance to axiomatics

A restriction on knowledge can reproduce qualitatively a vast array of quantum phenomena including
-Noncommutativity
-Interference
-No-cloning
-Features of entanglement
-…
(See Hardy quant-ph/9906123, Kirkpatrick quant-ph/0106072, Spekkens quant-ph/0401052)

Contextuality is one of the missing phenomena

Thus contextuality is a valuable clue for identifying the additional conceptual innovations required to derive quantum theory (if this is possible)

Relevance to quantum information

Many quantum information tasks that outperform their classical counterparts exist in local and noncontextual theories. Ex: key distribution, partially secure bit commitment, dense coding, etc.

However, Bell correlations are necessary to achieve better-than-classical results in certain communication complexity problems

Do any quantum information tasks rely on contextuality for their improvement over their classical counterpart?
-Random access codes? (See Galvao quant-ph/0212124)
-Quantum computation?